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DIFFUSION APPROXIMATION FOR A GI/G/m QUEUE WITH GROUP ARRIVALS

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Abstract This paper deals with the GI/G/m queueing system with group arrivals via diffusion approximation. We present approximate formulae for the distribution of the number of customers, the mean number of customers and the mean queue length. Moreover a modified approximation is considered to improve the accuracy of these formulae. The accuracy of the approximate formulae is numerically examined in some examples.

1. Introduction

This paper analyzes a many server queueing system with group arrivals denoted by $GI^X/G/m$ via diffusion approximation. The $GI^X/G/m$ system is of interest from the view point of practical applications. It is, however, extremely difficult to investigate such a system analytically. There are several studies about a certain subclass of $GI^X/G/m$ systems; see Chaudhry and Templeton [4]. The $M^X/M/m$ system was studied by Kabak [15] and Abol'nikov [1]. Cromie et al. [7] extended and corrected their results. Holman et al. [13] investigated the $E_K^X/M/m$ system and obtained the moments of the queue length. Baily and Neuts [3] and Baba [2] provided algorithmic methods for the $GI^X/M/m$ and $M^X/PH/m$ systems, respectively. Although these algorithmic methods give exact solutions, it often takes a very long time to execute computation. Therefore we need approximate methods such as diffusion approximation.

Let $Q(t)$ be the number of customers in the system at time t ($\geqq 0$). Then, roughly speaking, the diffusion approximation for $Q(t)$ is based on the idea that the discrete-state process $\{Q(t);\ t \geq 0\}$ can be approximated by an appropriate diffusion process $\{X(t);\ t \geq 0\}$. Since $Q(\cdot)$ should take nonnegative values, it is necessary to impose an impenetrable boundary at the origin of $X(\cdot)$. Usually, either the reflecting boundary (RB) or the elementary return boundary (ERB) has been used as such a boundary. There is a considerable amount of literature on diffusion approximations with these boundaries; see [6,8,11,12,18,20,21,22] for the RB, and [5,6,9,16,17] for the ERB. It is, however, known that diffusion approximations with the RB are only effective in heavy traffic, and that those with the ERB are essentially appropriate only for systems with Poisson arrivals.

Gelenbe [10] studied a diffusion process with the ERB where sojourn times have a Coxian distribution. In this paper, using this process, we will provide a diffusion approximation for queueing systems with general arrivals. In Section 2, we give a formulation by a diffusion process with the ERB. From this formulation, approximate formulae of some basic queueing characteristics are derived in Section 3. In Section 4, a modified approximation is considered to improve the accuracy of these formulae. Finally, the accuracy of these two diffusion approximations for the mean number of customers in the system is numerically examined in Section 5.

2. Formulation for the $GI^X/G/m$ System

We consider a many server queueing system with group arrivals denoted by $GI^X/G/m$. This queueing system is specified by the following assumptions. Customers arrive in groups of random sizes at a service facility with infinite waiting room. The group size X is a positive integer-valued i.i.d. random variable with a distribution $\{g_n,\ n = 1,2,\cdots\}$. The distribution $\{g_n\}$ has the mean γ (≥ 1) and the finite variance σ_g^2. The interarrival times T are general i.i.d. random variables with mean $1/\lambda$ and finite variance σ_a^2. Customers are served by one of m servers in order of arrivals. Their service times are also general i.i.d. random variables with the mean $1/\mu$ and the finite variance σ_s^2. Let $c_a = \lambda\sigma_a$, $c_s = \mu\sigma_s$ and $c_g = \sigma_g/\gamma$ be the coefficient of variation of interarrival times, service times and group sizes, respectively. Since we consider the system in the steady-state in this paper, we assume that $\rho = \lambda\gamma/m\mu < 1$.

Iglehart and Whitt [14] proved that the process $\{Q(t);\ t \geq 0\}$ in the unstable $GI^X/G/m$ system converges weakly to a Brownian motion process. It suggests the idea of an approximation for the stable queueing system, especially, in heavy traffic. Our diffusion approximation is based on the heavy traffic limit theorems.

We proceed to the diffusion approximation for the $GI^X/G/m$ system. Let T_n be the random variable with the Coxian distribution $\{\lambda_i, r_i\}_{i=1}^n$. The Coxian distribution $\{\lambda_i, r_i\}_{i=1}^n$ is represented as the distribution of a random time which a particle entered at point A spends in the network of Fig. 1. This network is composed of n phases at each of which, say i-th phase, the particle stays for an exponential distributed random time with the mean $1/\lambda_i$. Thereafter it leaves there and either enters the $(i+1)$-th phase with probability r_i or leaves the network with probability $1 - r_i$. We define $r_0 = 1$ and $r_n = 0$. Then the Laplace transform of T_n is given by

$$E[\exp(-sT_n)] = \sum_{i=1}^{n} (1 - r_i)u_i \prod_{k=1}^{i} \frac{\lambda_k}{\lambda_k + s}, \quad \mathrm{Re}(s) \geqq 0,$$

where

$$u_i = \prod_{k=0}^{i-1} r_k.$$

We can approximate the interarrival time T to T_n because the Coxian distribution can approximate a general distribution by means of matching its first K moments, where K can be arbitrarily large.

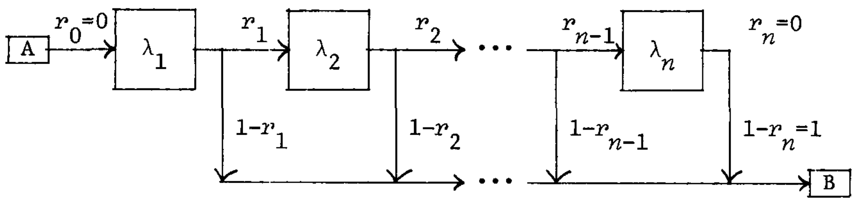

Fig. 1. Coxian distribution $\{\lambda_i, r_i\}_{i=1}^n$.

We next consider a diffusion process $\{X(t);\ t \geqq 0\}$ which approximates the process $\{Q(t);\ t \geqq 0\}$. The process $X(\cdot)$ is represented by two diffusion parameters $b(x)$ and $a(x)$ called infinitesimal mean and infinitesimal variance, respectively, which are defined as

$$b(x) = \lim_{\Delta t \downarrow 0} \frac{\mathrm{E}[X(t + \Delta t) - X(t) | X(t) = x]}{\Delta t},$$

and

$$a(x) = \lim_{\Delta t \downarrow 0} \frac{\mathrm{Var}[X(t + \Delta t) - X(t) | X(t) = x]}{\Delta t}.$$

Kimura and Ohsone [17] used

$$b(x) = \lambda\gamma - (\lceil x \rceil \wedge m)\mu,$$

and

$$a(x) = \lambda(\gamma^2 + \sigma_g^2) + (\lceil x \rceil \wedge m)\mu^3\sigma_s^2,$$

for the $M^X/G/m$ system, where $\wedge$ stands for minimum and $\lceil x \rceil$ denotes the smallest integer not smaller than x. The positive integer $\lceil x \rceil \wedge m$ corresponds to the number of busy servers. Considering the result of Chiamsiri and Leonard [5], we adopt

$$b(x) = \lambda\gamma - (\lceil x \rceil \wedge m)\mu, \tag{2.1}$$

and

$$a(x) = \lambda(c_a^2\gamma^2 + \sigma_g^2) + (\lceil x \rceil \wedge m)\mu c_s^2, \tag{2.2}$$

as the diffusion parameters for the $GI^X/G/m$ system. It is noted that both $b(x)$ and $a(x)$ are piecewise continuous functions having $(m - 1)$ first order discontinuity points.

Since $Q(\cdot)$ remains in the nonnegative region, it is necessary that we should impose a boundary at the origin of $X(\cdot)$. We adopt an ERB as the boundary at the origin, since this boundary is effective, especially, for queueing systems with group arrivals; see [6] and [17]. The trajectory of $X(\cdot)$ behaves as a free Brownian motion process on the open interval $(0, \infty)$. However, when it reaches the boundary at $x = 0$, it remains there for a random interval of time T_0 called a sojourn time at the origin. Thereafter, in the interval $(0, \infty)$ the trajectory jumps to a random point x whose p.d.f. is $f_0(x)$, and then starts from scratch. For the $GI^X/G/m$ system, the number of customers in the system increases instantaneously to k after an arrival of a new k-sized group to the empty queue. Therefore it is natural to define

$$(2.3) \qquad f_0(x) = \sum_{k=1}^{\infty} g_k \delta(x-k),$$

where $\delta(x-k)$ is Dirac's delta function concentrated at $x = k$.

We approximate the sojourn time T_0 to the stationary residual lifetime of the interarrival time T_n with the Coxian distribution $\{\lambda_i, r_i\}_{i=1}^{n}$. From the following proposition, it is found that the stationary residual lifetime distribution of the Coxian distribution $\{\lambda_i, r_i\}_{i=1}^{n}$ is the Coxian distribution $\{\lambda_i, R_i\}_{i=1}^{n}$ where R_i is a function of λ_j and r_j $(j = 1,2,\cdots,n)$.

Proposition. Let T_n be the random variable with the Coxian distribution $\{\lambda_i, r_i\}_{i=1}^{n}$, then the stationary residual lifetime of T_n has the Coxian distribution $\{\lambda_i, R_i\}_{i=1}^{n}$, where $R_0 = 1$, $R_n = 0$ and

$$R_i = \frac{\sum_{k=i+1}^{n} (u_k/\lambda_k)}{\sum_{k=i}^{n} (u_k/\lambda_k)}, \quad \text{for } i = 1,2,\cdots,n-1.$$

The proof is given in the Appendix.

Now we consider the diffusion process $\{X(t);\ t \geqq 0\}$ with the ERB, whose sojourn time T_n has the Coxian distribution $\{\lambda_i, R_i\}_{i=1}^{n}$, as the process expressing $\{Q(t);\ t \geqq 0\}$ approximately. As found in Gelenbe [10], the equations expressing the above process are given by

$$(2.4) \qquad \frac{\partial}{\partial t} p(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{a(x)p(x,t)\} - \frac{\partial}{\partial x}\{b(x)p(x,t)\} + \sum_{i=1}^{n} \lambda_i(1-R_i)P_i(t)f_0(x),$$

$$(2.5) \qquad \frac{d}{dt}P_1(t) = -\lambda_1 P_1(t) + \frac{1}{2}\frac{\partial}{\partial x}\{a(x)p(x,t)\} - b(x)p(x,t)\Big|_{x=0},$$

$$(2.6) \qquad \frac{d}{dt}P_i(t) = -\lambda_i P_i(t) + \lambda_{i-1}R_{i-1}P_{i-1}(t), \quad \text{for } i = 2,3,\cdots,n,$$

where $p(x,t)$ denotes the p.d.f. of $X(t)$ and $P_i(t)$ the probability that the trajectory of $X(t)$ is in the i-th phase of the sojourn time at the boundary at time t.

We consider the case that $X(t)$ is in the steady-state. Then the equations (2.4) $\sim$ (2.6) can be reduced to

$$(2.7)\qquad 0 = \frac{1}{2}\frac{d^2}{dx^2}\{a(x)p(x)\} - \frac{d}{dx}\{b(x)p(x)\} + \sum_{i=1}^{n}\lambda_i(1 - R_i)P_i f_0(x),$$

$$(2.8)\qquad 0 = -\lambda_1 P_1 + \frac{1}{2}\frac{d}{dx}\{a(x)p(x)\} - b(x)p(x)\Big|_{x=0},$$

$$(2.9)\qquad 0 = -\lambda_i P_i + \lambda_{i-1}R_{i-1}P_{i-1}, \quad \text{for } i = 2,3,\cdots,n,$$

where

$$p(x) = \lim_{t\uparrow\infty} p(x,t),$$

and

$$P_i = \lim_{t\uparrow\infty} P_i(t).$$

From the equation (2.9), we can derive

$$(2.10)\qquad \sum_{i=1}^{n}\lambda_i(1 - R_i)P_i = \sum_{i=1}^{n}\lambda_i P_i - \sum_{i=1}^{n}\lambda_i R_i P_i = \lambda_1 P_1,$$

and also we have for $i = 2,3,\cdots,n$

$$(2.11)\qquad P_i = \frac{\lambda_{i-1}R_{i-1}}{\lambda_i}\cdot\frac{\lambda_{i-2}R_{i-2}}{\lambda_{i-1}}\cdots\cdots\frac{\lambda_1 R_1}{\lambda_2}P_1 = \lambda_1 P_1 U_i/\lambda_i,$$

where

$$U_i = \prod_{j=0}^{i-1} R_j.$$

Let $\tilde{\pi}_0$ be the probability that the trajectory is at the boundary in the steady-state. Then it follows from (2.11) that

$$\tilde{\pi}_0 = \sum_{i=1}^{n} P_i = \lambda_1 P_1 \sum_{i=1}^{n}(U_i/\lambda_i).$$

Since $\Sigma_{i=1}^{n}(U_i/\lambda_i)$ is the mean of the stationary residual lifetime of T, we have

$$\sum_{i=1}^{n} (U_i/\lambda_i) = (1 + c_a^2)/2\lambda.$$

Thus we have

$$(2.12) \qquad \lambda_1 P_1 = 2\lambda\tilde{\pi}_0/(1 + c_a^2).$$

From (2.10) and (2.12), the equations (2.7) and (2.8) are finally reduced to

$$(2.13) \qquad \frac{1}{2}\frac{d^2}{dx^2}\{a(x)p(x)\} - \frac{d}{dx}\{b(x)p(x)\} = -\Lambda\tilde{\pi}_0 f_0(x),$$

$$(2.14) \qquad \frac{1}{2}\frac{d}{dx}\{a(x)p(x)\} - b(x)p(x)\Big|_{x=0} = \Lambda\tilde{\pi}_0,$$

where

$$\Lambda = 2\lambda/(1 + c_a^2).$$

It is noticed that these equations depend not on λ_i and R_i but only on the mean of the stationary residual lifetime of interarrival times.

3. A Solution of the Diffusion Equation

Before we proceed to solve (2.13) and (2.14), the normalizing condition and appropriate boundary conditions should be added:

$$(3.1) \qquad \tilde{\pi}_0 + \int_0^\infty p(x)dx = 1,$$

$$(3.2) \qquad \lim_{x \downarrow 0} p(x) = 0,$$

and

$$(3.3) \qquad \lim_{x \uparrow \infty} p(x) = 0.$$

Integrating (2.13) with respect to x and using (2.3) and (2.14), we have

$$(3.4) \qquad \frac{1}{2}\frac{d}{dx}\{a(x)p(x)\} - b(x)p(x) = \Lambda\tilde{\pi}_0\{1 - \sum_{k=1}^{\infty} g_k U(x-k)\},$$

where $U(\cdot)$ denotes the unit step function. From (2.1) and (2.2), we have the system of ordinary differential equations

$$(3.5)\qquad \frac{1}{2}a_k\frac{d}{dx}p_k(x) - b_kp_k(x) = \Lambda\tilde{\pi}_0\bar{g}_k, \quad \text{for } k = 1,2,\cdots,$$

where $a_k = a(k)$, $b_k = b(k)$, $\bar{g}_k = 1 - \Sigma_{i=1}^{k-1}g_i$ and $p_k(x)$ is the restriction of $p(x)$ to the interval $(k-1, k]$. For each k, a general solution of (3.5) is given by

$$(3.6)\qquad p_k(x) = \begin{cases} \tilde{\pi}_0\{C_k\exp(\frac{2b_k}{a_k}x) - \frac{\Lambda\bar{g}_k}{b_k}\}, & \text{if } b_k \neq 0 \\ \tilde{\pi}_0\{C_k + \frac{2\Lambda\bar{g}_k}{a_k}x\}, & \text{if } b_k = 0 \end{cases}$$

where C_k denotes an integration constant. In order to determine unknown integration constants, we impose the following conditions of the continuity of $p(x)$:

$$(3.7)\qquad p_k(k) = \lim_{x\downarrow k} p_{k+1}(x), \quad \text{for } k = 1,2,\cdots.$$

Let us denote $p_k(k)/\tilde{\pi}_0$ by q_k, then C_k is determined by (3.2) and (3.7):

$$C_k = \begin{cases} (q_k + \frac{\Lambda}{b_k}\bar{g}_k)\exp(-\frac{2b_k}{a_k}k), & \text{if } b_k \neq 0 \\ q_k - \frac{2\Lambda}{a_k}\bar{g}_k k, & \text{if } b_k = 0. \end{cases}$$

Therefore, we obtain

$$(3.8)\qquad p_k(x) = \begin{cases} \tilde{\pi}_0\left[(q_k + \frac{\Lambda}{b_k}\bar{g}_k)\exp\{\frac{2b_k}{a_k}(x-k)\} - \frac{\Lambda}{b_k}\bar{g}_k\right], & \text{if } b_k \neq 0 \\ \tilde{\pi}_0\{q_k + \frac{2\Lambda}{a_k}\bar{g}_k(x-k)\}, & \text{if } b_k = 0. \end{cases}$$

We can calculate $\{q_k\}$ by the recursive procedure:

$$q_0 = 0,$$

and

$$q_k = \begin{cases} (q_{k-1} + \frac{\Lambda}{b_k}\bar{g}_k)\exp(\frac{2b_k}{a_k}) - \frac{\Lambda}{b_k}\bar{g}_k, & \text{if } b_k \neq 0 \\ q_{k-1} + \frac{2\Lambda}{a_k}\bar{g}_k, & \text{if } b_k = 0. \end{cases}$$

If $b_k \neq 0$ for $k = 1,2,\cdots,m-1$, then $\tilde{\pi}_0$ is determined by (3.1):

$$(3.9) \qquad \tilde{\pi}_0 = \left[1 + \sum_{k=1}^{m-1} \left(\frac{a_k}{2b_k} - \frac{a_{k+1}}{2b_{k+1}} \right) q_k - \Lambda \left(\sum_{k=1}^{m} \frac{\bar{g}_k}{b_k} + \frac{1}{b_m}\{\gamma - m - \sum_{k=1}^{m-1}(k-m)g_k\} \right) \right]^{-1}.$$

When there exists a k such that $b_k = 0$, $\tilde{\pi}_0$ can be obtained by letting b_k tend to zero in (3.9).

Discretizing $p(x)$ provides approximate formulae for the distribution of the number of customers, the mean queue length and the mean number of customers in the system. Let Q denote the number of customers in the system in the steady-state and let $\pi_k = P\{Q = k\}$ $(k = 0,1,\cdots)$. Although there are several ways to discretize $p(x)$, we approximate π_k to $\tilde{\pi}_k = \int_{k-1}^{k} p_k(x)dx$ for $k = 1,2,\cdots$, and we do π_0 to $\tilde{\pi}_0$. Then, from (3.8), we have for $k = 1,2,\cdots$,

$$(3.10) \qquad \tilde{\pi}_k = \begin{cases} \tilde{\pi}_0\{\frac{a_k}{2b_k}(q_k - q_{k-1}) - \frac{\Lambda}{b_k}\bar{g}_k\}, & \text{if } b_k \neq 0 \\ \tilde{\pi}_0(q_k - \frac{\Lambda}{a_k}\bar{g}_k), & \text{if } b_k = 0. \end{cases}$$

Using $\tilde{\pi}_k$ $(k = 1,2,\cdots)$, we can obtain approximate formulae for the mean queue length $\tilde{E}[L]$ and the mean number of customers in the system $\tilde{E}[Q]$ as follows:

$$(3.11) \qquad \tilde{E}[L] = \sum_{k=m}^{\infty}(k-m)\tilde{\pi}_k$$

$$= -\tilde{\pi}_0 \left[\frac{a_m}{2b_m}\{1 - \exp(\frac{2b_m}{a_m})\}^{-1} q_m - \frac{\Lambda a_m}{2b_m^2}\{\gamma - m - \sum_{k=1}^{m-1}(k-m)g_k\} + \frac{\Lambda}{2b_m}\{\sigma_g^2 + (\gamma-m)(\gamma-m+1) - \sum_{k=1}^{m-1}(k-m)(k-m+1)g_k\} \right],$$

and

$$\text{(3.12)} \qquad \tilde{E}[Q] = \sum_{k=1}^{\infty} k\tilde{\pi}_k$$

$$= \tilde{E}[L] + \sum_{k=1}^{m-1} k\tilde{\pi}_k - \tilde{\pi}_0 m\Big(\frac{a_m}{2b_m} q_{m-1} + \frac{\Lambda}{b_m} \{\gamma - m + 1 - \sum_{k=1}^{m-1} (k-m+1) g_k\}\Big).$$

4. A Modified Diffusion Approximation

In this section we shall present a modified diffusion model for the $GI^X/G/m$ system. This model can be obtained by replacing $f_0(x)$ in (2.3) with the p.d.f.

$$\text{(4.1)} \qquad f_0(x) = \sum_{k=1}^{\infty} g_k \delta(x - k + \tfrac{1}{2}).$$

The above modification is based on an intuitive consideration that $x = k - \frac{1}{2}$ is more appropriate than $x = k$ as a representative point of $(k-1, k]$. From (2.13), we have

$$\text{(4.2)} \qquad \frac{1}{2}\frac{d^2}{dx^2}\{a(x)p(x)\} - \frac{d}{dx}\{b(x)p(x)\} = -\Lambda\tilde{\pi}_0 \sum_{k=1}^{\infty} g_k \delta(x - k + \tfrac{1}{2}).$$

Integrating (4.2) under the condition (2.14), it yields the system of ordinary differential equations

$$\text{(4.3)} \qquad \begin{aligned} \frac{1}{2} a_k \frac{d}{dx} p_k^-(x) - b_k p_k^-(x) &= \Lambda\tilde{\pi}_0 \bar{\bar{g}}_k, \\ \frac{1}{2} a_k \frac{d}{dx} p_k^+(x) - b_k p_k^+(x) &= \Lambda\tilde{\pi}_0 \bar{\bar{g}}_{k+1}, \end{aligned} \qquad \text{for } k = 1,2,\cdots,$$

where $p_k^-(x)$ and $p_k^+(x)$ are the restrictions of $p(x)$ in the interval $(k-1, k-0.5]$ and $(k-0.5, k]$, respectively. The general solutions of (4.3) are given by

$$p_k^-(x) = \begin{cases} \tilde{\pi}_0\{C_k^- \exp(\frac{2b_k}{a_k}x) - \frac{\Lambda \bar{g}_k}{b_k}\}, & \text{if } b_k \neq 0, \\ \tilde{\pi}_0\{C_k^- + \frac{2\Lambda \bar{g}_k}{a_k}x\}, & \text{if } b_k = 0, \end{cases}$$

and

$$p_k^+(x) = \begin{cases} \tilde{\pi}_0\{C_k^+ \exp(\frac{2b_k}{a_k}x) - \frac{\Lambda \bar{g}_{k+1}}{b_k}\}, & \text{if } b_k \neq 0, \\ \tilde{\pi}_0\{C_k^+ + \frac{2\Lambda \bar{g}_{k+1}}{a_k}x\}, & \text{if } b_k = 0, \end{cases}$$

where C_k^- and C_k^+ are integration constants. Using (3.2) and the continuity conditions

$$p_k^-(k - \frac{1}{2}) = \lim_{x \downarrow k - \frac{1}{2}} p_k^+(x), \quad \text{for } k = 1,2,\cdots,$$

and

$$p_k^+(k) = \lim_{x \downarrow k} p_{k+1}^-(x), \quad \text{for } k = 1,2,\cdots,$$

we have the recursive equations

$$q_k^- = \begin{cases} (q_{k-1}^+ + \frac{\Lambda}{b_k}\bar{g}_k)\exp(\frac{b_k}{a_k}) - \frac{\Lambda}{b_k}\bar{g}_k, & \text{if } b_k \neq 0, \\ q_{k-1}^+ + \frac{\Lambda}{a_k}\bar{g}_k, & \text{if } b_k = 0, \end{cases}$$

$$q_k^+ = \begin{cases} (q_k^- + \frac{\Lambda}{b_k}\bar{g}_{k+1})\exp(\frac{b_k}{a_k}) - \frac{\Lambda}{b_k}\bar{g}_{k+1}, & \text{if } b_k \neq 0, \\ q_k^- + \frac{\Lambda}{a_k}\bar{g}_{k+1}, & \text{if } b_k = 0, \end{cases}$$

$$q_0^+ = 0,$$

where $q_k^- = q_k^-(k - \frac{1}{2})$, $q_k^+ = q_k^+(k)$, $q_k^-(x) = p_k^-(x)/\tilde{\pi}_0$ and $q_k^+(x) = p_k^+(x)/\tilde{\pi}_0$. Moreover we have

$$q_k^-(x) = \begin{cases} (q_{k-1}^+ + \frac{\Lambda}{b_k}\bar{g}_k)\exp\{\frac{2b_k}{a_k}(x-k+1)\} - \frac{\Lambda}{b_k}\bar{g}_k, & \text{if } b_k \neq 0, \\ q_{k-1}^+ + \frac{2\Lambda}{a_k}\bar{g}_k(x-k+1), & \text{if } b_k = 0, \end{cases}$$

$$q_k^+(x) = \begin{cases} (q_k^- + \frac{\Lambda}{b_k}\bar{g}_{k+1})\exp\{\frac{2b_k}{a_k}(x-k+\frac{1}{2})\} - \frac{\Lambda}{b_k}\bar{g}_{k+1}, & \text{if } b_k \neq 0, \\ q_k^- + \frac{2\Lambda}{a_k}\bar{g}_{k+1}(x-k+\frac{1}{2}), & \text{if } b_k = 0, \end{cases}$$

and

$$\phi_k^- = \int_{k-1}^{k-\frac{1}{2}} q_k^-(x)dx$$

$$= \begin{cases} \frac{a_k}{2b_k}(q_k^- - q_{k-1}^+) - \frac{\Lambda}{2b_k}\bar{g}_k, & \text{if } b_k \neq 0, \\ \frac{1}{4}(q_k^- + q_{k-1}^+), & \text{if } b_k = 0, \end{cases}$$

$$\phi_k^+ = \int_{k-\frac{1}{2}}^{k} q_k^+(x)dx$$

$$= \begin{cases} \frac{a_k}{2b_k}(q_k^+ - q_k^-) - \frac{\Lambda}{2b_k}\bar{g}_{k+1}, & \text{if } b_k \neq 0, \\ \frac{1}{4}(q_k^+ + q_k^-), & \text{if } b_k = 0, \end{cases}$$

in the similar manner as in Section 3. Therefore we have a modified approximate formulae for the distribution of the number of customers in the system:

$$\tilde{\pi}_k = \tilde{\pi}_0(\phi_k^- + \phi_k^+)$$

$$= \begin{cases} \tilde{\pi}_0\{\frac{a_k}{2b_k}(q_k^+ - q_{k-1}^+) - \frac{\Lambda}{2b_k}(\bar{g}_{k+1} + \bar{g}_k)\}, & \text{if } b_k \neq 0, \\ \{\tilde{\pi}_0(q_k^+ + 2q_k^- + q_{k-1}^+)\}/4, & \text{if } b_k = 0, \end{cases}$$

where $\tilde{\pi}_0$ is determined by the normalizing condition

$$\sum_{k=0}^{\infty} \tilde{\pi}_k = 1.$$

Using $\{\tilde{\pi}_k\}$, we can obtain the modified formulae for the mean queue length and the mean number of customers in the system.

5. Numerical Examples

To examine the accuracy of the diffusion approximations, we shall numerically compare them with the exact solutions for the mean number of customers. For some $E_2^X/M/m$ systems, Tables 1 ∿ 4 show

EXACT : exact solutions,

DA : the diffusion approximations in Section 3,

and

MDA : the modified diffusion approximations in Section 4.

For notational convenience, we use in these tables the symbol $G(k)$ instead of X to denote the geometric distribution with mean k.

It is found from these tables that the MDA is more accurate than the DA for most cases, especially, when the number of servers is small and the mean group size is large. The MDA is, however, little effective when the number of servers is large. We can further observe that the accuracy of both DA and the MDA becomes much better as the mean group size decreases.

Table 1. The mean number of customers in the $E_2^{G(2)}/M/5$ system.

ρ	EXACT	DA	relative error(%)	MDA	relative error(%)
0.1	0.509	0.816	(60.314)	0.643	(26.326)
0.2	1.033	1.486	(43.853)	1.254	(21.394)
0.3	1.590	2.105	(32.390)	1.856	(16.730)
0.4	2.209	2.738	(23.947)	2.490	(12.721)
0.5	2.945	3.460	(17.487)	3.221	(9.372)
0.6	3.907	4.392	(12.414)	4.165	(6.604)
0.7	5.347	5.793	(8.341)	5.578	(4.320)
0.8	8.018	8.421	(5.026)	8.218	(2.494)
0.9	15.671	16.030	(2.291)	15.839	(1.072)
0.95	30.742	31.079	(1.096)	30.893	(0.491)
0.98	75.782	76.106	(0.428)	75.923	(0.186)

Table 2. The mean number of customers in the $E_2^{G(2)}/M/10$ system.

ρ	EXACT	DA	relative error(%)	MDA	relative error(%)
0.1	1.001	1.433	(43.157)	1.236	(23.477)
0.2	2.004	2.550	(27.246)	2.356	(17.565)
0.3	3.018	3.580	(18.622)	3.423	(13.419)
0.4	4.065	4.599	(13.137)	4.479	(10.185)
0.5	5.191	5.671	(9.247)	5.584	(7.571)
0.6	6.501	6.912	(6.322)	6.848	(5.338)
0.7	8.248	8.578	(4.001)	8.533	(3.455)
0.8	11.184	11.430	(2.200)	11.398	(1.913)
0.9	19.065	19.228	(0.855)	19.206	(0.740)
0.95	34.236	34.359	(0.359)	34.341	(0.307)
0.98	79.334	79.433	(0.125)	79.417	(0.105)

Table 3. The mean number of customers in the $E_2^{G(4)}/M/5$ system.

ρ	EXACT	DA	relative error(%)	MDA	relative error(%)
0.1	0.606	1.112	(83.498)	0.781	(28.878)
0.2	1.267	2.109	(66.456)	1.535	(21.152)
0.3	2.026	3.116	(53.801)	2.340	(15.499)
0.4	2.945	4.232	(43.701)	3.279	(11.341)
0.5	4.134	5.587	(35.148)	4.474	(8.224)
0.6	5.809	7.407	(27.509)	6.147	(5.819)
0.7	8.470	10.201	(20.437)	8.804	(3.943)
0.8	13.620	15.475	(13.620)	13.948	(2.408)
0.9	28.758	30.733	(6.868)	29.084	(1.134)
0.95	58.824	60.856	(3.454)	59.148	(0.551)
0.98	148.862	150.928	(1.388)	149.186	(0.218)

Table 4. The mean number of customers in the $E_2^{G(4)}/M/10$ system.

ρ	EXACT	DA	relative error(%)	MDA	relative error(%)
0.1	1.030	1.495	(45.146)	1.302	(26.408)
0.2	2.095	2.741	(30.835)	2.466	(17.709)
0.3	3.231	3.914	(21.139)	3.600	(11.421)
0.4	4.498	5.128	(14.006)	4.793	(6.558)
0.5	6.003	6.528	(8.746)	6.181	(2.965)
0.6	7.965	8.353	(4.871)	7.998	(0.414)
0.7	10.884	11.121	(2.178)	10.761	(-1.130)
0.8	16.265	16.346	(0.498)	15.982	(-1.740)
0.9	31.611	31.538	(-0.231)	31.169	(-1.398)
0.95	61.772	61.623	(-0.241)	61.253	(-0.840)
0.98	151.865	151.672	(-0.127)	151.301	(-0.371)

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Appendix

Proof of Proposition in Section 2

Let T_r and T_R be the random variables with the Coxian distributions $\{\lambda_i, r_i\}_{i=1}^n$ and $\{\lambda_i, R_i\}_{i=1}^n$, respectively, and let T_r° be the stationary residual lifetime of T_r. Furthermore, denote the p.d.f.s of T_r, T_r° and T_R by $g_r(x)$, $g_r^\circ(x)$ and $g_R(x)$, respectively, and denote their Laplace transform by $\xi_r(s)$, $\xi_r^\circ(s)$ and $\xi_R(s)$ ($\mathrm{Re}(s) \geqq 0$), respectively. From the definition of R_i, we have

$$\text{(A.1)} \qquad \xi_R(s) = \sum_{i=1}^n U_i(1 - R_i) \prod_{k=1}^i \frac{\lambda_k}{\lambda_k + s}$$

$$= \frac{\lambda_1}{\lambda_1 + s} - \left(\sum_{i=2}^n \frac{U_i}{\lambda_i} \prod_{k=1}^i \frac{\lambda_k}{\lambda_k + s} \right) s$$

$$= \left(\prod_{i=1}^n \frac{\lambda_i}{\lambda_i + s} \right) \left\{ \prod_{k=2}^n \left(1 + \frac{s}{\lambda_k}\right) - \sum_{i=2}^n \frac{U_i}{\lambda_i} \prod_{k=i+1}^n \left(1 + \frac{s}{\lambda_k}\right) s \right\},$$

where $U_i = \Pi_{k=0}^{i-1} R_k$, for $i = 1,2,\cdots,n$.

Let $\Lambda_n(i,j)$ ($i \geqq 1$, $j \geqq 0$) be the coefficient of s^j in the polynomial $\Pi_{k=i}^n (1 + \frac{s}{\lambda_k})$, that is,

$$\prod_{k=i}^n \left(1 + \frac{s}{\lambda_k}\right) = \sum_{j=0}^{n-i+1} \Lambda_n(i,j) s^j.$$

Then, it is easily verified that the following equations hold:

$$\Lambda_n(i,0) = 1,$$

$$\Lambda_n(i,j) - \Lambda_n(i+1,j) = \frac{1}{\lambda_i} \Lambda_n(i+1,j-1), \qquad \text{for } i = 1,2,\cdots,n-j,$$

and

$$\Lambda_n(i,j) = \frac{1}{\lambda_i} \Lambda_n(i+1,j-1), \qquad \text{for } i = n-j+1.$$

Using these equations, we can rewrite (A.1) as

$$\text{(A.2)} \qquad \xi_R(s) = \zeta_n(s)\{\sum_{j=0}^{n-1}\Lambda_n(2,j)s^j - \sum_{i=2}^{n}\frac{U_i}{\lambda_i}\sum_{j=0}^{n-i}\Lambda_n(i+1,j)s^{j+1}\}$$

$$= \zeta_n(s)\left[1 + \sum_{j=1}^{n-1}\{\Lambda_n(2,j) - \sum_{i=2}^{n-j+1}\frac{U_i}{\lambda_i}\Lambda_n(i+1,j-1)\}s^j\right],$$

where

$$\zeta_n(s) = \prod_{i=1}^{n}\frac{\lambda_i}{\lambda_i + s}.$$

On the other hand, we have

$$\text{(A.3)} \qquad \xi_r^\circ(s) = \frac{1 - \xi_r(s)}{s(1/\lambda)}$$

$$= \lambda\{\frac{1}{\lambda_1 + s} + \sum_{i=2}^{n}\frac{u_i}{\lambda_i}\prod_{k=1}^{i}\frac{\lambda_k}{\lambda_k + s}\}$$

$$= \lambda\zeta_n(s)\sum_{i=1}^{n}\frac{u_i}{\lambda_i}\prod_{k=i+1}^{n}(1 + \frac{s}{\lambda_k})$$

$$= \zeta_n(s)\left[1 + \lambda\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\frac{u_i}{\lambda_i}\Lambda_n(i+1,j)s^j\right].$$

From $U_i = \lambda\Pi_{k=i}^{n}(u_k/\lambda_k)$, we obtain

$$\text{(A.4)} \qquad \Lambda_n(2,j) - \sum_{i=2}^{n-j+1}\frac{U_i}{\lambda_i}\Lambda_n(i+1,j-1)$$

$$= \lambda\sum_{i=1}^{n-j}\Lambda_n(i+1,j)\sum_{k=i}^{n}\frac{u_k}{\lambda_k} - \lambda\sum_{i=2}^{n-j+1}\Lambda_n(i,j)\sum_{k=i}^{n}\frac{u_k}{\lambda_k}$$

$$= \lambda\sum_{i=1}^{n-j}\Lambda_n(i+1,j)\frac{u_i}{\lambda_i}.$$

Consequently, from (A.2), (A.3) and (A.4), we obtain

$$\xi_r^\circ(s) = \xi_R(s).$$

This completes the proof.□

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