Journal of the Operations Research
Society of Japan
Vol. 27, No. 1, March 1984

ON THE IMPACT OF UNCERTAINTY IN DYNAMIC JOB SEARCH

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(Received September 27, 1982: Revised August 23, 1983)

Abstract The object of this paper is to investigate the impact of uncertainty in a dynamic job search model where the states of the economy follow a Markov chain and the cost of search does not only depend on the state but on the period. Especially, we explore whether or not increasing the uncertainty about the states of the economy, the wage distribution, the number of job offered at a time and other search environments is beneficial to the searcher.

1. Introduction

In this paper we consider a dynamic job search model where the states of the economy follow a Markov chain and the cost of search does not only depend on the state but on the period. Most of papers [1], [5], [7], [8], [10], related to job search models, are actually one period or stationary with constant search cost and identical wage distribution. This paper treats with a dynamic version of [7] and especially emphasizes the impact of uncertainty on the state of the economy, the wage distribution, the number of jobs offered at a time and other search environments. First of all, analytic properties of an optimal search strategy are investigated. Next, we explore whether increasing the riskiness of search environments is beneficial or not to the searcher.

The paper [7] proves in the job search of the one period model that (i) an optimal policy is myopic and has a reservation wage, (ii) increasing the riskiness of the wage offer distribution is beneficial to the searcher, (iii) increasing the number of job offers tendered per period is detrimental to the searcher. This paper shows in the job search model of a dynamic economy that an optimal policy is not myopic but has a reservation wage which increases as the economy is improved, and that the above properties of (ii), (iii) still hold true. Furthermore, we give an explicit form of reservation wage and show that the expected duration of search is independent of the uncertainty on wage when wages are identically and independently normally distributed.

2. A Dynamic Job Search Model

Suppose that an individual, called the searcher, is seeking a job opportunity over discrete time periods $n, n=1,2,\ldots,N$. At the beginning of each period he is offered a job and then has to make a decision of whether he accepts the offer and stops searching or continues to search. He must also pay the cost of search at each period. On the other hand, the state of the economy changes according to a Markov chain. At each period the probability that the searcher receives an offer of the wage is independent of all past offers and of time, but depends on the state of last period. We use the following notations:

$c_n(i)$ = the cost of search at period n when the economy is in state i,

P_{ij} = the state transition probabilities,

X_n = the wage of a job offered at period n,

$F_i(\cdot)$ = the probability distribution of X_{n+1} when the economy is in state i at period n,

β = a discount factor, $0 \leq \beta < 1$,

where $n=1,2,\ldots,N, i,j=1,2,\ldots,S$. We assume that no recall is allowed.

Definition. The following stopping rule is called a reservation wage policy:

(1) Stop searching if $X_n > R_n(i)$

Continue to search, otherwise.

Such values $R_n(i)$ characterizing the above policy are called reservation wages. A job offer, or simply a wage, should be interpreted as the discounted present value of the searcher's life time earnings from a job. Once he accepts a job offered, the search process terminates and he stays in the permanent position of employment. Let $V_n(i,x)$ be the maximum value of the discounted expected return attainable from period n on when the economy is in state i and the current job offer is x at period n. It follows from the principle of optimality that $V_n(i,x)$ satisfies the recursive equation

$$V_n(i,x) = \max\{x, -c_n(i) + \beta \sum_{j=1}^{S} P_{ij} \int_0^\infty V_{n+1}(j,y)\,dF_i(y)\} \tag{2}$$

$$\equiv \max\{x, R_n(i)\}$$

where

$$(3)\qquad R_n(i) = -c_n(i) + \beta \sum_{j=1}^{S} p_{ij}\int_0^\infty V_{n+1}(j,y)dF_i(y)$$

and $\quad V_{N+1}(\cdot,\cdot) \equiv 0,\ n=N,\ldots,2,1.$

It follows from (1) that an optimal policy is the reservation wage policy given by (1). Note in this model that the reservation wage $R_n(i)$ depends on i and the remaining periods $(N-n)$ and hence the optimal policy is neither myopic nor one period look ahead.

Assumptions. (A) $F_1(x) \geq F_2(x) \geq \cdots \geq F_S(x)$ for all x.

(B) $c_n(i)$ is decreasing in i and increasing in n.

(C) $\sum_{j=k}^{S} p_{ij}$ is increasing in i for each k.

Remarks: Suppose that the higher number of the states implies the better states of the economy, that is, the state 1 is the worst and the state S is the best. The assumption (A) can be interpreted as follows: the better state stochastically dominates the worse one in terms of wage probability distribution. The assumption (B) is self-explaining. The assumption (C) asserts that the probability of a transition into any subset of the states $\{k,k+1,\ldots,S\}$ starting from the better state is larger than the probability of a transition starting from the worse state.

Theorem 1. Under assumptions (A), (B), (C) the following statements hold.

(i) $R_n(i)$ is increasing in i and decreasing in n.

(ii) $V_n(i,x)$ is increasing in i.

(iii) $V_n(i,x)$ is piecewise linear, increasing and convex in x.

Proof: The proof is by induction on n. For $n=N$ the statements (i) and (ii) certainly hold. Assume that $V_{n+1}(j,y)$ is increasing in j.

$$R_n(i) = -c_n(i) + \beta\int_0^\infty \sum_{k=1}^{S} [V_{n+1}(k,y) - V_{n+1}(k-1,y)] \sum_{j=k}^{S} p_{ij}dF_i(y)$$

which is the sum of increasing functions of i. $R_n(i)$ is increasing in i. Hence, $V_n(i,x)$ is also increasing in i. Similarly, it can be shown that $R_n(i)$ is decreasing in n. The statement (iii) follows from the fact that $V_n(i,x)$ is actually the maximum of linear and constant functions. (Q.E.D.)

Theorem 2. Assume that $c_n(i) = c(i)$ for all n. Then $\lim_{N\to\infty} V_n(i,x) \equiv V(i,x)$ exists and satisfies the recursive equation

$$(4)\qquad \begin{aligned} V(i,x) &= \max\{x,\ -c(i) + \beta\sum_j p_{ij}\int_0^\infty V(j,y)\,dF_i(y)\} \\ &\equiv \max\{x,\ R(i)\} \end{aligned}$$

where $R(i)$ is also the limit of $R_n(i)$ in equation (3). Furthermore, $R(i)$ is increasing in i and $V(i,x)$ is increasing in i and convex in x.

The proof is straightforward and then is omitted.

3. The Impact of Uncertainty

It may be often too expensive or hard to estimate the true values of transition probability P_{ij}, wage distribution $F_i(\cdot)$ and the cost of search $c_n(i)$. Such estimates of P_{ij}, $F_i(\cdot)$ and $c_n(i)$ inevitably include some errors. It is, therefore, interesting and important to pay some attention to the impact of uncertainty about probability distributions P_{ij}, $F_i(\cdot)$ and $c_n(i)$. It is obvious from theorem 1 that the better state of the economy the higher reservation wage, and the shorter the remaining periods the lower reservation wage. How about the searcher's benefit if uncertainty increases? First of all, to investigate the effect of uncertainty about transition probabilities we write $P^1 > P^2$ whenever $\sum_{j=k}^{S} P^1_{ij} \geq \sum_{j=k} P^2_{ij}$ for all k and define

$$(5) \qquad \begin{aligned} V_n(i,x,P) &= \max\{x,\ -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,P)dF_i(y)\} \\ &\equiv \max\{x,\ R_n(i,P)\} \end{aligned}$$

where $R_n(i,P) = -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,P)dF_i(y)$.

Theorem 3. If $P^1 > P^2$, then we have

(i) $V_n(i,x,P^1) \geq V_n(i,x,P^2)$ for all i,x,n.

(ii) $R_n(i,P^1) \geq R_n(i,P^2)$ for all i,n.

Proof: The proof is again by induction on n. For $n = N$ the assertions hold with equalities. Assume for $n + 1$ that $V_{n+1}(\cdot,\cdot,P^1) \geq V_{n+1}(\cdot,\cdot,P^2)$ for $P^1 > P^2$. Then, we have

$$\begin{aligned} R_n(i,P^1) &\geq -c_n(i) + \beta\sum_j P^1_{ij}\int^\infty V_{n+1}(j,y,P^2)dF_i(y) \\ &= -c_n(i) + \beta\int_0^\infty \sum_{k=1}^{S} [V_{n+1}(k,y,P^2) - V_{n+1}(k-1,y,P^2)]dF_i(y) \sum_{j=k}^{S} P^1_{ij} \\ &\geq -c_n(i) + \beta\int_0^\infty \sum_{k=1}^{S} [V_{n+1}(k,y,P^2) - V_{n+1}(k-1,y,P^2)] \sum_{j=k}^{S} P^2_{ij} dF_i(y) \\ &= -c_n(i) + \beta\int_0^\infty \sum_j P^2_{ij}\ V_{n+1}(j,y,P^2)dF_i(y) \\ &= R_n(i,P^2). \end{aligned}$$

Therefore, $V_n(i,x,P^1) = \max\{x,R_n(i,P^1)\} \geq \max\{x,R_n(i,P^2)\} = V_n(i,x,P^2)$.

(Q.E.D.)

Corollary. If P^1 and P^2 are the lower and upper bounds of P, respectively with $P^1 < P < P^2$, then we have $R_n(i,P^1) \leq R_n(i,P) \leq R_n(i,P^2)$ and $V_n(i,x,P^1) \leq V_n(i,x,P) \leq V_n(i,x,P^2)$ for all n,i,x. The higher probability that the economy goes to any subset of the states $\{k,k+1,\ldots,S\}$ gives the searcher the higher benefit and reservation wage.

Secondly, we explore the effect of increasing uncertainty about wage distribution $F_i(\cdot)$. To this end we write

$$\begin{aligned} V_n(i,x,F) &= \max\{x,-c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,F)dF_i(y)\} \\ &\equiv \max\{x,R_n(i,F)\} \end{aligned}$$

where

$$R_n(i,F) = -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,F)dF_i(y).$$

Theorem 4. For two distinct wage distributions F^1, F^2 such as $EX^1 = EX^2$ and $\int_0^x F_i^1(y)dy \leq \int_0^x F_i^2(y)dy$ for all x, we have

(i) $R_n(i,F^1) \leq R_n(i,F^2)$ for each i,n

(ii) $V_n(i,x,F^1) \leq V_n(i,x,F^2)$ for each i,n,x.

Proof: The proof is again by induction on n. For $n = N$ theorem 4 holds with equalities. Since from theorem 1 $V_n(i,x,F)$ is convex in x, it can be shown (see [11]) that the second degree of stochastic dominance implies $\int_0^\infty V_n(j,y,F^1)dF_i^1(y) \leq \int_0^\infty V_n(j,y,F^2)dF_i^2(y)$. Therefore,

$$\begin{aligned} R_n(i,F^1) &\equiv -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,F^1)dF_i^1(y) \\ &\leq -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,F^2)dF_i^2(y) \\ &= R_n(i,F^2). \end{aligned}$$

This also implies $V_n(i,x,F^1) \leq V_n(i,x,F^2)$. (Q.E.D.)

Remarks: Since $EX^1 = EX^2$ and $\int_0^x F_i^1(y)dy \leq \int_0^x F_i^2(y)dy$ implies $Var(X^1) \leq Var(X^2)$, theorem 4 asserts that increasing the riskiness of the wage distribution is beneficial to the searcher. As a result of this, he turns out to set the higher reservation wage for increased uncertainty about the wage distribution.

Suppose that the searcher receives the random number of job offers N_n at period n. To facilitate comparison with the case of $N_n = 1$ for all n we assume that $E[N_n] = 1$ for all n, and that the N_n are independent and possess the same wage distribution $F_i(x)$, depending only on the state of the economy.

Theorem 5. If $E[N_n] = 1$ for all n, then we have

(i) $R_n(i,G) \le R_n(i,F)$

(ii) $V_n(i,x,G) \le V_n(i,x,F)$

where $G_i(x)$ is the distribution of the best offer received when the random offers N_n are tendered and the economy is in state i.

Proof: It is easy to see that $G_i(x) = E[F_i(x)^{N_n}] \ge F_i(x)^{EN_n} = F_i(x)$ by applying Jensen's inequality. Since $V_n(i,x)$ is increasing in x, we have

$$R_n(i,G) \equiv -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,G)dG_i(y)$$

$$\le -c_n(i) + \beta\sum_j P_{ij}\int_0^\infty V_{n+1}(j,y,F)dF_i(y) \equiv R_n(i,F).$$

Hence, $V_n(i,x,G) = \max\{x,R_n(i,G)\} \le \max\{x,R_n(i,F)\} = V_n(i,x,F)$. (Q.E.D.)

Remarks: Theorem 5 simply states that it is favorable for the searcher to have exactly one offer at each period rather than a random number of offers with the mean of one at each period. Therefore, the searcher cannot take advantage of the possibility of receiving more than one offer at each period.

It is worthwhile to note that even if the cost of search were negligible to the searcher, he would still have an incentive to searching because he has no perfect information about where his best job is located, and because he is unable to distinguish jobs *ex ante*. Therefore, we consider a case in which the search cost vanishes, that is, $c_n(i) \equiv 0$. Furthermore, we assume that X_n is distributed with the mean μ_i and common variance σ^2. Then, equation (2) can be rewritten as follows:

$$\text{(7)}\qquad \begin{aligned} V_n(i,x) &= \max\{x,\beta\int_0^\infty\sum_j P_{ij}V_{n+1}(j,y)dF_i(y)\} \\ &\equiv \max\{x,r_n(i)\},\ nn=1,2,\ldots,N-1. \end{aligned}$$

At period N, $r_N(i) = \beta\sum_j P_{ij}\mu_j$ since the searcher must accept whatever job is offered at period N. Also, note that we use lower case letters for the case of no search cost.

Theorem 6. For all n $r_n(i) = a_n(i) + b_n(i)\cdot\sigma$ where $b_n(i) > 0$ for all i and n but $n = N$.

Proof: By induction on n, we have for period N

$$r_N(i) = \beta\sum_j P_{ij}\mu_j = a_N(i) + b_N(i)\cdot\sigma$$

where $a_N(i) = \beta\sum_j P_{ij}\mu_j$ and $b_N(i) = 0$ for all i. Assume for $(n+1)$ that $r_{n+1}(i) = a_{n+1}(i) + b_{n+1}(i)\cdot\sigma$. For n we have

$$r_n(i) = \beta\sum_j P_{ij}\, E \max\{X_{n+1},\ r_{n+1}(j)\}$$
$$= \beta\sum_j P_{ij}[r_{n+1}(j) + \int_{r_{n+1}(j)}^{\infty}(x-r_{n+1}(j))dF_i(x)]$$
$$= \beta\sum_j P_{ij}[a_{n+1}(j)+b_{n+1}(j)\cdot\sigma+ \int_{a_{n+1}(j)+b_{n+1}(j)\cdot\sigma}^{\infty}(x-a_{n+1}(j)-b_{n+1}(j)\sigma)dF_i(x)]$$

where the second equality follows from [4,p.333]. Putting $a_n(i) = \beta\sum_j P_{ij}a_{n+1}(j)$ and $y = (x-a_{n+1}(j))/\sigma$, we obtain

$$r_n(i) = a_n(i) + \beta\sum_j P_{ij}[b_{n+1}(j)+\sigma\int_{b_{n+1}(j)}^{\infty}(y-b_{n+1}(j))dF_i(y)]\sigma$$
$$= a_n(i) + b_n(i)\cdot\sigma$$

where $b_n(i) = \beta\sum_j P_{ij}[b_{n+1}(j)+\sigma\int_{b_{n+1}(j)}^{\infty}(y-b_{n+1}(j))dF_i(y)] > 0.$ (Q.E.D.)

A more interesting case is of that the wages are identically and independently normally distributed with the common mean μ and variance σ^2 and that $\beta=1$. In this case, equation (7) reduces to as follows:

(7)' $v_n(x) = \max\{x, r_n\}$

(8) $r_n = E\ v_{n+1}(X_{n+1}) = E\max\{X_{n+1},\ r_{n+1}\},\ n=1,2,\ldots,N-1.$

At period N $r_N = \mu$. Then, we obtain the following corollary.

Corollary. Assume that $c_n(i) = 0$ for all i, n and $\beta = 1$, and that a wage is identically and independently normally distributed with the common mean μ and variance σ. Then, $n = 1,2,\ldots,N$, $r_n = \mu + b_n\cdot\sigma$, where b_n is positive but for $n = N$ and independent of σ.

Proof: By induction on n, we have for period N

$$r_N = \mu = \mu + b_N\cdot\sigma \text{ with } b_N = 0.$$

Assume for $(n+1)$ that $r_{n+1} = \mu + b_{n+1}\cdot\sigma$. For n we have

$$r_n = E\max\{X,\ \mu+b_{n+1}\cdot\sigma\}$$
$$= \mu + b_{n+1}\cdot\sigma + \int_{\mu+b_{n+1}\cdot\sigma}^{\infty}(x-\mu-b_{n+1}\cdot\sigma)dF(x)$$

Putting $y = (x-\mu)/\sigma$, we obtain

$$r_n = \mu + [b_{n+1} + \int_{b_{n+1}}^{\infty}(y-b_{n+1})d\Phi(y)]\sigma$$
$$\equiv \mu + b_n\cdot\sigma$$

where $\Phi(\cdot)$ is the standard normal distribution and $b_n \equiv E \max\{Y, b_{n+1}\} > 0$.

(Q.E.D.)

Remarks: Using σ as a measure of uncertainty, the higher the uncertainty on the wage, the higher the reservation wage is set. Hence, we may be led to the conjecture that the expected number of search periods is increased by the increased uncertainty on the wage. However, this conjecture does not hold true, provided that jobs offer from a normally distributed sample space.

Theorem 6. Assume that $\beta = 1$ and $c_n(i) = 0$ for all i, n, and that the wages are identically and independently normally distributed with the common mean μ and variance σ^2. Then, the expected number of search periods is independent of the uncertainty on the wage.

Proof: Let π_n be the probability of stopping the search at period n. Under our reservation wage policy, π_n is given by

$$\pi_n = [1 - F(r_n)] \prod_{t=1}^{n-1} F(r_t).$$

Let Φ be the normal distribution with the mean 0 and variance 1. However, from Lemma 1 above we know

$$\begin{aligned} F(r_t) &= \Phi((r_t - \mu)/\sigma) \\ &= \Phi((\mu + b_n \cdot \sigma - \mu)/\sigma) = \Phi(b_n) \end{aligned}$$

which is independent of σ as a measure of uncertainty. The expected number of search periods, denoted by N^*, is

$$N^* = \sum_{n=1}^{N} n\pi_n = \sum_{n=1}^{N} n[1 - \Phi(b_n)] \prod_{t=1}^{n-1} \Phi(b_n)$$

which is also independent of σ. (Q.E.D.)

Remarks: Note that the reservation wage does not depend on n for an *infinite* period model. So, put $r_n = r$ for all $n = 1, 2, \ldots$. Then, the expected number of search periods is as follows;

$$\begin{aligned} N^* &= \sum_{n=1}^{\infty} n\pi_n = \sum_{n=0}^{\infty} n[1 - F(r)][F(r)]^{n-1} \\ &= 1/[1 - \Phi(b)]. \end{aligned}$$

Therefore, an increased uncertainty on wages has no effect on the expected duration of search.

5. Some Concluding Remarks

In this paper we have extended the theory of job search to include the case of the dynamic nonstationary search environment. We proved under some specific conditions that (1) an optimal search strategy is no longer myopic but is characterized by a reservation wage, which depends on the state of the economy and on the remaining periods of searching, (2) the maximum return and the reservation wage both increase as the economy is improved and (3) increasing the riskiness of the wage offer distribution is beneficial to the searcher, but he cannot take advantage of the random number of job offers tendered per period. (4) We also give an explicite form to the reservation wage for the case of no search cost. Especially when wages are identically and independently normally distributed, such reservation wage has a positive linear form of the variance and the expected duration of search is independent of the variance as a measure of uncertainty on the wage. What the paper intended to do is explicitly to discuss the impact of increased uncertainty about the state transition probability, the wage offer distribution, the number of job offers per period and the cost of search. Future research should be encouraged to include the case of that the searcher may revise his wage offer distribution as he learns from the past experience of searching. A possible approach is to assume that past observations are emitted from some other probability distributions and the wage distribution is revised according to a Bayesian rule.

Acknowledgement

This paper was presented at the 11th Management Science Colloquim held in Osaka, 1982 summer. The authers wish to express their sincere gratitude to Professor M. Sakaguchi, Osaka University, for his many helpful comments and suggestions.

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