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OPTIMAL SEARCH AND STOP IN CONTINUOUS SEARCH PROCESS

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Abstract This paper investigates an optimal search policy with stopping for a stationary target being in one of n boxes. It is assumed that the search is conducted continuously with a total search cost C per unit time and the search in box i costs c_i per unit search effort. The conditional probability of detecting the target with unit search effort is α_i and a reward R_i is given to the searcher when he successfully detects the target in box i. We derive conditions for the optimal search and stop policy which minimizes the expected risk of the search (the expected search cost minus the expected reward). The physical meaning of the conditions and several properties of the optimal policy are elucidated. The optimal policy for two-box case is examined in detail, and necessary and sufficient conditions for the optimal policy and the closed form risk function are obtained.

1. Introduction

Suppose we wish to find an object, the target, which is known to be in a given region. The region consists of n disjoint subregions, say boxes, and the probability of the target being in box i is known to be $p_i(>0)$, $i=1,\cdots,n$, $\sum_i p_i = 1$. The search in box i costs $c_i(>0)$ per unit search effort and the conditional probability of detecting the target with unit search effort is assumed to be $\alpha_i(>0)$, irrespective of the history of the past search. This assumption implies that the random search is conducted in each box and if a search effort $\psi(i,t)$ is applied to box i until t, the conditional detection probability when the target exists really in the box is given by $1-\exp(-\alpha_i\psi(i,t))$. It is assumed that the search is carried out continuously in time subject to the restriction that a definite search cost C per unit time is available to the searcher. It is also assumed that the search cost $C\Delta t$ can be arbitrarily divided and allocated to boxes, that is, the search effort is considered as a continuous variable instead of a quantized one. (Because of these assumptions, the search is said "continuous" which contrasts with the quantized search model.) The searcher is given a reward R_i when he successfully detects the target in box i.

When the reward R_i is sufficiently large, we should continue searching until detection. But, in some class of search, the search cost is relatively expensive compared with the reward. In this case, attention must be given to the problem of stopping the search prior to detection. This paper deals with this situation by assuming that the measure of effectiveness of the search policy is the expected risk defined as the expected value of the difference between the search cost until detection or stopping, whichever comes first, and the reward.

An optimal stopping problem of a search process was investigated first by Chew, Jr. [1,2] and later by Ross [3] in a more general form. They dealt with the problem in a quantized search model and formulated it by use of dynamic programming argument. Ross obtained several sufficient conditions for the optimal policy which minimizes the expected risk, and further discussed some approximations to the optimal policy. Thereafter, several authors investigated special two-box case of Ross' problem [4,5]. Up to the present, however, the optimal policies in general cases are not yet obtained explicitly even for the two-box problem. Richardson and Belkin [6] investigated a stopping rule for continuous search assuming a continuous target space. They assumed a homogeneous target space with respect to both the search cost and the reward. In this case, as is shown later, the problem is so simple that they could investigate the model with monotonic functions in time for the search cost and the reward. Nakai [7] dealt with a stopping problem in a section of his paper concerning the optimal search for a target with a random lifetime. In his paper, he also assumed the homogeneity of the search cost in both time and the target space. He obtained necessary and sufficient conditions for the optimal search policy and a necessary condition for the optimal stopping time when the lifetime density function of the target is differentiable.

In this paper, we deal with such a case that the search cost and the reward vary over the search space but do not depend on time and the target is assumed to be immortal and stationary.

In Section 2 of this paper, the problem described above is formulated in a functional equation. Section 3 presents necessary and sufficient conditions for the conditionally optimal search policy when a stopping time is given, and several properties of the policy are elucidated. Section 4 deals with the optimal stopping time which minimizes the expected risk of the conditionally optimal policy. In Section 5, two-box problem is examined in detail. Necessary and sufficient conditions for the optimal policy and the closed form risk functions are derived explicitly. Finally, in Section 6, we discuss the physical meaning of the conditions for the optimal policy and present results for some special cases. Some extentions of our model are also considered.

2. Formulation of the Problem

The notations used in this paper are given in the following.

R_i : The reward when the target is detected in box i.

c_i : The search cost per unit search effort in box i, $c_i > 0$.

C : The available total search cost rate. (Search cost per unit time). $C > 0$.

T_i : The stopping time of the search in box i.

T : The stopping time of search process. $T = \max_i T_i$. T^* is the optimal stopping time. If the search should not be started, $T^* = 0$, and if it should not be stopped, $T^* = \infty$.

T_i^0 : The start time of the search in box i.

t^0 : The start time of the parallel search when $T^* = \infty$.

α_i : The conditional detection probability per unit search effort in box i, $\alpha_i > 0$.

$\Phi = \{\phi(i,t)\}$: The search policy. $\phi(i,t)\Delta t$ is the search effort allocated to box i in time interval $[t, t+\Delta t]$. If $t > T$, $\phi(i,t) = 0$ for all i.

$\Phi^*(s) = \{\phi^*(i,t|s)\}$: The conditionally optimal search policy when the duration of the search, s, is given. Sometimes, $\phi^*(i,t|s)$ is abridged as $\phi^*(i,t)$ if any confusion does not arise.

$\psi(i,t)$: The cumulative search effort allocated to box i until t, $\psi(i,t) = \int_0^t \phi(i,\tau)d\tau$, and also $\psi^*(i,t) = \int_0^t \phi^*(i,\tau)d\tau$.

$P = \{p_i\}$: The initial probability vector of the target distribution. p_i is the initial probability of the target being in box i, $i = 1, \cdots, n$, $p_i > 0$, $\sum_i p_i = 1$.

$P(t) = \{p_i(t)\}$: The posterior probability vector of the target distribution when the initial probability vector of the target is P and the target is not found until t.
$p_i(t) = p_i \exp(-\alpha_i \psi(i,t)) / \sum_i p_i \exp(-\alpha_i \psi(i,t))$.
We shall sav that the process is in state $P(t)$ at time t.

$f(P(t),\Phi,T)$: The expected risk of the search between t and T when the state is in $P(t)$ at t and the policy Φ is employed until the stopping time T. Since $P(t) = P$ at $t = 0$, the expected risk of the search between 0 and T is denoted by $f(P,\Phi,T)$.

$I_D(t) = \{i \mid \psi(i,t) > 0\}$.

$I_S(t) = \{i \mid \phi(i,t) > 0\}$.

$I_D(t)$ is the set of boxes in which the search effort has ever been allocated until t and $I_S(t)$ is the set of boxes which are being searched at t.

$\lambda(t)$: Lagrange multiplier which controls the effort allocation at t.

In the section of two-box problem, the following notations are used.

Φ_0 : The search policy which is stopped immediately.

Φ_i : The search policy searching only box i until the stopping time.

Φ_i^∞ : The search policy which is started searching in box i and is continued until a switching time, and thereafter, both boxes are searched in parallel until detection.

T^i : The conditionally optimal stopping time when the search policy Φ_i is employed.

t_i^0 : The conditionally optimal switching time when the search policy Φ_i^∞ is employed.

Here we formulate the problem as follows.

Considering the situation where the state is in P at $t = 0$ and the policy $\Phi = \{\phi(i,t)\}$ is employed until T, the expected risk $f(P,\Phi,T)$ is written down as follows,

$$f(P,\Phi,T) = \sum_i p_i\left[\int_0^T \alpha_i\phi(i,t)\{\sum_i c_i\psi(i,t)-R_i\}e^{-\alpha_i\psi(i,t)}dt + e^{-\alpha_i\psi(i,T)}\sum_i c_i\psi(i,T) \right].$$

The integrand in the first term in the bracket is the product of the risk, $\{\sum_i c_i\psi(i,t)-R_i\}$, when the target is detected in box i at t, and the probability density of detecting the target at t, $\alpha_i\phi(i,t)\exp(-\alpha_i\psi(i,t))$, which is deduced from the assumption of the random search. Hence, the first term in the bracket, $\int_0^T\alpha_i\phi(i,t)\{\sum_i c_i\psi(i,t)-R_i\}\exp(-\alpha_i\psi(i,t))dt$, is the conditional expected risk when the target is in box i and is detected at some time in $[0,T]$. The second term is the expected risk when the target is not detected until T. The sum of these two terms is the conditional expected risk of the search policy given that the target is in box i. Therefore, the expected risk, $f(P,\Phi,T)$, can be obtained by the summation of the conditional expected risk weighted by p_i. Here, $\phi(i,t)$ is the search effort allocation to box i at t, therefore, the restriction $C\Delta t \geq \sum_i c_i\phi(i,t)\Delta t$ for all $t \in [0,T]$, is imposed. However, in this problem, since the target is assumed to be stationary and the reward and the cost do not vary in time, the conversion of the restriction, $C\Delta t \geq \sum_i c_i\phi(i,t)\Delta t$, to $C\Delta t = \sum_i c_i\phi(i,t)\Delta t$ does not change the nature of the problem at all. Therefore, for the simplicity of the problem, we assume $C\Delta t = \sum_i c_i\phi(i,t)\Delta t$ for all $t \in [0,T]$. Then, substituting $\sum_i c_i\phi(i,t)\Delta t = C\Delta t$ into the above relation and integrating by parts, we obtain the next simplified expression,

(1)
$$f(P,\phi,T) = \sum_i p_i[C\int_0^T e^{-\alpha_i\psi(i,t)}dt - R_i(1-e^{-\alpha_i\psi(i,T)})]$$
$$= \sum_i p_i[\int_0^T\{C-\alpha_i\phi(i,t)R_i\}e^{-\alpha_i\psi(i,t)}dt].$$

Hence, the problem is formulated as a variational problem to find the optimal functions $\{\Phi(i,t)\}$ and the optimal stopping time T which minimize the functional $f(P,\Phi,T)$ subject to the following restrictions,

(2)
$$\phi(i,t) \geq 0 \quad \text{for all } i \text{ and } t \in [0,T],$$
$$C\Delta t = \sum_i c_i\phi(i,t)\Delta t \text{ for all } t \in [0,T],$$
$$\phi(i,t) = 0 \quad \text{for all } i \text{ and } t > T.$$

3. Allocation of Search Effort

The search policy which minimizes the expected risk when the stopping time is given is called the conditionally optimal search policy. In this section, we derive necessary and sufficient conditions for the conditionally optimal search policy and investigate the properties of the policy. The conditionally optimal search policy and the cumulative search effort are denoted by $\{\phi^*(i,t)\}$ and $\{\psi^*(i,t)\}$ respectively. Since these are obtained by conditioning the initial state P and the stopping time T, $\{\phi^*(i,t|P,T)\}$ and $\{\psi^*(i,t|P,T)\}$ should be used rigorously instead of $\{\phi^*(i,t)\}$ and $\{\psi^*(i,t)\}$. But in this paper, these abbrivations do not cause any confusions, therefore, $\{\phi^*(i,t)\}$ and $\{\psi^*(i,t)\}$ are used simply and an annotation will be given if necessary.

Theorem 1. Necessary and sufficient conditions for the conditionally optimal search policy $\{\phi^*(i,t)\}$ are that,

(3)
if $\phi^*(i,t) > 0$,
$$\frac{\alpha_i p_i}{c_i}\left[R_i e^{-\alpha_i\psi^*(i,T)} + C\int_t^T e^{-\alpha_i\psi^*(i,\tau)}d\tau\right] = \lambda(t),$$
if $\phi^*(i,t) = 0$,
$$\frac{\alpha_i p_i}{c_i}\left[R_i e^{-\alpha_i\psi^*(i,T)} + C\int_t^T e^{-\alpha_i\psi^*(i,\tau)}d\tau\right] \leq \lambda(t),$$

for all i and almost every t in $[0,T]$. ("Almost every t in $[0,T]$" means "all t in $[0,T]$ except for those in a set of measure 0.") Here, $\lambda(t)$ is the so-called Lagrange multiplier which is positive and is determined by $C\Delta t = \sum_i c_i\phi^*(i,t)\Delta t$.

Proof: Since the constraints on $\phi(i,t)$ are somewhat severe, the theorems for the standard problem of the calculus of variations can not be applied directly to derive the above theorem. The proof of the necessity is given by following a pattern of reasoning similar to that of De Guenin [8], considering the variation of $\phi^*(i,t)$ satisfying the constraints (2) in both time and boxes. In our problem, since the total search cost C is defined as the search cost rate and the constraint is $C\Delta t = \sum_i c_i\phi(i,t)\Delta t$, $\phi(i,t)$ is a density of search effort per unit time and, if $\phi(i,t)$ is dense at t, the conditional detection probability is zero at that time. Therefore, the optimal search policy $\phi^*(i,t)$ is not always dense in any time interval and we can find box i such that $\phi^*(i,t) > 0$ in $[t_1-\varepsilon/2,\ t_1+\varepsilon/2]$ for any t_1, $t_1 < T$. Then, consider the following policy $\{\phi(i,t)\}$ which differs from $\{\phi^*(i,t)\}$ only in the $\varepsilon/2$-neighborhood of t_1, if $|t - t_1| \le \varepsilon/2$,

$$\tilde{\phi}(i,t) = \phi^*(i,t) - h/c_i,$$

$$\tilde{\phi}(j,t) = \phi^*(j,t) + h/c_j,$$

$$\tilde{\phi}(k,t) = \phi^*(k,t) \quad k \neq i \text{ and } j,$$

and if $|t - t_1| > \varepsilon/2$,

$$\tilde{\phi}(i,t) = \phi^*(i,t)$$

for all i. Namely, the policy $\{\tilde{\phi}(i,t)\}$ is obtained from $\{\phi^*(i,t)\}$ by transferring some small search budget h from box i to some box j in the time interval $[t_1-\varepsilon/2,\ t_1+\varepsilon/2]$. Since $\phi^*(i,t) > 0$ in this interval, this transfer is always possible if h is sufficiently small, and it satisfies the restrictions (2). We obtain from (1)

$$f(P,\tilde{\Phi},T) - f(P,\Phi^*,T) = p_i(e^{\alpha_i h\varepsilon/c_i} - 1)[R_i e^{-\alpha_i\psi^*(i,T)} + C\int_{t_1}^{T} e^{-\alpha_i\psi^*(i,\tau)}d\tau]$$

$$+ p_j(e^{-\alpha_j h\varepsilon/c_j} - 1)[R_j e^{-\alpha_j\psi^*(j,T)} + C\int_{t_1}^{T} e^{-\alpha_j\psi^*(j,\tau)}d\tau] \ge 0.$$

If $h\varepsilon \ll 1$, the above relation can be written as

$$\frac{\alpha_i p_i}{c_i}[R_i e^{-\alpha_i\psi^*(i,T)} + C\int_{t_1}^{T} e^{-\alpha_i\psi^*(i,\tau)}d\tau]$$

$$\ge \frac{\alpha_j p_j}{c_j}[R_j e^{-\alpha_j\psi^*(j,T)} + C\int_{t_1}^{T} e^{-\alpha_j\psi^*(j,\tau)}d\tau].$$

It should be noted that the above result was obtained under the single condition $\phi^*(i,t) > 0$, $t \in [t_1-\varepsilon/2,\ t_1+\varepsilon/2]$. If $\phi^*(j,t) > 0$ in the interval, we can consider the other policy which is obtained by transferring the small budget of search cost h from box j to i. We can prove the following relation in exactly the same way mentioned above.

$$\frac{\alpha_j p_j}{c_j}\left[\ R_j e^{-\alpha_j \psi^*(j,T)} + C\int_{t_1}^{T} e^{-\alpha_j \psi^*(j,\tau)} d\tau\ \right]$$
$$\geq \frac{\alpha_i p_i}{c_i}\left[\ R_i e^{-\alpha_i \psi^*(i,T)} + C\int_{t_1}^{T} e^{-\alpha_i \psi^*(i,\tau)} d\tau\ \right].$$

Hence, if $\phi^*(i,t) > 0$ and $\phi^*(j,t) > 0$, $t \in [t_1-\varepsilon/2,\ t_1+\varepsilon/2]$, then

$$\frac{\alpha_i p_i}{c_i}\left[\ R_i e^{-\alpha_i \psi^*(i,T)} + C\int_{t_1}^{T} e^{-\alpha_i \psi^*(i,\tau)} d\tau\ \right] = \lambda(t_1).$$

And if $\phi^*(i,t) = 0$, whereas $\phi^*(j,t) > 0$, then

$$\frac{\alpha_i p_i}{c_i}\left[\ R_i e^{-\alpha_i \psi^*(i,T)} + C\int_{t_1}^{T} e^{-\alpha_i \psi^*(i,\tau)} d\tau\ \right] \leq \lambda(t_1).$$

The proof of the sufficiency. The convexity of $f(P,\Phi,T)$ in Φ is proved easily as follows. Here, we define $\theta\Phi$ and $\Phi^1+\Phi^2$ as $\theta\Phi = \{\theta\phi(i,t)\}$ and $\Phi^1+\Phi^2 = \{\phi^1(i,t)+\phi^2(i,t)\}$ respectively. By considering arbitrary effort allocations Φ^1 and Φ^2 which satisfy the restrictions (2) and setting $\Phi^0 = (1-\theta)\Phi^1+\theta\Phi^2$, $0 \leq \theta \leq 1$, we obtain

$$\begin{aligned} f(P,\Phi^0,T) &= \sum_i p_i\Big[\ C\int_0^T e^{-\alpha_i\{(1-\theta)\psi^1(i,\tau)+\theta\psi^2(i,\tau)\}} d\tau \\ &\qquad - R_i(1-e^{-\alpha_i\{(1-\theta)\psi^1(i,T)+\theta\psi^2(i,T)\}})\Big] \\ &\leq \sum_i p_i\Big[\ C\int_0^T\{(1-\theta)e^{-\alpha_i\psi^1(i,\tau)} + \theta e^{-\alpha_i\psi^2(i,\tau)}\}d\tau \\ &\qquad - R_i\{(1-\theta)(1-e^{-\alpha_i\psi^1(i,T)}) + \theta(1-e^{-\alpha_i\psi^2(i,T)})\}\ \Big] \\ &= (1-\theta)f(P,\Phi^1,T) + \theta f(P,\Phi^2,T). \end{aligned}$$

Suppose that $\Phi^* = \{\phi^*(i,t)\}$ is a search policy which satisfies the relation (3) and $\Phi = \{\phi(i,t)\}$ is an arbitrary search policy subject to the restrictions (2). Setting $\Phi^1 = \Phi^*$ and $\Phi^2 = \Phi$ in the above relation and considering a sufficiently small θ, we obtain

$$\theta\{f(P,\Phi^*,T) - f(P,\Phi,T)\} \le f(P,\Phi^*,T) - f(P,\Phi^0,T)$$

$$= \sum_i p_i[C\int_0^T (e^{-\alpha_i\psi^*(i,t)} - e^{-\alpha_i\psi^0(i,t)})dt + R_i(e^{-\alpha_i\psi^*(i,T)} - e^{-\alpha_i\psi^0(i,T)})]$$

$$= \sum_i p_i\alpha_i\theta[C\int_0^T e^{-\alpha_i\psi^*(i,t)}\{\int_0^t(\phi(i,\tau)-\phi^*(i,\tau))d\tau\}dt$$

$$+ R_i e^{-\alpha_i\psi^*(i,T)}\int_0^T(\phi(i,\tau)-\phi^*(i,\tau))d\tau]$$

$$= \theta\sum_i c_i[\frac{\alpha_i p_i}{c_i}\int_0^T\{R_i e^{-\alpha_i\psi^*(i,T)} + C\int_\tau^T e^{-\alpha_i\psi^*(i,t)}dt\}\{\phi(i,\tau)-\phi^*(i,\tau)\}d\tau]$$

$$\le \theta\sum_i c_i[\int_0^T\lambda(\tau)\{\phi(i,\tau)-\phi^*(i,\tau)\}d\tau]$$

$$= \theta\int_0^T\lambda(\tau)[\sum_i c_i\phi(i,\tau) - \sum_i c_i\phi^*(i,\tau)d\tau] = 0$$

from (1)(2) and (3). Therefore, since $f(P,\Phi^*,T) \le f(P,\Phi,T)$ holds for arbitrary policy Φ, a search policy Φ^* satisfying the relation (3) is optimal. (Q.E.D.)

The following corollary is obtained directly by setting $t = T$ in Theorem 1.

Corollary 1.1.

If $\phi^*(i,T) > 0$, $\quad \frac{\alpha_i p_i}{c_i} R_i\, e^{-\alpha_i\psi^*(i,T)} = \lambda(T)$.

If $\phi^*(i,T) = 0$, $\quad \frac{\alpha_i p_i}{c_i} R_i\, e^{-\alpha_i\psi^*(i,T)} \le \lambda(T)$.

As for $\lambda(t)$ in Theorem 1, we have the following corollary.

Corollary 1.2. *$\lambda(t)$ is a continuous, strictly decreasing and strictly convex function of t.*

Proof: For arbitrary t and $\delta(>0)$, $t+\delta \le T$, we can always find box i and j such that $\phi^*(i,t) > 0$ and $\phi^*(j,t+\delta) > 0$. Then, the next inequalities are obtained from Theorem 1.

$$C\frac{\alpha_j p_j}{c_j}\int_t^{t+\delta} e^{-\alpha_j\psi^*(j,\tau)}d\tau \le \lambda(t) - \lambda(t+\delta) \le C\frac{\alpha_i p_i}{c_i}\int_t^{t+\delta} e^{-\alpha_i\psi^*(i,\tau)}d\tau.$$

Therefore, $\lambda(t) > \lambda(t+\delta)$ and $\lim_{\delta\to 0}\lambda(t+\delta) = \lambda(t)$.

The convexity of $\lambda(t)$ is proved as follows.

We consider arbitrary time points t_1 and t_2 where $t_1 < t_2 \le T$ and $t = (1-\theta)t_1 + \theta t_2$, $0 \le \theta \le 1$. Here we can always find box i with positive search effort at t. Then the followings are obtained from Theorem 1.

$$\lambda(t) = \frac{\alpha_i p_i}{c_i}[\, R_i e^{-\alpha_i\psi^*(i,T)} + C\int_t^T e^{-\alpha_i\psi^*(i,\tau)}d\tau \,]$$

$$= (1-\theta)\frac{\alpha_i p_i}{c_i}[\, R_i e^{-\alpha_i\psi^*(i,T)} + C\int_{t_1}^T e^{-\alpha_i\psi^*(i,\tau)}d\tau \,]$$

$$+ \theta\frac{\alpha_i p_i}{c_i}[\, R_i e^{-\alpha_i\psi^*(i,T)} + C\int_{t_2}^T e^{-\alpha_i\psi^*(i,\tau)}d\tau \,]$$

$$+ \frac{\alpha_i p_i}{c_i}[\, \theta\int_t^{t_2} e^{-\alpha_i\psi^*(i,\tau)}d\tau - (1-\theta)\int_{t_1}^t e^{-\alpha_i\psi^*(i,\tau)}d\tau \,]$$

$$\le (1-\theta)\lambda(t_1) + \theta\lambda(t_2) + \frac{\alpha_i p_i}{c_i}C[\, \theta\int_t^{t_2} e^{-\alpha_i\psi^*(i,\tau)}d\tau - (1-\theta)\int_{t_1}^t e^{-\alpha_i\psi^*(i,\tau)}d\tau \,].$$

As $\psi^*(i,\tau)$ is a non-decreasing function of τ, we can easily prove that the last term, $[\, \theta\int_t^{t_2}\exp(-\alpha_i\psi^*(i,\tau))d\tau - (1-\theta)\int_{t_1}^t\exp(-\alpha_i\psi^*(i,\tau))d\tau \,]$, is non-positive. Therefore,

$$\lambda(t) \le (1-\theta)\lambda(t_1) + \theta\lambda(t_2).$$

The equality holds only when $\theta = 0$ or $\theta = 1$, therefore, $\lambda(t)$ is a strictly convex function of t. (Q.E.D.)

The following corollaries elucidate the properties of the conditionally optimal search policy.

Corollary 1.3. Suppose $0 \le t_1 < t_2 \le T$.

1. If $\phi^*(i,t)$ is positive in the interval $[t_1,t_2)$ and is zero at t_2, $\phi^*(i,t)$ is zero throughout the interval $[t_2,T]$.
2. If $\phi^*(i,t)$ is positive at both t_1 and t_2, and is not dense at these points, then $\phi^*(i,t)$ is always positive in the interval $[t_1,t_2]$.

Proof: 1. *We assume that there exists the smallest time point* t_3 *at which* $\phi^*(i,t_3) > 0$, $t_2 < t_3 \le T$. Applying Theorem 1 to t_3 and to t, $t_1 \le t < t_2$, we obtain

$$\lambda(t_3) = \frac{\alpha_i p_i}{c_i}[R_i e^{-\alpha_i\psi^*(i,T)} + C\int_{t_3}^T e^{-\alpha_i\psi^*(i,\tau)}d\tau],$$

$$\lambda(t) = \frac{\alpha_i p_i}{c_i}C\int_t^{t_3} e^{-\alpha_i\psi^*(i,\tau)}d\tau + \lambda(t_3).$$

Approaching t to t_2, we have

$$\lim_{t\to t_2-0}\lambda(t) = \frac{\alpha_i p_i}{c_i}C\, e^{-\alpha_i\psi^*(i,t_2)}(t_3 - t_2) + \lambda(t_3) = \lambda(t_2).$$

Therefore,

$$\frac{\lambda(t_3) - \lambda(t_2)}{t_3 - t_2} = -\frac{\alpha_i p_i}{c_i} C\, e^{-\alpha_i \psi^*(i,t_2)}.$$

Similarly, the following relation is obtained from Theorem 1 at some t, $t_2 < t < t_3$,

$$\lambda(t) \geq \frac{\alpha_i p_i}{c_i} C\, e^{-\alpha_i \psi^*(i,t_2)}(t_3 - t) + \lambda(t_3).$$

$$\frac{\lambda(t_3) - \lambda(t)}{t_3 - t} \leq -\frac{\alpha_i p_i}{c_i} C\, e^{-\alpha_i \psi^*(i,t_2)}.$$

Therefore,

$$\frac{\lambda(t_3) - \lambda(t_2)}{t_3 - t_2} \geq \frac{\lambda(t_3) - \lambda(t)}{t_3 - t}.$$

This result contradicts the strict convexity of $\lambda(t)$, and therefore, we can conclude $\phi^*(i,t) = 0$ in the interval $[\, t_2, T\,]$.

2. If we assume $\phi^*(i,t_3) = 0$ at some t_3, $t_1 < t_3 < t_2$, $\phi^*(i,t) = 0$ in the interval $[\, t_3, T\,]$ from the above result. This contradicts $\phi^*(i,t_2) > 0$. (Q.E.D.)

Corollary 1.4. If both $\phi^*(i,t) > 0$ and $\phi^*(j,t) > 0$ hold in an interval $[t_1, t_2]$, then

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,t)} = \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,t)}, \tag{4}$$

and

$$\alpha_i \phi^*(i,t) = \alpha_j \phi^*(j,t) = \frac{C}{\sum_{I_s^*(t)} (c_i/\alpha_i)} \tag{5}$$

at any time in $[t_1,\, t_2]$.

Proof: From Theorem 1, we have

$$\lambda(t_1) - \lambda(t) = \frac{\alpha_i p_i}{c_i} C \int_{t_1}^{t} e^{-\alpha_i \psi^*(i,\tau)} d\tau = \frac{\alpha_j p_j}{c_j} C \int_{t_1}^{t} e^{-\alpha_j \psi^*(j,\tau)} d\tau.$$

Since the above relation holds at any time t in $[t_1, t_2]$, we can conclude (4). Applying (4) to the time $t+\Delta t$ in $[t_1, t_2]$, we obtain

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,t) - \alpha_i \phi^*(i,t)\Delta t} = \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,t) - \alpha_j \phi^*(j,t)\Delta t}.$$

Therefore,

$$\alpha_i \phi^*(i,t) = \alpha_j \phi^*(j,t).$$

From the constraint $\sum_i c_i \phi^*(i,t) = C$, we obtain (5). (Q.E.D.)

Suppose $i \in I_D^*(T)$, i.e. the search effort has ever been allocated to box i until T. Then, from Corollary 1.3, box i is searched in the interval $[T_i^{0*}, T_i^*]$. The start time T_i^{0*} and the stopping time T_i^* in box i satisfy the following relations.

Corollary 1.5. Suppose both i and j are elements of $I_D^*(T)$.

1. Iff $\alpha_i p_i \,/\, c_i \geq \alpha_j p_j \,/\, c_j$, $T_i^{0*} \leq T_j^{0*}$.
2. Iff $T_i^* \geq T_j^*$, $\alpha_i p_i \exp(-\alpha_i \psi^*(i,T)) \,/\, c_i \leq \alpha_j p_j \exp(-\alpha_j \psi^*(j,T)) \,/\, c_j$.
3. Iff $R_i \geq R_j$, $T_i^* \geq T_j^*$. Therefore, iff $i \in I_S^*(T)$, box i is the box which has the maximum reward in $I_D^*(T)$.

Proof: 1. From Theorem 1, we obtain the following inequalities at T_i^{0*} and T_j^{0*}.

$$\frac{\alpha_i p_i}{c_i}\Big[\, R_i e^{-\alpha_i \psi^*(i,T)} + C \int_{T_i^{0*}}^{T} e^{-\alpha_i \psi^*(i,\tau)} d\tau \,\Big]$$

$$\geq \frac{\alpha_j p_j}{c_j}\Big[\, R_j e^{-\alpha_j \psi^*(j,T)} + C \int_{T_i^{0*}}^{T} e^{-\alpha_j \psi^*(j,\tau)} d\tau \,\Big]$$

$$\frac{\alpha_i p_i}{c_i}\Big[\, R_i e^{-\alpha_i \psi^*(i,T)} + C \int_{T_j^{0*}}^{T} e^{-\alpha_j \psi^*(j,\tau)} d\tau \,\Big]$$

$$\leq \frac{\alpha_j p_j}{c_j}\Big[\, R_j e^{-\alpha_j \psi^*(j,T)} + C \int_{T_j^{0*}}^{T} e^{-\alpha_j \psi^*(j,\tau)} d\tau \,\Big].$$

Substracting the upper inequality from the lower, we have

$$\text{(6)} \qquad \frac{\alpha_i p_i}{c_i} \int_{T_j^{0*}}^{T_i^{0*}} e^{-\alpha_i \psi^*(i,\tau)} d\tau \leq \frac{\alpha_j p_j}{c_j} \int_{T_j^{0*}}^{T_i^{0*}} e^{-\alpha_j \psi^*(j,\tau)} d\tau.$$

Here we assume $T_i^{0*} > T_j^{0*}$. Noting $\psi^*(i,t) = 0$ and $\psi^*(j,t) \geq 0$ in any $t \in [T_j^{0*}, T_i^{0*})$, the following inequality is derived from (6).

$$\frac{\alpha_i p_i}{c_i} \int_{T_j^{0*}}^{T_i^{0*}} e^{-\alpha_i \psi^*(i,\tau)} d\tau = \frac{\alpha_i p_i}{c_i} (T_i^{0*} - T_j^{0*})$$

$$\leq \frac{\alpha_j p_j}{c_j} \int_{T_j^{0*}}^{T_i^{0*}} e^{-\alpha_j \psi^*(j,\tau)} d\tau < \frac{\alpha_j p_j}{c_j} (T_i^{0*} - {}_j^{0*}).$$

Hence we have

$$\frac{\alpha_i p_i}{c_i} < \frac{\alpha_j p_j}{c_j}.$$

Since this inequality contradicts the assumption of the corollary, $\alpha_i p_i/c_i \geq \alpha_j p_j/c_j$, we can conclude $T_i^{0*} \leq T_j^{0*}$.

The sufficiency is proved as follows.

If $T_i^{0*} \leq T_j^{0*}$, then $\psi^*(i,t) \geq 0$ and $\psi^*(j,t) = 0$ in the interval $[T_i^{0*}, T_j^{0*})$. Therefore, we obtain from (6)

$$\frac{\alpha_i p_i}{c_i}(T_j^{0*}-T_i^{0*}) \geq \frac{\alpha_i p_i}{c_i}\int_{T_i^{0*}}^{T_j^{0*}} e^{-\alpha_i \psi^*(i,\tau)} d\tau$$

$$\geq \frac{\alpha_j p_j}{c_j}\int_{T_i^{0*}}^{T_j^{0*}} e^{-\alpha_j \psi^*(j,\tau)} d\tau = \frac{\alpha_j p_j}{c_j}(T_j^{0*}-T_i^{0*}).$$

Consequently,

$$\frac{\alpha_i p_i}{c_i} \geq \frac{\alpha_j p_j}{c_j}.$$

2. Since $\phi^*(i,T_i^*) > 0$ and $\phi^*(j,T_j^*) > 0$, we obtain the following inequality from Theorem 1.

$$\frac{\alpha_i p_i}{c_i}\int_{T_j^*}^{T_i^*} e^{-\alpha_i \psi^*(i,\tau)} d\tau \leq \frac{\alpha_j p_j}{c_j}\int_{T_j^*}^{T_i^*} e^{-\alpha_j \psi^*(j,\tau)} d\tau . \tag{7}$$

Since $\psi^*(.,t)$ is a non-decreasing function of t, if $T_i^* \geq T_j^*$, the inequality becomes

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T_i^*)}(T_i^*-T_j^*) \leq \frac{\alpha_i p_i}{c_i}\int_{T_j^*}^{T_i^*} e^{-\alpha_i \psi^*(i,\tau)} d\tau$$

$$\leq \frac{\alpha_j p_j}{c_j}\int_{T_j^*}^{T_i^*} e^{-\alpha_j \psi^*(j,\tau)} d\tau = \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T_j^*)}(T_i^*-T_j^*).$$

Because $\psi^*(i,T_i^*) = \psi^*(i,T)$ for any i, the above inequality implies

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T)} \leq \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T)}. \tag{8}$$

The sufficient condition is proved by confirming $T_i^* \geq T_j^*$ when $\alpha_i p_i \exp(-\alpha_i \psi^*(i,T))/c_i \leq \alpha_j p_j \exp(-\alpha_j \psi^*(j,T))/c_j$. We assume $T_i^* < T_j^*$. Then we have

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T_i^*)}(T_j^*-T_i^*) \le \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T_j^*)}(T_j^*-T_i^*).$$

Since $\phi^*(i,t)=0$ $t\varepsilon(T_i,T_j)$, the left-hand side of this inequality is $\int_{T_i^*}^{T_j^*}\alpha_i p_i \exp(-\alpha_i\psi^*(i,\tau))d\tau \;/\; c_i$. And the right-hand side of the above is smaller than $\int_{T_i^*}^{T_j^*}\alpha_j p_j \exp(-\alpha_j\psi^*(j,\tau))d\tau \;/\; c_j$, because $\psi^*(j,t)$ is a increasing function of t, $t\ \varepsilon\ [T_i^*,T_j^*]$. Therefore,

$$\int_{T_j^*}^{T_i^*} \frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,\tau)} d\tau > \int_{T_j^*}^{T_i^*} \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,\tau)} d\tau.$$

This result contradicts the inequality (7) which is valid for any case where $\phi^*(i,T_i^*) > 0$, $\phi^*(j,T_j^*) > 0$. Therefore, $T_i^* \ge T_j^*$ if $\alpha_i p_i \exp(-\alpha_i\psi^*(i,T))/c_i \le \alpha_j p_j \exp(-\alpha_j\psi^*(j,T))/c_j$.

3. Suppose $R_i \ge R_j$. We assume $T_i^* < T_j^*$. From the above corollary, we have

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T)} > \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T)}.$$

By the assumption, $\phi^*(i,T_j^*) = 0$ and $\phi^*(j,T_j^*) > 0$, we obtain the following inequality from Theorem 1.

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T)} [R_i + C(T-T_j^*)] \le \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T)} [R_j + C(T-T_j^*)].$$

From these inequalities, we have $R_i < R_j$. This result contradicts the condition of this corollary. Therefore, if $R_i \ge R_j$, then $T_i^* \ge T_j^*$ is concluded.

We shall prove that $R_i \ge R_j$ if $T_i^* \ge T_j^*$. In this case, the inequality (8) holds. Considering the fact that $\phi^*(i,T_i^*) > 0$, $\phi^*(j,T_i^*) = 0$, we have

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,T_i^*)} [R_i + C(T-T_i^*)] \ge \frac{\alpha_j p_j}{c_j} e^{-\alpha_j \psi^*(j,T_j^*)} [R_j + C(T-T_i^*)].$$

From these inequalities, we obtain $R_i \ge R_j$. (Q.E.D.)

The well-known theorem which elucidates the optimal policy when searching is never stopped until detection is also derived from Theorem 1 easily. The next corollary is given without showing the derivation. In this case, we shall denote the conditionally optimal search policy as $\{\phi^*(i,t|\infty)\}$ in order to distinguish it from the policy with a finite stopping time.

Corollary 1.6. If the optimal search never stops searching until detection, then for all i and $t \in [t^0,\infty)$,

$$\alpha_i \phi^*(i,t|\infty) = \frac{C}{\sum_i c_i/\alpha_i}$$

where t^0 is given by

$$(9) \qquad t^0 = \frac{1}{C} \left[\sum_i \frac{c_i}{\alpha_i} \left(\log \frac{\alpha_i p_i/c_i}{\alpha_j p_j/c_j} \right) \right],$$

and j is the box having the minimum value of $\alpha_i p_i/c_i$. The state $P(t)$, $t \geq t^0$, is

$$(10) \qquad \{p_i(t)\} = \left\{ \frac{c_i/\alpha_i}{\sum_i c_i/\alpha_i} \right\} .$$

4. The Optimal Stopping Time

In this section, we derive the condition for the optimal stopping time. For this purpose, hereafter, we deal with the stopping time T as a variable. Let $P(T)$ denote the posterior probability vector of the target distribution when the conditionally optimal search with stopping time T fails to detect the target. We denote the optimal stopping time as T^*, and therefore, the optimal policy is determined from Theorem 1 with $T = T^*$.

Theorem 2. Let $Q(T)$ be the non-detection probability of the conditionally optimal search with stopping time T, $Q(T) = \sum_i p_i \exp(-\alpha_i \psi^*(i,T))$. A necessary condition for the optimal stopping time T^* is $\lambda(T^*) = Q(T^*)$, $\lambda(T^*-\delta) \geq Q(T^*-\delta)$ and $\lambda(T^*+\delta) \leq Q(T^*+\delta)$, where $0 < \delta << 1$ and $\lambda(T)$ is given by Corollary 1.1.

Proof: Suppose $T \leq T^*$. Since $f(P,\Phi^*,T)$ is a non-increasing function of T, in the neighborhood of T^*, the following inequalities are obtained,

$$\begin{aligned} 0 &\geq f(P,\Phi^*(T),T) - f(P,\Phi^*(T-\delta),T-\delta) \\ &\geq f(P,\Phi^*(T),T) - f(P,\Phi^*(T),T-\delta) \\ &= C\,\delta\,[\,Q(T) - \lambda(T)\,] + o(\delta). \end{aligned}$$

Therefore, $Q(T) \leq \lambda(T)$, in the neighborhood of T^* such as $T \leq T^*$.

Suppose $T \geq T^*$. The following relation is valid for $\delta > 0$

$$f(P,\Phi^*(T+\delta),T+\delta) \leq f(P,\Phi^*(T),T) + Q(T)f(P(T),\Phi^*(\delta),\delta),$$

where $\Phi^*(\delta)$ is the conditionally optimal search policy when the initial state is $P(T)$ and the duration of the search is δ. Then we obtain

$$Q(T)f(P(T),\Phi^*(\delta),\ \delta) = C\ \delta\ [\ Q(T) - \lambda(T)\ ] + o(\delta).$$

Therefore, in the neighborhood of T^*,

$$0 \leq f(P,\Phi^*(T+\delta),T+\delta) - f(P,\Phi^*(T),T) \leq C\ \delta\ [\ Q(T) - \lambda(T)\ ] + o(\delta),$$

hence, if $T \geq T^*$,

$$Q(T) \geq \lambda(T).$$

From these relations, the theorem was deduced. (Q.E.D.)

The following corollaries are deduced from Theorem 2 immediately.

Corollary 2.1. A necessary condition for the optimal stopping time T^* is

$$(11)\qquad \frac{\alpha_i p_i(T)R_i}{c_i} \begin{cases} \geq 1 & \text{if } T < T^*, \\ = 1 & \text{if } T = T^*, \\ \leq 1 & \text{if } T > T^*, \end{cases}$$

for any T in the neighborhood of T^* and any i such as $\phi^*(i,T) > 0$.

Corollary 2.2. A necessary condition for $T^* = 0$ is $\max \alpha_i p_i R_i / c_i \leq 1$.

Corollary 2.3. If $\max \alpha_i p_i(T) R_i / c_i > 1$, the search should not be stopped at T.

Corollary 2.4. A sufficient condition for $T^* = \infty$ is $\sum_i \{c_i / \alpha_i R_i\} \leq 1$.

The last one deduced from Corollary 2.3 corresponds to Ross' Theorem 1.2.

From these corollaries, if $\max \alpha_i p_i R_i / c_i \geq 1$, we should start the search. This seems to be reasonable, because the expected reward per unit search effort at $t = 0$ is larger than the cost of the search. But it is interesting to note that, in some cases, even if $\alpha_i p_i R_i / c_i < 1$ for all boxes, the search should be started too. The following example is the case.

[Example 1]

We consider the two box search having the following parameters.

i	1	2
α_i	1	1
c_i	1	0.45
R_i	2	0.8
p_i	0.45	0.55
C	1	

In this case, max $\alpha_i p_i R_i / c_i = 0.9778$ for $i = 2$. But since $f(P,\Phi^*(\infty),\infty) = -0.03517$, to stop immediately is not optimal. (As is shown in Section 5, the optimal policy for this example is to search box 2 until $t^0 = 0.4496$, and thereafter, search both boxes with effort distribution $c_1\phi^*(1,t|\infty) : c_2\phi^*(2,t|\infty) = 0.690 : 0.310$ until the target is found.)

Theorem 3. If $(\sum_i c_i/\alpha_i)^2 > \sum_i c_i R_i/\alpha_i$, the optimal stopping time is finite.

Proof: The following relation is valid for any T^1, T^2, $T^1 \geq T^2$.

$$f(P,\Phi^*(T^1),T^1) - f(P,\Phi^*(T^1),T^2)$$
$$= \sum_i \int_{T^2}^{T^1} p_i e^{-\alpha_i \psi^*(i,\tau|T^1)} [C - \alpha_i \phi^*(i,\tau|T^1) R_i] d\tau.$$

Let $T^1 = \infty$ and $T^2 = t^0$ given by (9), we have

$$\alpha_i \phi^*(i,t|\infty) = \frac{C}{\sum_i c_i/\alpha_i}$$

and

$$p_i e^{-\alpha_i \psi^*(i,t|\infty)} = \frac{c_i}{\alpha_i} \lambda(t^0) e^{-\frac{C}{\sum_i c_i/\alpha_i}(t-t^0)}$$

for $t \geq t^0$ and for all i, as is shown in Corollary 1.6. Then,

$$f(P,\Phi^*(\infty),\infty) - f(P,\Phi^*(\infty),t^0) = \lambda(t^0)[(\sum_i \frac{c_i}{\alpha_i})^2 - \sum_i \frac{c_i R_i}{\alpha_i}].$$

Therefore, if $(\sum_i c_i/\alpha_i)^2 > \sum_i (c_i R_i/\alpha_i)$, we obtain

$$f(P,\Phi^*(\infty),\infty) > f(P,\Phi^*(\infty),t^0) \geq f(P,\Phi^*(t^0),t^0).$$

Namely, non-stopping policy is not optimal. (Q.E.D.)

The next corollary is the contrapositive statement of Theorem 3.

Corollary 3.1. A necessary condition for $T^* = \infty$ is $(\sum_i c_i/\alpha_i)^2 \leq \sum_i c_i R_i/\alpha_i$.

As is shown in the next section, Example 1 mentioned before corresponds to the case $T^* = \infty$, and actually satisfies the above condition, but it does not satisfy Corollary 2.4 which is a sufficient condition for $T^* = \infty$.

Theorem 4. If $R_i \leq c_i/\alpha_i$ for all i, the search should not be started.

Proof: We obtain the following inequality from (1) and Corollary 1.4.

$$f(P,\Phi^*,T) = \int_0^T Q(t)[C - \sum_j p_j(t)\alpha_j \phi^*(j,t) R_j] dt$$

$$= \int_0^T Q(t)[C - \frac{p_i(t)\alpha_i}{c_i} \sum_j c_j \phi^*(j,t) R_j] dt$$

$$\geq \int_0^T Q(t)C[1 - \frac{\alpha_i p_i(t)R_i}{c_i} \Big| R_i = \max R_j,\ j \in I_S(t)]dt \ .$$

Here, if $R_i \leq c_i/\alpha_i$, the integrand is positive at any time $t \in [0,T]$. Therefore, $f(P,\Phi^*,T) > 0$ for any $T(T > 0)$, and to stop immediately is optimal.

(Q.E.D.)

Theorem 5. Suppose $0 < T^* < \infty$.

1. If $\alpha_i p_i R_i/c_i > 1$, box i is an element of $I_D(T^*)$.
2. If $i \in I_S(T^*)$ and $\alpha_i p_i R_i/c_i < \alpha_j p_j R_j/c_j$, box j is an element of $I_D(T^*)$.
3. If $i \in I_D(T^*)$ and $\alpha_i p_i R_i/c_i < \alpha_j p_j R_j/c_j$ and $\alpha_i p_i/c_i < \alpha_j p_j/c_j$, box j is an element of $I_D(T^*)$.
4. If $i \in I_S(T^*)$ and $R_i > 0$, R_i is larger than $\sum_{I_S(T^*)} c_i/\alpha_i$.

Proof: 1. Since $\alpha_i p_i R_i/c_i > 1$, to stop immediately is not optimal from Corollary 2.3. Here we assume $i \notin I_D(T^*)$ and let k be an element of $I_S(T^*)$. From Corollary 1.1 and Theorem 2, we obtain

$$\frac{\alpha_i p_i R_i}{c_i} \leq \frac{\alpha_k p_k R_k}{c_k} e^{-\alpha_k \psi(k,T^*)} = \lambda(T^*) = \sum_j p_j e^{-\alpha_j \psi^*(j,T^*)} .$$

This implies $\sum_j p_j \exp(-\alpha_j \psi^*(j,T^*)) > 1$. The contradiction was deduced from the assumption $i \notin I_D(T^*)$.

2. We assume $j \notin I_D(T^*)$. From Corollary 1.1, we obtain

$$\frac{\alpha_j p_j R_j}{c_j} \leq \frac{\alpha_i p_i R_i}{c_i} e^{-\alpha_i \psi^*(i,T^*)} = \lambda(T^*).$$

Since $\alpha_i p_i R_i/c_i < \alpha_j p_j R_j/c_j$, the above inequality leads to a contradiction $\exp(-\alpha_i \psi^*(i,T^*)) > 1$, $\psi^*(i,T^*) > 0$.

3. We assume $j \notin I_D(T^*)$. Applying Theorem 1 at time t such as $\phi^*(i,t) > 0$, we obtain

$$\frac{\alpha_j p_j}{c_j} [R_j + C(T^* - t)] \leq \frac{\alpha_i p_i}{c_i} [R_i e^{-\alpha_i \psi^*(i,T^*)} + C\int_t^{T^*} e^{-\alpha_i \psi^*(i,\tau)} d\tau].$$

This inequality is rearranged as follows

$$\frac{\alpha_j p_j R_j}{c_j} - \frac{\alpha_i p_i R_i}{c_i} e^{-\alpha_i \psi^*(i,T^*)} \leq C\int_t^{T^*} (\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,\tau)} - \frac{\alpha_j p_j}{c_j}) d\tau.$$

The left-hand side of this inequality is positive and the right side is negative by the conditions. The assumption $j \notin I_D(T^*)$ brought this contradiction.

4. If $i \in I_S(T^*)$, then $\alpha_i R_i p_i(T^*)/c_i = 1$ from Theorem 2 and R_i is the same for all $i \in I_S(T^*)$ from Corollary 1.5 (3). We have

$$\sum_{I_S(T^*)} \frac{c_i}{\alpha_i R_i} = \frac{1}{R_i} \sum_{I_S(T^*)} \frac{c_i}{\alpha_i} = \sum_{I_S(T^*)} p_i(T^*) < 1.$$

Therefore,

$$R_i \,\Big|\, i \in I_S(T^*) > \sum_{I_S(T^*)} \frac{c_i}{\alpha_i}.$$

(Q.E.D.)

5. Optimal Policy for Two-box Search

In this section, we derive the optimal policy and its risk function for two-box case explicitly.

Applying Corollary 1.3 and 1.4 to the two-box problem, we can easily find that, when the optimal stopping time T^* is finite, the optimal policy searches only one box, another box being left unsearched.

$$\psi^*(i,T^*) = \frac{CT^*}{c_i}, \quad \psi^*(j,T^*) = 0, \quad j \neq i, \quad T^* < \infty.$$

From this, the type of the optimal policy at a state P for two-box search is one of the following.

1) Search is stopped immediately (this policy is denoted as Φ_0).
2) Search is started in box i (i=1 or 2) and is continued until a stopping time T ($<\infty$), if the target is not found. Another box is never searched (denoted as Φ_i).
3) Search is started in box i and is continued until t, and thereafter, box 1 and 2 are searched in parallel in such a way that the state variable remains unchanged at $P(t)$ (denoted as Φ_i^∞).

Let T^i and t_i^0 be the optimal value of T and t which minimize the conditional risk function given that Φ_i and Φ_i^∞ are employed respectively. T^i and t_i^0 are readily derived from Corollary 2.1 and 1.6 as follows.

$$(12) \qquad T^i = \frac{c_i}{\alpha_i C} \log\left[\frac{\alpha_i p_i}{c_i p_j}\left(R_i - \frac{c_i}{\alpha_i}\right)\right], \quad \text{if } R_i > \frac{c_i}{\alpha_i}, \quad p_i > \frac{c_i}{\alpha_i R_i}.$$

$$(13) \qquad t_i^0 = \frac{c_i}{\alpha_i C} \log \frac{\alpha_i p_i c_j}{\alpha_j p_j c_i}, \quad \text{if } p_i > p_i^\infty, \quad \text{where } p_i^\infty = \frac{c_i/\alpha_i}{c_1/\alpha_1 + c_2/\alpha_2}.$$

Since the policies have been determined, we can calculate the risk functions corresponding to Φ_0, Φ_i and Φ_i^∞.

(14) $f(P,\Phi_0,0) = 0,$

(15) $f(P,\Phi_i,T^i) = \frac{c_i}{\alpha_i} - p_i R_i + p_j \frac{c_i}{\alpha_i} \log \left[\frac{\alpha_i p_i}{c_i p_j}\left(R_i - \frac{c_i}{\alpha_i}\right)\right]$

where $R_i > \frac{c_i}{\alpha_i}$ and $p_i > \frac{c_i}{\alpha_i R_i} = p_i(T^i),$

(16) $f(P,\Phi_i^\infty,\infty) = -(p_1 R_1 + p_2 R_2) + \frac{c_i}{\alpha_i} + p_j\left[\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} + \frac{c_i}{\alpha_i}\log\frac{\alpha_i p_i c_j}{\alpha_j p_j c_i}\right]$, if $p_i > p_i^\infty$.

Therefore, the optimal risk function and the optimal policy are determined by the relation

$$f(P,\Phi^*,T^*) = \min\,[f(P,\Phi_0,0),\ \min_i f(P,\Phi_i,T^i),\ \min_i f(P,\Phi_i^\infty,\infty)].$$

Since there are only two boxes in this case, the initial state can be represented by a scalar p_i, $i = 1$ or 2, instead of the vector P. Consider the state p_i is varied from zero to unity for a set of search parameters. The type of the optimal policy is unchanged for some interval of p_i, but in passing a point the type makes change. Here, an optimal policy region (abbreviated as OPR hereafter) is defined as an interval of p_i in which a definite type of policy is optimal, and an OPR is named after the type of its optimal policy.

Using the above relation, we shall examine the structure of OPR when p_i is varied.

Since $f((p_i=1,p_j=0),\Phi_i,T^i) = f((p_i=1,p_j=0),\Phi_i^\infty,\infty) = \frac{c_i}{\alpha_i} - R_i$, Φ_i OPR or Φ_i^∞ OPR exists in the neighborhood of $p_i \simeq 1$, if and only if

(17) $R_i - \frac{c_i}{\alpha_i} > 0.$

From $f(P,\Phi_i,T^i) > f(P,\Phi_i^\infty,\infty)$, we obtain

(18) $\frac{c_i}{\alpha_i}\log\left[\frac{\alpha_j}{c_j}\left(R_i - \frac{c_i}{\alpha_i}\right)\right] \le \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} - R_j.$

If (17) and (18) hold, there exists Φ_i^∞ OPR, and therefore, Φ_j^∞ OPR exists too, because Φ_j^∞ OPR must always be adjacent to Φ_i^∞ OPR. Since $f(P(T^i),\Phi_i,T^i)$ is zero from (15), if (17) (18) and $f(P(T^i),\Phi_j^\infty,\infty) > 0$ are established, OPR has a structure such as $[\cdots,\Phi_0,\Phi_i,\Phi_j^\infty,\Phi_i^\infty]$. The condition $f(P(T^i),\Phi_j^\infty,\infty) > 0$ results an inequality

(19) $$R_j(R_i - \frac{c_i}{\alpha_i}) \le R_i(\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}) + \frac{c_i}{\alpha_i}[\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} + \frac{c_j}{\alpha_j} \log\{\frac{\alpha_j}{c_j} (R_i - \frac{c_i}{\alpha_i})\}].$$

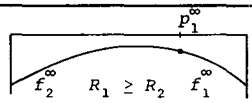
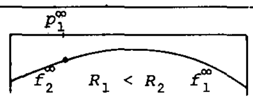
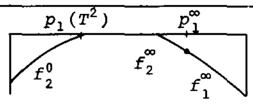
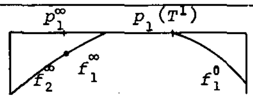
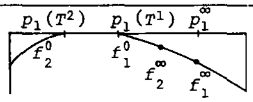
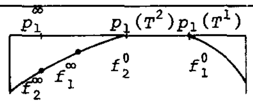
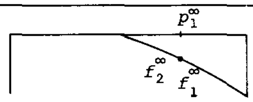
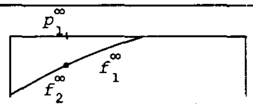
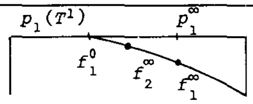
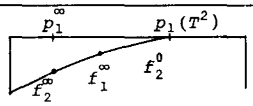
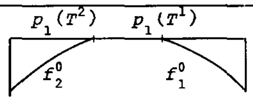
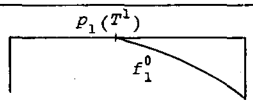
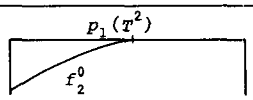
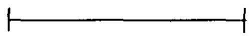
Here, we define ($\tilde{17}$) ($\tilde{18}$) and ($\tilde{19}$) as the relations corresponding to (17) (18) and (19) respectively with opposit inequality signs. Then, the condition specifying a structure of OPR is derived by considering combinations of the relations (17) (18) (19) ($\tilde{17}$) ($\tilde{18}$) and ($\tilde{19}$) for $i = 1,2$. Hereafter, without any loss of generality, we presume that c_1/α_1 is not smaller than c_2/α_2. Let the probability of the target being in box 1 be the state variable p_1 instead of P. Then, the structure of OPR are classified in eight types and the conditions of search parameters corresponding to each type are obtained as is shown in Table 1. In Table 1, the notation, $[\Phi_2^\infty, \Phi_1^\infty]$ for example, implies that policy Φ_2^∞ is optimal in the interval $0 \le p_1 \le p_1^\infty$ and Φ_1^∞ in $p_1^\infty < p_1 \le 1$. Furthermore, the risk functions corresponding to Φ_i and Φ_i^∞, (15) and (16), are abridged as f_i^0 and f_i^∞ respectively.

Table 1 tells that what conditions on the search parameters are necessary for the OPR structure being of a given type. In Fig.1, the conditions are visualized by use of $(R_1\text{-}R_2)$-plane. In Fig.1, the notation, [17(1)] for example, means that the indicated curve is given by formula (17) with $i = 1$ and its inequality replaced by an equality sign. The numbers I ∿ VIII correspond to the Type No. in Table 1.

As is shown in Fig.1, the $(R_1\text{-}R_2)$-plane is covered exhaustively by disjoint regions each of which corresponds to a type of OPR structure. This implies that all structure of OPR are listed up exhaustively with the conditions given by Table 1. Thus, the optimal policy and the associated risk function are completely determined explicitly as are given in Table 1 for any set of search parameters, p_i, c_i, α_i, and R_i, $i = 1,2$.

As for Example 1 described in previous section, we can easily ascertain that the initial state P belong to Φ_2^∞ OPR of the structure type I, and therefore, the optimal policy is to search box 2 until t_2^0, and thereafter, to continue the parallel search given by Corollary 1.6 until the target is found.

Table 1. Structure of the optimal policy region

Type No.	Condition	OPR structure	Optimal $f(\cdot)$ vs. p_1 curve
I	(17) $(\tilde{18})$ for both i = 1 and 2.	$[\ \Phi_2^\infty\ ,\ \Phi_1^\infty\ ]$	 
II	(18) for i = 2, $(\tilde{18})$ and $(\tilde{19})$ for i = 1, and (17) for both i = 1,2.	$[\ \Phi_2, \Phi_0, \Phi_2^\infty, \Phi_1^\infty\ ]$	
	As above with box numbers 1 and 2 interchanged.	$[\ \Phi_2^\infty, \Phi_1^\infty, \Phi_0, \Phi_1\ ]$	
III	(18) for i = 2, $(\tilde{18})$ and (19) for i = 1, and (17) for both i = 1,2.	$[\Phi_2, \Phi_0, \Phi_1, \Phi_2^\infty, \Phi_1^\infty]$	
	As above with box numbers 1 and 2 interchanged.	$[\Phi_2^\infty, \Phi_1^\infty, \Phi_2, \Phi_0, \Phi_1]$	
IV	(17) $(\tilde{18})$ and $(\tilde{19})$ for i = 1, and $(\tilde{17})$ for i = 2.	$[\ \Phi_0, \Phi_2^\infty, \Phi_1^\infty\ ]$	
	As above with box numbers 1 and 2 interchanged.	$[\ \Phi_2^\infty, \Phi_1^\infty, \Phi_0\ ]$	
V	(17) $(\tilde{18})$ and (19) for i = 1, $(\tilde{17})$ for i = 2.	$[\ \Phi_0, \Phi_1, \Phi_2^\infty, \Phi_1^\infty\ ]$	
	As above with box numbers 1 and 2 interchanged.	$[\ \Phi_2^\infty, \Phi_1^\infty, \Phi_2, \Phi_0\ ]$	
VI	(17) for both i = 1,2, and (18) for both i = 1,2.	$[\ \Phi_2, \Phi_0, \Phi_1\ ]$	
VII	$(\tilde{17})$ for i = 2, (17) and (18) for i = 1.	$[\ \Phi_0\ ,\ \Phi_1\ ]$	
	As above with box numbers 1 and 2 interchanged.	$[\ \Phi_2\ ,\ \Phi_0\ ]$	
VIII	$(\tilde{17})$ for both = 1,2.	$[\ \Phi_0\ ]$	

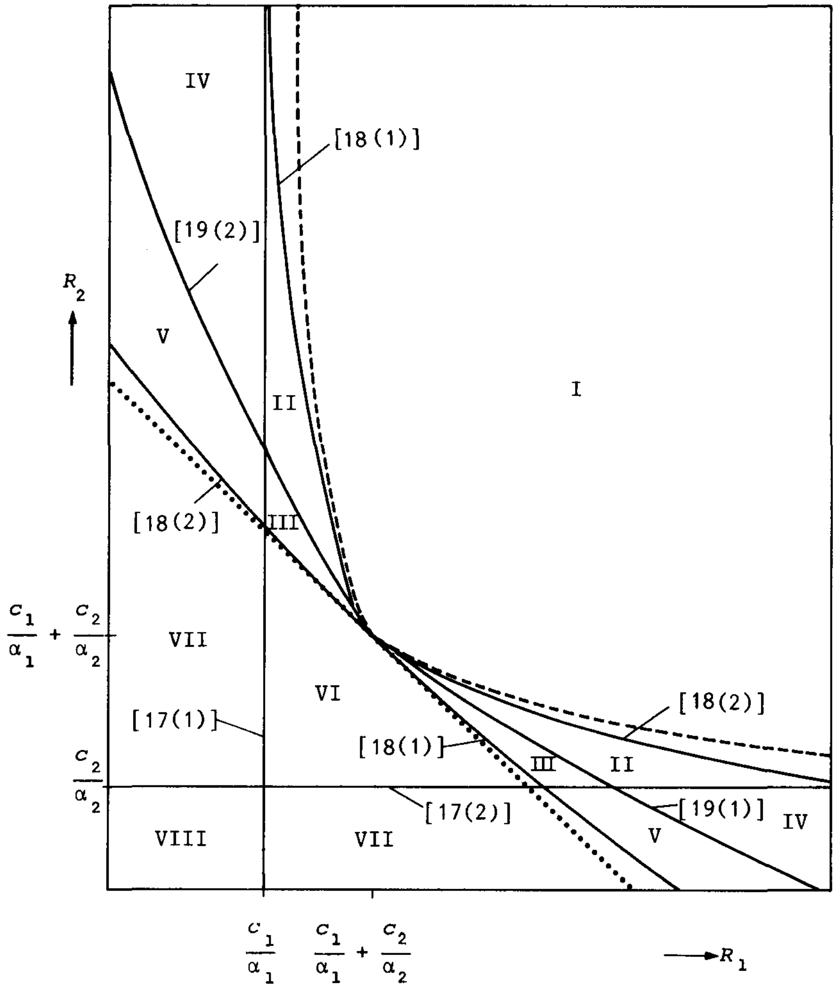


Fig. 1. The optimal policy region in (R_1-R_2)-plane

6. Discussions

In this section, discussions are presented to the results obtained in the previous sections. First, the conditions for the optimal policy are given their physical meaning, and then the structure of the optimal policy is examined for the identical reward case. Possible extensions of our model are also discussed, and some comments on the special cases are given.

6.1. The physical meanings of the conditions for the optimal policy

Let us consider the physical meaning of Theorem 1. Denoting the search effort allocated to box i in the interval $(t,\tau]$ as $\eta^*(i,t,\tau) = \int_t^\tau \phi^*(i,\xi)d\xi$, we can rewrite the condition (3) as follows.

$$(3)' \qquad \frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,t)} [R_i + \int_t^T e^{-\alpha_i \eta^*(i,t,\tau)} \{C - \alpha_i \phi^*(i,\tau)R_i\} d\tau] = \lambda(t).$$

The term $\int_t^T [C - \alpha_i \phi^*(i,\tau)R_i] \exp(-\alpha_i \eta^*(i,t,\tau))d\tau$ is interpreted as the expected risk of the optimal search from t to T conditional on that the target is in box i and is not detected until t. Here we recall that the search is to be continued until T if the target is not detected, since the stopping time is T. Therefore, if we detect the target in box i at t, we earn the reward R_i and also save the risk of the search otherwise to be incurred in $[t, T]$. The sum of R_i and the conditional risk can be interpreted as the shadow price which motivates the search in box i at t. Meanwhile, the term $\alpha_i p_i \exp(-\alpha_i \psi^*(i,t))$ is the detection probability of the target in box i at t when the unit search effort is allocated, and the denominator, c_i, is the cost of the unit search effort. Hence, the left-hand side of (3)′ means the expected shadow price - cost ratio when the unit search effort is allocated to box i at t. Therefore, Theorem 1 means that, if search effort is to be allocated to box i at t, the amount of the search effort should be determined in such a way that the expected shadow price - cost ratio mentioned above is balanced to $\lambda(t)$ among the boxes being searched at t, and if search effort should not be allocated to box i at t, the box i does not have a larger expected shadow price - cost ratio than $\lambda(t)$.

Theorem 2 states the behavior of $\lambda(t)$ at the neighborhood of the optimal stopping time T^*. As mentioned before, $\lambda(t)$ is a strictly decreasing function of t (Corollary 1.2) and it reaches the smallest value $\lambda(T)$ at the stopping time T. If $\lambda(T) > Q(T)$, the optimal stopping time T^* is larger than T (Corollary 2.3). Furthermore, at the optimal stopping time T^*, $\lambda(T^*)$ is equal to $Q(T^*)$. The physical meaning of this Theorem is elucidated by Corollary 2.1 as follows. The largest value of the expected marginal reward-cost ratio $\alpha_i p_i(T)R_i/c_i$ decreases across unity when the stopping time T increases across T^*. Therefore, the search should be stopped when the expected marginal reward becomes smaller than the cost.

Here, we consider the case in which $T^* = \infty$. The following equation is deduced from (3) and $\lim_{T\to\infty} \psi^*(i,T) = \infty$.

$$\frac{\alpha_i p_i}{c_i} e^{-\alpha_i \psi^*(i,t)} C \int_t^\infty e^{-\alpha_i \eta^*(i,t,\tau|\infty)} d\tau = \lambda(t).$$

When $t \geq t^0$, the integration of the left-hand side of this equation is calculated by using Corollary 1.6 and we obtain

$$Q(t) = \lambda(t), \quad t \geq t^0.$$

In this case, as the state remains unchanged at $\{p_i(t)\} = \{(c_i/\alpha_i)/(\sum_i c_i/\alpha_i)\}$ in $[t^0,\infty)$, the above relation is maintained in this interval, and therefore, the search should not be stopped.

It is worthwhile to point out that the infinitesimal look-ahead policy is not optimal in this problem. As mentioned above, the optimal policy is determined by balancing the expected shadow price - cost ratio given by (3) at each time t. Apparently, the expected shadow price at t is affected by both the past actions before t and the future actions until T^*. This is one of the reason that the infinitesimal look-ahead policy is not optimal. But, as is shown later, when $R_i = R$ for all i, the condition (3) relates only to the past actions. Therefore, in this case, the infinitesimal look-ahead policy becomes the optimal policy.

Here we elucidate the meaning of the condition of Theorem 3. From Corollary 1.6, the optimal policy $\{\phi^*(i,t|\infty)\}$, $t \in [t^0,\infty)$, is to search all boxes in parallel with the balanced effort distribution $\alpha_i\phi^*(i,t|\infty) = C/(\sum_i c_i/\alpha_i)$, and consequently, the state remains unchanged at $\{p_i^0\} = \{(c_i/\alpha_i)/(\sum_i c_i/\alpha_i)\}$. Let $E(C|P^0)$ be the expected cost of the search when the initial state is $\{p_i^0\}$ and the policy $\{\phi^*(i,t|\infty)\}$ is employed. We obtain the following recurrence relation by considering the search in $[t,t+\delta]$, $t \geq t^0$.

$$E(C|P^0) = C\delta + \sum_i p_i^0(1 - \alpha_i\phi^*(i,t|\infty)\delta)E(C|P^0).$$

From this, we obtain $E(C|P^0) = \sum_i c_i/\alpha_i$. The expected reward $E(R|P^0)$ of the balanced search is $E(R|P^0) = \sum_i p_i^0 R_i$.

On the other hand, by using (10) the condition of Theorem 3 is rewritten as

$$\sum_i c_i/\alpha_i > \sum_i p_i^0 R_i.$$

Therefore, the condition, $(\sum_i c_i/\alpha_i)^2 > \sum_i c_i R_i/\alpha_i$, means $E(C|P^0) > E(R|P^0)$, and it is reasonable that the search under this conditoin should be stopped at a finite stopping time.

6.2. The optimal policy when the rewards are identical for all boxes

We can easily see that the problem is extremely simplified when the rewards are identical for all boxes, $R_i = R$. The reason is explained as follows.

If the rewards R_i are identical for all boxes, $T_i = T$ for all boxes $i \in I_D(T)$ from Corollary 1.5.(3). Therefore, $I_D(T) = I_S(T)$, and $\alpha_i p_i \exp(-\alpha_i \psi^*(i,t))/c_i$ = constant for all $i \in I_S(T)$ from Corollary 1.4. Consequently, Theorem 1 is reduced to the following simple form.

$$\text{If} \quad \phi^*(i,t) > 0, \quad \frac{\alpha_i p_i R}{c_i} e^{-\alpha_i \psi^*(i,t)} = \lambda(t).$$

$$\text{If} \quad \phi^*(i,t) = 0, \quad \frac{\alpha_i p_i R}{c_i} \leq \lambda(t).$$

Therefore, as expected, the conditionally optimal search policy is identical with the policy which minimizes the expected search cost. The condition for the optimal stopping time T^* is $\alpha_i p_i(T^*)R/C_i = 1$ from Corollary 2.1. Therefore, the optimal policy can be determined by the infinitesimal look-ahead policy.

If $\max \alpha_i p_i R/c_i < 1$, the search should not be started.

If $\max \alpha_i p_i R/c_i > 1$, then the search should be continued in such a way that $\alpha_i p_i(t)R/c_i$ is balanced among boxes and is stopped when the value $\alpha_i p_i(t)R/c_i$ reaches unity, namely, $T^* = \min \{t | \alpha_i p_i(t)R/c_i = 1\}$.

If $\alpha_i p_i(t^0)R/c_i > 1$ (t^0 is given by (9)), the search should not be stopped until detection.

These characteristics of the problem are shown finely in two-box case. If R_1 is equal to R_2, Table 1 is extremely simplified and possible OPR structures are reduced to Type I, VI, VII and VIII as is observed from Fig.1. Then necessary and sufficient conditions for the optimal policy are obtained as Table 2.

From Table 2, it is concluded that if $\max \alpha_i p_i R/c_i > 1$ and $\alpha_i p_i/c_i > \alpha_j p_j/c_j$, the optimal policy searches first box i. The condition $\max_i \alpha_i p_i R/c_i > 1$ implies searching should be continued at P from Corollary 2.3, and $\alpha_i p_i/c_i$ means the marginal detection probability in box i. Therefore, the observation mentioned above implies that if searching is continued, the box having the maximum value of marginal detection probability should be searched first. This search policy is identical with the optimal policy minimizing the expected search cost to find the target without stopping, and also, it corresponds to Ross' Theorem 4.3.

Table 2. Optimal policy when $R_1 = R_2 = R$

Type No.	Condition		Optimal Policy	$\max_i \alpha_i p_i R/c_i$
I	$R > \frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2}$	$\frac{\alpha_1 p_1}{c_1} \geq \frac{\alpha_2(1-p_1)}{c_2}$	Φ_1^∞	≥ 1 for $i=1$
		$\frac{\alpha_1 p_1}{c_1} < \frac{\alpha_2(1-p_1)}{c_2}$	Φ_2^∞	≥ 1 for $i=2$
VI	$\frac{c_1}{\alpha_1} + \frac{c_2}{\alpha_2} \geq R > \frac{c_1}{\alpha_1}$	$\frac{\alpha_1 p_1}{c_1} \geq \frac{1}{R} \left(> \frac{\alpha_2(1-p_1)}{c_2}\right)$	Φ_1	≥ 1 for $i=1$
		$\frac{\alpha_1 p_1}{c_1} < \frac{1}{R}$ $\frac{\alpha_2(1-p_1)}{c_2} < \frac{1}{R}$	Φ_0	less than 1
		$\frac{\alpha_2(1-p_1)}{c_2} \geq \frac{1}{R} \left(> \frac{\alpha_1 p_1}{c_1}\right)$	Φ_2	≥ 1 for $i=2$
VII	$\frac{c_1}{\alpha_1} \geq R > \frac{c_2}{\alpha_2}$	$\frac{\alpha_1 p_1}{c_1} \geq \frac{1}{R} \left(> \frac{\alpha_2(1-p_1)}{c_2}\right)$	Φ_1	≥ 1 for $i=1$
		$\frac{\alpha_1 p_1}{c_1} < \frac{1}{R}$	Φ_0	less than 1
VIII	$\frac{c_2}{\alpha_2} \geq R$		Φ_0	less than 1

6.3. Extentions

1. In this paper, we investigate a search process under the assumption of the exponential detection law, but this assumption is not essential to deal with the problem. The model is generalized easily by replacing the exponential detection function, $1 - \exp(-\alpha_i \psi(i,t))$, with a more general function $g(\psi(i,t))$ which is a strictly increasing, concave and differentiable function of $\psi(i,t)$. As discussed by De Guenin [8], these assumptions for $g(\psi(i,t))$ are needed to guarantee the solvability of the allocation of the total cost rate C to boxes at any time. By replacing $\exp(-\alpha_i \psi(i,t))$ and $\alpha_i \phi(i,t)\exp(-\alpha_i \psi(i,t))$ with

$1-g(\psi(i,t))$ and $dg(\psi(i,t))/dt$ respectively, all theorems obtained in this paper are valid.

2. The assumptions of the continuity (or the discontinuity) respect to the target space and the time space are also easily generalized without any difficulty. A model in which both the target space and the time space are discrete is of special importance when we calculate the optimal search strategy numerically. In this model, most of the theorems derived here are valid if the integration with respect to t is interchanged with the summation. However, the equation in some theorems, for example $\lambda(T^*) = Q(T^*)$ in Theorem 2, is not valid generally in consequence of the discontinuity of time.

The assumption of the continuous divisibility for C is very important for our model. If the total search cost rate C can not be divided arbitrarily, our model which has been investigated by the culculas of variation must be analyzed by the approach of the integer programming. Then, the conditions for the optimal policy should be complicated drastically.

3. In this paper, we assume that the search is stopped, if necessary, without any activity. But in some class of search, we may stop the search by conducting some activity to the probable target position which is guessed from the posterior probability distribution of the target. This search process is called as whereabouts search. It has been studied as a problem maximizing the whereabouts probability (the detection probability added to the correct guess probability for the target location when the search fails to detect the target), subject to a total search cost. Here, we extend our model to a whereabouts search model in which the objective of the search is to minimize the expected risk.

Consider a situation in which the initial state is P, a search strategy Φ_k is employed, and if it fails to detect the target, the search is stopped at T_k by guessing box k. Introducing a guess cost W_k for box k and a reward R'_k gained by a correct guess in box k, the expected risk is obtained as follows.

$$(20) \qquad f(P,\Phi_k,T_k) = \sum_i p_i[R_{ik}e^{-\alpha_i\psi(i,T_k)} - R_i + C\int_0^{T_k} e^{-\alpha_i\psi(i,\tau)}d\tau]$$

where $R_{ik} = R_i+W_k-\delta_{ik}R'_k$, $\delta_{ik} = 1$ if $i=k$, $\delta_{ik} = 0$ if $i\neq k$.

Therefore, our problem is formulated as a problem to find a triplet $(\Phi^*_{k^*},T^*_{k^*},k^*)$ which minimizes the expected risk $f(P,\Phi_k,T_k)$ given by (20), subject to the following restrictions.

$$T_k \geq 0 \quad \text{for all } k,$$

(21) $$\phi(i,t) \geq 0 \quad \text{for all } i \text{ and } t \in [0,T_k],$$

$$\sum_i c_i\phi(i,t)\Delta t = C\Delta t \quad \text{for any } t \in [0,T_k].$$

We can apply the same approach as mentioned in this paper to derive the optimal policy and the similar theorems for the conditionally optimal search strategy $\phi^*(i,t)$ and the conditionally optimal guess time T_k^* are obtained [9].

6.4. Some comments on the special case

Our model does not exclude the special case such as $c_i = \infty$ or $R_i \leq 0$ for some boxes.

If $c_i = \infty$, the expected shadow price-cost ratio of box i is always zero from (3). Consequently, as expected, search effort is never allocated to box i. In this case, as the inequality $(\sum_i c_i/\alpha_i)^2 > \sum_i c_iR_i/\alpha_i$ is always satisfied, the search should be stopped sooner or later (Theorem 3). This case corresponds to Chew's model in which an unsearchable box is assumed [1,2].

It is remarkable that a box, box i, is searched in some case, even if $R_i \leq 0$. If $p_i\alpha_i(R_i + CT^*)/c_i$ is large notwithstanding $R_i \leq 0$, $\psi^*(i,T^*) > 0$ from Theorem 1. In this case, detection of the target in box i does not bring any merits, but it can save the cost in searching the other boxes if the target is in box i. In other words, the search in box i contributes to minimize the expected risk by confirming the probable absence of the target in box i. The following example is the case.

[Example 2]

Let us consider the following two box search problem.

i	1	2
α_i	1	1
c_i	4	1
R_i	10	-0.4
p_i	0.75	0.25
C	1	

In this case, the search parameters correspond to Φ_2^∞ OPR of the structure type V in Table 1 and the optimal search policy is to start by searching in box 2 and it is continued until $t_2^0 = 0.2877$. Thereafter, both boxes are searched with the effort distribution $c_1\phi^*(1,t|\infty) : c_2\phi^*(2,t|\infty) = 0.8 : 0.2$ until the target is detected. Then, we obtain $f(P,\Phi^*(\infty),\infty) = -5.3137$.

In two-box case, there exists the optimal policy region searching in box

i when $R_i \leq 0$ as is shown in the OPR structure type IV and V. In these cases, it should be noted that searching in box i is conducted when R_j and $p_j (j \neq i)$ are considerably large, and always the parallel search follows when searching in box i fails to detect the target. Therefore, searching in box i is interpreted as a sort of preliminary search conducted before more fruitful search in box j.

6.5. Supplementary comment on two-box case

Kan [5] dealt with the quantized two-box search problem under the assumption $R_1 = R_2$, and pointed out the existence of the structure of the OPR $[\Phi_2^\infty, \Phi_1^\infty, \Phi_0, \Phi_1]$ and $[\Phi_2, \Phi_0, \Phi_2^\infty, \Phi_1^\infty]$. In our continuous search problem, there exist the policy regions characterized by the structures $[\Phi_j, \Phi_0, \Phi_j^\infty, \Phi_i^\infty]$ and $[\Phi_0, \Phi_i, \Phi_j^\infty, \Phi_i^\infty]$ as are seen in Table 1, Type II and V. The former is just the structure pointed out by Kan. But strangely, this type can exist only when $R_1 \neq R_2$ in our continuous search model. Furthermore, there exists a structure of OPR having five regions such as $[\Phi_2, \Phi_0, \Phi_1, \Phi_2^\infty, \Phi_1^\infty]$ or $[\Phi_2^\infty, \Phi_1^\infty, \Phi_2, \Phi_0, \Phi_1]$. (Type III in Table 1.) These discrepancies may be deduced from the difference of the model structure; Kan's is a quantized model and our model is continuous.

Finally, some additional explanations will be given to Fig. 1. The broken line in Fig. 1 shows the condition for $T^* = \infty$ (the outside region of the broken line) given by Corollary 2.4 ; $\sum_i c_i/\alpha_i R_i \leq 1$. This corollary corresponds to Ross' Theorem 1.2 which gives a sufficient condition for $T^* = \infty$, and as expected, is included in the region of Type I. The dotted line in Fig. 1 is the boundary of the condition for $T^* < \infty$ given by Theorem 3 ; $(\sum_i c_i/\alpha_i)^2 < \sum_i c_i R_i/\alpha_i$, (the inside region of the dotted line). It should be noted that the structure of OPR is complicated when the search parameters belong to the region between these two lines.

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