

ON CERTAINTY EFFECT IN EXPECTED UTILITY THEORY

Yoshio Tabata
Osaka University

(Received April 21, 1983; Revised August 18, 1983)

Abstract A phenomenon of certainty effect is well known as a typical paradox in expected utility theory and is often observed in the laboratory gambling to test the axiomatic system of utility theory. A necessary and sufficient condition that the certainty effect is observed is demonstrated by means of a generalized utility function and some properties of individual's behavior derived from the condition are discussed.

1. Introduction

It is recognized that the theory of individual's behavior under uncertainty constitutes a fundamental field of operations research, management science, economics and so on. An expected utility model plays an important part in studies on this field and the copious literature on the decision-making under uncertainty has its origin in the expected utility model[9]. This model is generally accepted as a normative model of rational choice. Thus, it is assumed that all the rational individuals should make their decisions according to a set of axioms such as transitivity, independence and so on. After a skillfully planned paradox against the axiomatic system was first introduced by a French economist Allais[1], many authors[2, 4] posed various examples that violate the so-called independence axiom systematically by means of laboratory experiments called laboratory gambings. Among others, Kahneman and Tversky[4] generalized the Allais' paradox and observed a so-called *certainty effect* (or common ratio effect), but they threw no light on the mathematical condition under which the certainty effect is observed.

In the present paper, we consider situations of decision-making under uncertainty and provide a necessary and sufficient condition that the certainty effect occurs by means of a generalized utility function introduced by Machina [6, 7]. In Sections 2 and 3 we summarize the certainty effect and the gener-

alized utility function which plays an important role to explain the certainty effect theoretically. Section 4 demonstrates a necessary and sufficient condition that the certainty effect occurs. Concluding remarks are discussed in the last section.

2. Certainty Effect

In the conventional utility theory, it is widely accepted that the utilities of outcomes are weighted by their probabilities. As was mentioned in the previous section, many authors described that people's preferences systematically violate this principle. The most famous counter-example to expected utility theory is a phenomenon of certainty effect labeled by Kahneman and Tversky. This effect is explained in a word that people overweight outcomes with certain, relative to outcomes with merely probable.

In this section we consider the certainty effect for the sake of the future discussion. To motivate our discussion on such a counter-example, let us consider the following Hagen-type simple laboratory gambling (or lotteries) as is usually employed to deal with the decision-making problem[1].

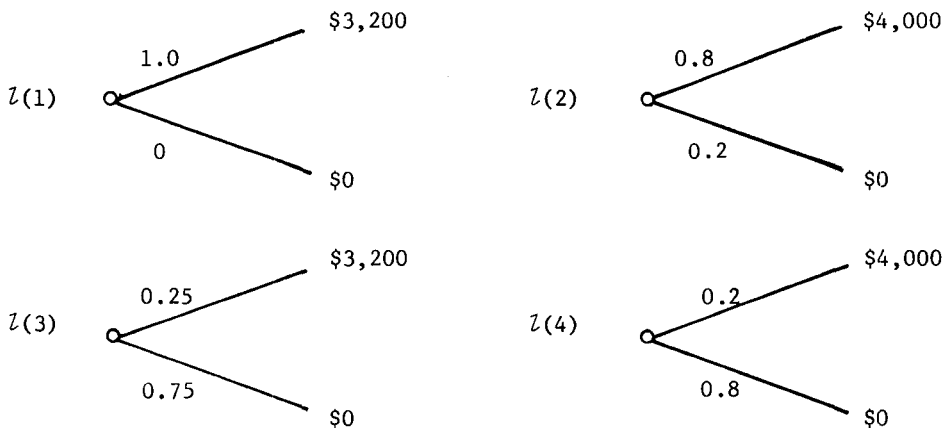


Fig. 1

We ask the following question to you; Which of the lottery would you prefer, $l(1)$ or $l(2)$ ($l(3)$ or $l(4)$) ? Most people prefer $l(1)$ to $l(2)$ ($l(4)$ to $l(3)$) and reason as follows; " In the choice problem between $l(1)$ and $l(2)$, we must choose between \$3,200 for certain and a gamble with \$0. Why gamble? On the other hand, in the choice problem between $l(3)$ and $l(4)$, the chance of

getting \$3,200 is almost the same as getting \$4,000, so we had better choose $l(4)$ over $l(3)$."

It is clear that the results violate the expected utility hypothesis since for a von Neumann-Morgenstern utility function u satisfying the set of axioms,

$$l(1) \succ l(2) \leftrightarrow V(l(1)) > V(l(2)) \leftrightarrow u(3,200) > 0.8u(4,000) + 0.2u(0)$$

and

$$\begin{aligned} l(4) \succ l(3) &\leftrightarrow V(l(4)) > V(l(3)) \\ &\leftrightarrow 0.2u(4,000) + 0.8u(0) > 0.25u(3,200) + 0.75u(0) \\ &\leftrightarrow 0.8u(4,000) + 0.2u(0) > u(3,200) \end{aligned}$$

where $V(l(\cdot))$ is an expected utility value for the lottery $l(\cdot)$.

As was stated by Tversky and Kahneman[10], these results imply that a reduction of the probability of an outcome by a constant factor has more impact when the outcome was initially certain than when it was merely probable.

In order to deal with this phenomenon more generally, we introduce the following Hagen-type lotteries F_1, F_2 ;

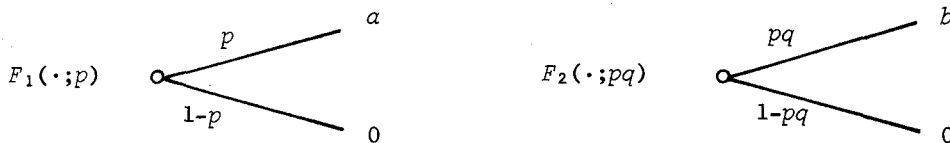


Fig. 2

where $0 \leq p \leq 1, 0 \leq q \leq 1$ and $0 < a \leq qb$.

Assume that an individual prefers F_1 to F_2 for $p = p_1$, that is,

$$F_1(\cdot; p_1) \succ F_2(\cdot; p_1q) \longleftrightarrow V(F_1(\cdot; p_1)) > V(F_2(\cdot; p_1q)).$$

Decreasing the value of p from p_1 gradually, we often observe that he changes his preference order into $F_2 \succ F_1$ for all $p < p_0$ ($< p_1$). In this case it is called that the certainty effect occurs at $p = p_0$. Figure 3 illustrates the situation. Since the result derived above is inconsistent with the expected utility hypothesis (or set of axioms in expected utility theory) and is obviously inexplicable within the framework of expected utility theory, his choice is usually explained as irrational. As the phenomenon is often observed, we cannot but conclude that it is impossible to explain the cer-

tainty effect that way. In other words, this phenomenon is very troublesome and it has not yet been made clear theoretically.

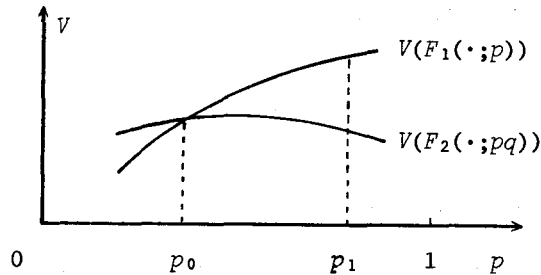


Fig. 3

3. Generalized Utility Function

As was described in the previous section, the phenomenon of certainty effect is not illustrated successfully in the framework of conventional expected utility theory. We can, however, deal with the phenomenon successfully from a mathematical view point by means of a generalized utility function recently introduced by Machina[6, 7]. In the present paper, we shall provide a necessary and sufficient condition that the certainty effect occurs. The concept of the generalized utility function is outlined as follows[3, 6]: Let $D[0, M]$ be the set of all probability distributions $F(\cdot)$ over the closed interval $[0, M]$, where $M < \infty$ may be an arbitrarily large monetary value. If the individual's preferences satisfy the axioms of expected utility hypothesis, we can express the preference functional as

$$(1) \quad V(F(\cdot)) = \int u(x) dF(x)$$

where $u(\cdot)$ is the von Neumann-Morgenstern utility function and all integrals are over $[0, M]$ unless otherwise specified. If $V(\cdot)$ is a differentiable function of $F(\cdot)$ (This means that the preferences are smooth), we can omit the assumption that $V(\cdot)$ is a linear functional on $D[0, M]$. Introducing the topology of weak convergence on the choice set $D[0, M]$ and supposing that the preference functional $V(\cdot)$ is Fréchet differentiable with respect to $F(\cdot)$, it is implied by using Hahn-Banach Theorem and Riesz' Representation Theorem that there exists a local (or generalized) utility function $U(\cdot; F(\cdot))$ on $[0, M]$ which satisfies

$$(2) \quad V(F^*) - V(F) = \int U(\xi; F(\xi)) [dF^*(\xi) - dF(\xi)] + o(\| F^* - F \|)$$

for any other distribution $F^*(\cdot) \in D[0, M]$, where $o(\cdot)$ denotes a function of higher order than its argument and $\| \cdot \|$ denotes the standard L^1 norm[5]. Simple examples of this functional form are

$$V(F) = \int \xi dF(\xi) \pm \frac{1}{2} [\int e^{-\xi} dF(\xi)]^2$$

with local utility function

$$U(x; F) = x \pm e^{-x} \int e^{-\xi} dF(\xi)$$

and

$$V(F) = \int \log(x + 1) dF(x) + \log [\int (x + 1) dF(x)]$$

with local utility function

$$U(x; F) = \log(x + 1) + \frac{x + 1}{\int (x + 1) dF(x)}$$

It should be noted that various local properties of the preference functional $V(\cdot)$ are derived by the properties of the local utility function $U(\cdot; \cdot)$. Moreover, some properties can be extended to global properties[6].

In the next section, we shall discuss the certainty effect by using this function.

4. Main Theorem and Corollaries

We now state and prove the major result of this paper. For simplicity we assume that the generalized utility function $U(x; F)$ is increasing in x for any $F(\cdot)$. It was proved by Machina that this assumption is equivalent to $V(F^*) > V(F)$ for $F^* \succ F$.

Theorem. For two simple Hagen-type lotteries described in Figure 2, a necessary and sufficient condition that the certainty effect is observed at $p = p_0$ is that the generalized utility function $U(x; F(\cdot; \alpha))$ satisfies the following inequalities: For $p \gtrless p_0$,

$$(3) \quad \int_0^1 \{ U(\alpha; F(\cdot; \alpha)) - U(0; F(\cdot; \alpha)) \} d\alpha \gtrless q \int_0^1 \{ U(b; F(\cdot; \alpha)) - U(0; F(\cdot; \alpha)) \} d\alpha$$

where

$$F(\cdot; \alpha) = \alpha F_1(\cdot; p) + (1 - \alpha) F_2(\cdot; pq) \quad (0 \leq \alpha \leq 1).$$

Proof: To prove this theorem, the use of path integral in the choice set $D[0, M]$ will succeed in our attempt. The outline of the method is given as follows: if the path $\{F(\cdot; \alpha) \mid \alpha \in [0, 1]\}$ is smooth enough so that the $\|F(\cdot; \alpha) - F(\cdot; \alpha^*)\|$ is differentiable in α at $\alpha = \alpha^*$, then the equation (2) may be rewritten as

$$(4) \quad \frac{d}{d\alpha} \{V(F(\cdot; \alpha))\}_{\alpha=\alpha^*} = \frac{d}{d\alpha} \left\{ \int U(x; F(\cdot; \alpha^*)) dF(x; \alpha) \right\}_{\alpha=\alpha^*}$$

since the derivative of higher order term $o(\cdot)$ will be zero at zero. Applying the fundamental theorem of integral calculus to the equation (4) derived above, we have

$$(5) \quad V(F(\cdot; 1)) - V(F(\cdot; 0)) = \int_0^1 \left[\frac{d}{d\alpha} \left\{ \int U(x; F(\cdot; \alpha^*)) dF(x; \alpha) \right\}_{\alpha=\alpha^*} \right] d\alpha^*.$$

It is noted that the lotteries F_1 and F_2 can be rewritten as

$$F_1(\cdot; p) = pG_a + (1 - p)G_0$$

$$F_2(\cdot; pq) = pqG_b + (1 - pq)G_0$$

where G_a denotes the probability distribution which assigns unit probability to the point a . Thus, it is clear that

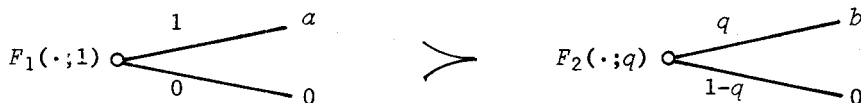
$$F(\cdot; 1) = F_1(\cdot; p)$$

$$F(\cdot; 0) = F_2(\cdot; pq)$$

and

$$(6) \quad \begin{aligned} F(\cdot; \alpha) &= \alpha F_1(\cdot; p) + (1 - \alpha) F_2(\cdot; pq) \\ &= \alpha p G_a + (1 - p) \alpha G_0 + (1 - \alpha) pq G_b + (1 - \alpha)(1 - pq) G_0. \end{aligned}$$

Now, without loss of generality, we assume that an individual prefers the lottery $F_1(\cdot; 1)$ to the lottery $F_2(\cdot; q)$, that is,



(Necessity) The assumption that the certainty effect occurs at $p = p_0$ implies that

$$F(\cdot;1) \succeq F(\cdot;0) \quad \text{for any } p \geq p_0$$

and

$$F(\cdot;1) \prec F(\cdot;0) \quad \text{for any } p < p_0,$$

that is,

$$(7) \quad V(F(\cdot;1)) - V(F(\cdot;0)) = \Phi(p) \begin{cases} \geq 0 & \text{for any } p \geq p_0 \\ < 0 & \text{for any } p < p_0. \end{cases}$$

On the other hand, from equations (5) and (6),

$$\begin{aligned} (8) \quad \Phi(p) &= V(F(\cdot;1)) - V(F(\cdot;0)) \\ &= \int_0^1 \left[\frac{d}{d\alpha} \left\{ \int U(x;F(\cdot;\alpha^*)) dF(x;\alpha) \right\}_{\alpha=\alpha^*} \right] d\alpha^* \\ &= \int_0^1 \left[\frac{d}{d\alpha} \left\{ \int U(x;F(\cdot;\alpha^*)) (\alpha p dG_\alpha + (1-p)\alpha dG_0 + (1-\alpha)pq dG_b \right. \right. \\ &\quad \left. \left. + (1-\alpha)(1-pq) dG_0) \right\}_{\alpha=\alpha^*} \right] d\alpha^* \\ &= \int_0^1 \left[\frac{d}{d\alpha} \left\{ \alpha p U(a;F(\cdot;\alpha^*)) + (1-p)\alpha U(0;F(\cdot;\alpha^*)) + (1-\alpha)pq U(b;F(\cdot;\alpha^*)) \right. \right. \\ &\quad \left. \left. + (1-\alpha)(1-pq) U(0;F(\cdot;\alpha^*)) \right\}_{\alpha=\alpha^*} \right] d\alpha^* \\ &= p \int_0^1 \left[U(a;F(\cdot;\alpha^*)) - qU(b;F(\cdot;\alpha^*)) - (1-q)U(0;F(\cdot;\alpha^*)) \right] d\alpha^*. \end{aligned}$$

It should be noted that $\Phi(p)$ is a continuous function of p and $\Phi(0) = 0$.

Moreover

$$\Phi(1) = V(F_1(\cdot;1)) - V(F_2(\cdot;q)) > 0$$

since $F_1(\cdot;1) \succ F_2(\cdot;q)$. Thus, as is shown in Figure 4, there exists some p_0 such that $\Phi(p_0) = 0$. These facts on equation (8) and inequality (7) imply that equation (3) is satisfied if the certainty effect occurs since $p > 0$.

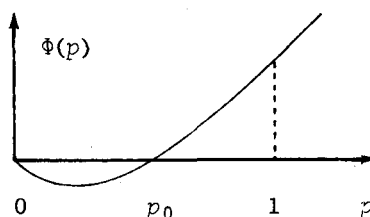


Fig. 4

(Sufficiency) It is clear that the following inequality holds if equation (3) is satisfied:

$$\begin{aligned} V(F_1(\cdot;p)) - V(F_2(\cdot;pq)) &= V(F(\cdot;1)) - V(F(\cdot;0)) \\ &= p \int_0^1 [U(a;F(\cdot;\alpha^*)) - qU(b;F(\cdot;\alpha^*)) - (1-q)U(0;F(\cdot;\alpha^*))] d\alpha^* \\ &= \Phi(p) \stackrel{>}{<} 0 \quad \text{for } p \stackrel{>}{<} p_0. \end{aligned}$$

This implies that the certainty effect occurs at $p = p_0$. \square

From this theorem we can easily derive the following Corollary 1. The corollary is concerned with a sufficient condition of the theorem which is more convenient to applications than condition (2). We define a generalized Arrow-Pratt measure of absolute risk aversion $A_U(x;F)$ for $U(x;F(\cdot;\alpha))$ as

$$A_U(x;F(\cdot;\alpha)) = -U''(x;F(\cdot;\alpha)) / U'(x;F(\cdot;\alpha)).$$

We say that an individual with generalized utility function U is more risk averse than an individual with generalized utility function W if $A_U > A_W$. Having specified a measure of the individual's aversion to risk, we give the corollary.

Corollary 1. If the generalized Arrow-Pratt measure $A_U(x;F)$ is negative for all $x \in [0, b]$, all $\alpha \in [0, 1]$ and some p^* , then the certainty effect occurs.

Proof: If $A_U(x;F) < 0$, then $U'' > 0$. This means that the generalized utility function $U(x;F(\cdot;\alpha))$ is a convex function of x in $[0, b]$. Thus, from the monotonicity and the convexity of $U(x;F(\cdot;\alpha))$, the inequality

$$qU(b;F(\cdot;\alpha)) + (1-q)U(0;F(\cdot;\alpha)) \geq U(bq;F(\cdot;\alpha)) > U(a;F(\cdot;\alpha))$$

holds for any $q, \alpha \in [0, 1]$ and some p^* . Thus,

$$U(a;F(\cdot;\alpha)) - qU(b;F(\cdot;\alpha)) < (1-q)U(0;F(\cdot;\alpha)),$$

and the integrand of equation (2) is negative. This result means that $\Phi(p^*) < 0$. On the other hand, $\Phi(1) > 0$, so the certainty effect is observed. \square

The condition of this corollary is sufficient but not necessary for theorem to hold. It should be pointed out that the condition $A_U(x;F) < 0$ or the convexity of $U(x;F)$ plays a key role for the occurrence of the certainty effect. This means that the risk lover (or plunger) always undergoes the

certainty effect. Moreover, Corollary 1 provides a valuable information to evaluate the class of generalized utility functions. The following Corollary 2 is the contraposition of Corollary 1.

Corollary 2. If $A_U(x;F(\cdot;\alpha)) > 0$ for all $x \in [0, b]$, then the certainty effect does never occur.

This corollary says that a risk averter whose Arrow-Pratt measure $A_U(x;F(\cdot;\alpha))$ is nonnegative or equivalently whose generalized utility function $U(x;F(\cdot;\alpha))$ is concave in x , never leads to the certainty effect.

5. Concluding Remarks

This paper is contributed to the certainty effect which is often observed in the laboratory gambling to test the expected utility theory and is inconsistent with the set of axioms. A necessary and sufficient condition under which the certainty effect is observed is provided by means of the generalized utility function defined on the choice set. It is shown that the certainty effect is not contradictory to an individual's rationality although it is rejected in the expected utility theory for the reason of individual's irrationality. It is presented that the convexity of the generalized utility function is a fundamental property of the sufficient condition and corresponds to the well-known properties of the theory of risk aversion in the expected utility hypothesis[8]. In Corollaries 1 and 2, we find that a plunger with negative generalized Arrow-Pratt measure always shows the phenomenon of certainty effect and a risk averter with positive one never does. This result suggests that the risk averter is faithful to the expected utility hypothesis and so long as a model with a concave von Neumann-Morgenstern utility function is employed, the expected utility hypothesis never breaks down. It is, however, clear that the individual's behavior may not be discussed fully by the model with concavity assumption only.

Acknowledgments

The author had many discussions with Prof. Y. Fukuba of Osaka University and Prof. K. Ito of Kobe University in preparing this paper. Their contributions to this paper have been very great and the author takes pleasure in acknowledging the important part played by them. He also thanks the referees for their helpful comments on this paper.

References

- [1] Allais, M. and Hagen, O. Eds.: *Expected Utility Hypotheses and the Allais Paradox*. D. Reidel Publishing Company, Dordrecht, Holland, 1979.
- [2] Borch, K. H.: *The Economics of Uncertainty*. Princeton University Press, New Jersey, 1968.
- [3] Fukuba, Y., Ito, K., Tabata, Y. and Sakagami, Y.: Two Modified Models of Expected Utility Theory. *Osaka Economic Papers*, Vol.32, Nos.2·3 (1982), 20-52, (In Japanese).
- [4] Kahneman, D. and Tversky, A.: Prospect Theory, An Analysis of Decision under Risk. *Econometrica*, Vol.47, No.2 (1979), 263-291.
- [5] Klambauer, G.: *Real Analysis*. American Elsevier, New York, 1973.
- [6] Machina, M. J.: "Expected Utility" Analysis without the Independence Axiom. *Econometrica*, Vol.50, No.2 (1982), 277-323.
- [7] Machina, M. J.: A Stronger Characterization of Declining Risk Aversion. *Econometrica*, Vol.50, No.4 (1982), 1069-1079.
- [8] Pratt, J. W.: Risk Aversion in the Small and in the Large. *Econometrica*, Vol.32, Nos.1·2 (1964), 122-136.
- [9] Schoemaker, J. P. H.: The Expected Utility Model: its Variants, Purposes, Evidence and Limitations. *Journal of Economic Literature*, Vol.20, (June 1982), 529-563.
- [10] Tversky, A. and Kahneman, D.: The Framing of Decisions and the Psychology of Choice. *Science*, Vol.211, No.30 (1981), 453-458.

Yoshio Tabata: Department of Applied
Physics, Faculty of Engineering,
Osaka University, Suita, Osaka, 565,
Japan.