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REPLACEMENT POLICIES IN THE CASE THAT FAILURE DISTRIBUTIONS DEPEND ON THE NUMBER OF FAILURES

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Abstract This paper treats the replacement problems with incomplete repairs. In many papers, replacement problems have been studied under the assumption of minimal repairs or complete repairs, and not considered the number of failures as the deteriorating measure of units. In this paper, we introduce and formalize some basic model as an approach to that case. In the basic model it is shown that a non-randomized policy is optimal.

1. Introduction

Consider a system having a single unit. If the unit fails, the system is down. We must go on operating the system for an infinite time horizon by either repair or replacement when the system is down. It is assumed that failures are instantly detected and the time for repair and replacement can be neglected.

In general, if a unit that has failed at age x is repaired then the failure distribution (FD) of the unit depends on all its histories H_x. We denote it by $F(t|H_x)$. Practically, the age of a unit, the number of failures and etc. are considered as dependencies of the history. Many of papers treat the case in which the FD depends on only the unit's age. Moreover they often assume that the failures and subsequent repair activities do not affect the unit's failure rate. This corresponds to the case that $F(t|H_x)=\frac{G(x+t)-G(x)}{1-G(x)}$, where $G(\cdot)$ represents the FD of a new unit. (The action of restoring a failed unit to operating without affecting its failure rate is often called minimal repair.) In this case, the deteriorating of units is described by its failure rate (e.g. Barlow and Hunter [1], Makabe and Morimura [3]). This assumption seems to be reasonable for describing models of a system with very expensive and complicated units (cf. Pierskalla [4]).

In this paper we study the case in which the FD of a unit having a

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history H_x depends on only the number of failures at the time epoch, say n, i.e. $F(t|H_x) = F_n(t)$. This seems to be suitable for some practical problems.

If the system is down, we must immediately decide whether we repair or replace the failed unit. Corresponding to the decision, repair cost C_1 or replacement C_0 is suffered. $C_1 < C_0$ is assumed. Then the problem is to find an optimal replacement policy in the sense that it minimizes the expected average cost per unit time.

In section 2 we introduce a basic model and formalize it. It is shown that the optimal replacement policy is stationary. In section 3, by three lemmas and a theorem, it is proved that the optimal policy is a non-randomized one.

2. A Basic Model

Let S be a countable set of states $\{0, 1, 2, \ldots\}$. A unit whose number of failures is n is said to be at state n. A new unit is supposed to be always at state 0. We often call S state spece. Our assumption is that the unit at state n has the FD $F_n(\cdot)$. Also let D be decision space $\{s, \bar{s}\}$, where s, $\bar{s}$ means replacement and repair, respectively.

If a process starts at time 0 with the new unit, the first operating time is distributed by $F_0(\cdot)$. When the unit fails, we must select a decision at the time epoch of the failure. If the decision $\bar{s}$ is selected the process goes to state 1. In the case of decision s, a new unit is taken and state of the process remains at state 0. When the process starts with the unit at state n, the operating time is obeyed to $F_n(\cdot)$. At failure, if we select $\bar{s}$ the process goes to state $(n+1)$, and otherwise it goes to state 0. Therefore the process is regenerated whenever the decision s (replacement) is selected. The time interval between successive replacement is often called one cycle.

Let a sequence of decisions call a policy. The problem is to find the optimal policy in the sense that it minimizes the expected average cost per unit time. From the theory of regenerative stochastic processes, the effectiveness can be rewritten by the expected average cost over one cycle (e.g., see Ross [5], Proposition 5.9 in p.98). Hence it suffices to consider only one cycle. In other words, the optimal policy is the policy that minimizes the expected average cost over one cycle. Also if the replacement is considered as the termination of one cycle, the problem corresponds to a stopping problem.

A replacement policy is said to be stationary iff any decision depends on only current state. A stationary policy is a randomized one iff any decision is done by a probabilistic law $(d_{ik})_{i\varepsilon S, k\varepsilon D}$ where $\sum_{k\varepsilon D} d_{ik} = 1$ for all $i \varepsilon S$. When $d_{ik} = 0$ or 1, it is called a non- randomized policy. In this case, $d_{ik} = 1$ or 0 means that we always take the decision k or never take the decision k at state i, respectively.

Here we show that the optimal stopping policy is stationary. An artificial absorbing state θ is added to be convenient. If the process is stopped it goes to state θ. Let Z_n be state at time n, we define Δ, X^{δ} and Y^{δ} as similar to Bergman [2]. In this case Δ_i is either s or $\bar{s}$ for all $i \varepsilon S$. Furthermore, when $Z_n \neq \theta$, $X^{\Delta_n}(Z_n)$ equals to C_0 or C_1 according to $\Delta_n = s$ or $\bar{s}$ respectively, and $Y^{\Delta_n}(Z_n)$ is a random variable with the FD $F_{Z_n}(\cdot)$. Also we set $X^{\Delta_n}(\theta) = Y^n(\theta) = 0$. Bergman [2] shows.

PROPOSITION. Assume that we have two sequences of r.v.s $(X^{\Delta_1}(Z_1), X^{\Delta_1}(Z_1), \ldots)$ and $(Y^{\Delta_1}(Z_1), Y^{\Delta_1}(Z_1), \ldots)$ such that

$$E(X^{\Delta_n}(Z_n) \mid \mathscr{H}_{n-1}) = M(\Delta_n) > 0,$$

$$E(Y^{\Delta_n}(Z_n) \mid \mathscr{H}_{n-1}) = S(\Delta_n),$$

where $\mathscr{H}_k$ is the Borel field of events with respect to the first k stage and, with probability one,

$$X^{\Delta_n}(Z_n) \leq \xi_n,$$

$$Y^{\Delta_n}(Z_n) \leq \eta_n$$

for two sequences of independent identically distributed r.v.s $(\xi_1, \xi_2, \ldots)$ and $(\eta_1, \eta_2, \ldots)$. Then a stationary policy is optimal.

Since the above conditions are clearly satisfied in our case, the optimal stopping policy is stationary. Thus the optimal policy is stationary.

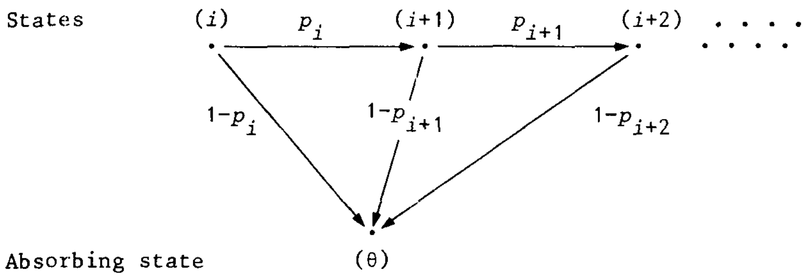

Fig. 1

Our main purpose is to show that the optimal policy is a non-randomized one when the process starts from state i. Let p_k be the probability that we do not stop the process at state k, i.e. that it moves from state k to next state $(k+1)$. This means our decision is repair with probability p_k. Also, $(1-p_k)$ represents the probability that we stop it at state k (see Fig. 1). Then we construct a stationary randomized stopping policy $\Delta(i)$ as the row vector $(p_i, p_{i+1}, p_{i+2}, \ldots)$. Fig. 1 shows that without loss of generality we can assume if $p_j = 0$ then $p_k = 0$ for all $k > j$. In what follows, we identify $\Delta(i)$ with the row vector, so that we denote $\Delta(i) = (p_i, p_{i+1}, \ldots)$. For the policy $\Delta(i)$, set

(1) $$S_\Delta(i) = 1 + p_i + p_i p_{i+1} + \cdots = \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} p_{k+i},$$

where we define $\prod_{k=0}^{n-1} p_k = 1$ for convenience. Let $T(i)$ be the function giving the number of times (including the original position) in which the process starting from state i is in a transient state and n_j the function giving the total number of times that the process is in state j. We have

$$E[T(i)] = \sum_{j=i}^{\infty} E[n_j] = \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} g_j(k),$$

where $g_j(k)$ is the probability that the process goes to state j after k steps. Here, since $g_j(k) = \prod_{m=0}^{k-1} p_{i+m}$ $(j=k+i)$ and $= 0$ (otherwise), we obtain

$$E[T(i)] = \sum_{k=0}^{\infty} \sum_{j=i}^{\infty} g_j(k) = \sum_{k=0}^{\infty} \prod_{m=0}^{k-1} p_{i+m} = S_\Delta(i).$$

Hence $S_\Delta(i)$ is the expected time until absorbing to state θ under the policy $\Delta(i)$.

Let m_n be the mean of the distribution function $F_n(\cdot)$ and suppose that m_n is positive and finite for all $n \in S$. Also let $S(i)$ be the class of policies that $S_\Delta(i) < \infty$, i.e. $S(i) = \{\Delta(i) : S_\Delta(i) < \infty\}$. Note that $\lim_{N\to\infty} \prod_{k=i}^{N} p_k = 0$ for any policy in $S(i)$. We restrict ourselves to policies in $S(i)$. Then the expected average cost until absorbing to state θ per unit time under the policy $\Delta(i)$ is easily obtained as

(2) $$G^\Delta(i) = \frac{C_0 + (S_\Delta(i)-1)C_1}{\sum_{n=0}^{\infty} \prod_{k=0}^{n-1} p_{i+k} m_{i+n}}.$$

The denominator of (2) is the expected time interval and the numerator is the expected incurred cost until absorbing to state θ. Set

$$(3) \qquad \pi_k(i) = \prod_{j=i}^{k-1} p_{j+i}/S_\Delta(i) \quad (k=0,\ 1,\ 2,\ \ldots).$$

For all $k \in S$, $\pi_k(i)$ is finite and non-negative. It is easily seen that the row vector $\pi(i) = (\pi_0(i),\ \pi_1(i),\ \ldots)$ has the following properties: (i) For all $i \in S$, $\pi_n(i) = p_{i+n-1}\pi_{n-1}(i)$. (ii) $\sum_{n=0}^{\infty} \pi_n(i)(1-p_{i+n}) = \pi_0(i)$ since $\prod_{k=i}^{N} p_k \to 0$ as $N \to \infty$. (iii) If there is some n such that $\pi_n(i) = 0$ then $\pi_m(i) = 0$ for all $m > n$. (iv) $\pi_k(i) \geqq \pi_{k+1}(i)$. (v) $\pi_0(i) \neq 0$ and $\sum_{n=0}^{\infty} \pi_n(i) = 1$.

Dividing both the denominator and the numerator of (2) by $S_\Delta(i)$, we can rewrite $G^\Delta(i)$ as

$$(4) \qquad G^\Delta(i) = \frac{C_1+(C_0-C_1)\pi_0(i)}{\sum_{n=0}^{\infty} \pi_n(i)m_{i+n}} .$$

By $\Pi(i)$ we denote the class in which any $\pi(i)$ has the properties that (I) $\pi_k(i) \geqq \pi_{k+1}(i)$ $(k=0,\ 1,\ \ldots)$ and (II) $\sum_{k=0}^{\infty} \pi_k(i) = 1$. If we set

$$(5) \qquad p_{i+n} = \frac{\pi_{n+1}(i)}{\pi_n(i)} \quad (n=0,\ 1,\ \ldots)$$

for all $i \in S$, we can uniquely determine the policy $\Delta(i)$ for any $\pi(i) \in \Pi(i)$. (For convenience' sake, we define $\frac{0}{0} = 0$.) And we have that $p_i = \pi_1(i)/\pi_0(i)$, $p_i p_{i+1} = \pi_2(i)/\pi_0(i)$ and so on. It follows from $\pi_0(i) \neq 0$ that

$$S_\Delta(i) = \sum_{n=0}^{\infty} \frac{\pi_n(i)}{\pi_0(i)} = \frac{1}{\pi_0(i)} < \infty.$$

Hence, since the policy $\Delta(i)$ that is uniquely derived from $\pi(i)$ is contained is $S(i)$, there is one-one correspondence between $\Pi(i)$ and $S(i)$ under the relation (5). Therefore the problem to find the optimal policy $\Delta^*(i)$ is equivalent to the problem to find the optimal $\pi^*(i) \in \Pi(i)$ that minimizes $G^\Delta(i)$ of (4). In what follows the policy $\Delta(i)$ and $\pi(i)$ are identified.

For any $\pi(i) \in \Pi(i)$, we set $f_\pi(i) = \sum_{n=0}^{\infty} \pi_n(i)m_{i+n}$. It is clearly finite and positive. Thus we can rewrite (4) as

$$(6)\qquad G^{\Delta}(i) = \frac{C_1+(C_0-C_1)\pi_0(i)}{f_\pi(i)} .$$

3. The Optimal Policy

We assume that the sequence $\{m_n\}_{n=0}^{\infty}$ is decreasing, where m_n is the mean of the distribution $F_n(\cdot)$. This assumption does not refer to the shape of distribution. Even if the time to failure at each stage is ecponentially distributed, i.e. $F_n(t) = 1 - e^{-m_n^{-1}t}$, where $\{m_n\}$ is a decreasing sequence, our results hold. No restrictions for distributions are assumed more than that. Since the estimation of means is relatively easy, our loose assumption is useful.

Here we have to prepare next three lemmas for getting the optimal policy.

Lemma 1. For fixed k and ℓ such that $0 < k < \ell$, let $\pi(i) = (\pi_0, \ldots, \pi_{k-1}, \pi_k, \ldots, \pi_\ell, \pi_{\ell+1}, \ldots)$ with $\pi_{k-1} > \pi_k > \pi_\ell > \pi_{\ell+1}$ be a element of $\Pi(i)$. If we set $\pi'(i) = (\pi_0, \ldots, \pi_k+\varepsilon, \ldots, \pi_\ell-\varepsilon, \ldots)$ for any ε satisfying $0 < \varepsilon \leqq \min\{(\pi_{k-1}-\pi_k), (\pi_\ell-\pi_{\ell+1})\}$ then $\pi'(i) \in \Pi(i)$ and $G^{\Delta}(i) \geqq G^{\Delta'}(i)$.

Proof. Taking notice of the domain of ε, it is obvious that $\pi'(i) \in \Pi(i)$. From (6) we directly have

$$G^{\Delta}(i) - G^{\Delta'}(i) = \frac{1}{f_\pi(i)f_{\pi'}(i)}\{\pi_0 C_0 + (1-\pi_0)C_1\}\{f_{\pi'}(i) - f_\pi(i)\}.$$

Using the relation $f_\pi(i) = \sum_{n=0}^{\infty} \pi_n m_{i+n}$, it is seen that

$$\begin{aligned} f_{\pi'}(i) - f_\pi(i) &= (\pi_k + \varepsilon)m_{i+k} - \pi_k m_{i+k} + (\pi_\ell - \varepsilon)m_{i+\ell} - \pi_\ell m_{i+\ell} \\ &= (m_{i+k} - m_{i+\ell})\varepsilon \geqq 0. \end{aligned}$$

Since $\pi_0(i) < 1$, lemma is proved.

This lemma shows that for any $\pi(i) \in \Pi(i)$ we may concentrate the value of each component toward a lower dimensional component as possible as we can. By using lemma 1 at most countable times, the row vector $(\rho_0, \ldots, \rho_{n-1}, \rho_n, \rho_{n+1}, \ldots)$ such that $\rho_0 = \ldots = \rho_{n-1}$, $\rho_n = 1 - n\rho_0$ and $\rho_{n+1} = 0$ for suitable n is consequently obtained.

For fixed i, we shall define that

$$M_i(n) = \frac{m_{i+1}}{m_{i+n}} + (\frac{m_{i+1}}{m_{i+n}} - 1) + \ldots + (\frac{m_{i+n-1}}{m_{i+n}} - 1) \quad (n = 1, 2, \ldots). \tag{7}$$

Since the sequence $\{m_k\}_{k=0}^{\infty}$ is decreasing, $M_i(n)$ is not less than 1 and increasing on n.

Lemma 2. Let $\pi(i) = (\pi_0, \ldots, \pi_{n-1}, \pi_n, 0, \ldots)$ be a vector in which $\pi_0 = \ldots = \pi_{n-1} = \rho$ and $\pi_n = 1 - n\rho$. If $M_i(n) < C_0/C_1$ then $G^{\Delta}(i) \geqq G^{\Delta'}(i)$ where $\pi'(i) = (\rho-\varepsilon, .., \rho-\varepsilon, 1-n(\rho-\varepsilon), 0, \ldots)$ for any ε satisfying $0 < \varepsilon \leqq \rho - \frac{1}{n+1}$. Also, if $M_i(n) \geqq C_0/C_1$ then $G^{\Delta}(i) \geqq G^{\Delta''}(i)$ where $\pi''(i) = (\rho+\varepsilon, \ldots, \rho+\varepsilon), 0, \ldots)$ for any ε satisfying $0 < \varepsilon \leqq \frac{1}{n} - \rho$.

Proof. First we assume $M_i(n) < C_0/C_1$. Since $\pi_0(i) = \rho$, $\pi'_0(i) = \rho - \varepsilon$, we have

$$\{G^{\Delta}(i) - G^{\Delta'}(i)\}f_{\pi}(i)f_{\pi'}(i) = C_0\{\rho f_{\pi'}(i) - (\rho - \varepsilon)f_{\pi}(i)\} + C_1\{(1 - \rho)f_{\pi'}(i) - (1 - \rho + \varepsilon)f_{\pi}(i)\}. \tag{8}$$

From the assumption of lemma, we have that

$$f_{\pi}(i) = \rho \sum_{k=0}^{n-1} m_{i+k} + (1 - n\rho)m_{i+n} > 0,$$

and that

$$f_{\pi'}(i) = (\rho - \varepsilon) \sum_{k=0}^{n-1} m_{i+k} + \{1 - n(\rho - \varepsilon)\}m_{i+n} > 0.$$

Inserting $f_{\pi}(i)$ and $f_{\pi'}(i)$ into (8), the right-hand side of (8) becomes

$$\varepsilon m_{i+n}C_0 - \varepsilon\{m_i + \ldots + m_{i+n-1}) - (n - 1)m_{i+n}\}C_1.$$

Also we have

$$\frac{1}{m_{i+n}}\{(m_i + \ldots + m_{i+n-1}) - (n - 1)m_{i+n}\}$$

$$= \frac{m_i}{m_{i+n}} + \frac{m_{i+1} - m_{i+n}}{m_{i+n}} + \ldots + \frac{m_{i+n-1} - m_{i+n}}{m_{i+n}} = M_i(n).$$

Hence the right-hand side of (8) is positive since $M_i(n) < C_0/C_1$. This completes the first assertion. The second assertion will be proved analogously.

From this lemma, there exists a vector, whose all non-zero components are equal, that dominates the vector obtained by lemma 1. The vector such that there exists a non-randomized policy that dominates any randomized policy.

Lemma 3. Let $\pi(i) = (\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots)$ and $\pi'(i) = (\frac{1}{k+1}, \ldots, \frac{1}{k+1}, 0,$

...) for any fixed $k > 0$. If there exists a n such that (i) $n > k$ and (ii) $M_i(n) < C_0/C_1$ then $G^{\Delta}(i) > G^{\Delta'}(i)$. If there is not n satisfying both (i) and (ii) then $G^{\Delta}(i) \leqq G^{\Delta'}(i)$.

Proof. By simple calculations, we have

$$k(k+1)f_{\pi}(i)f_{\pi'}(i)\{G^{\Delta}(i) - G^{\Delta'}(i)\}$$

$$= C_0 m_{i+k} - C_1\{m_i + \dots + m_{i+k-1}) - (k-1)m_{i+k}\}.$$

Noting the monotonicity of M_i, a similar proof of lemma 2 implies the first assertion of lemma. The second assertion is immediate.

By using above three lemmas, we shall prove next theorem that gives the optimal policy. Let $L(i)$ be the set of integers that do not satisfy the conditions of lemma 3, i.e. $L(i) = \{k : M_i(k) \geqq C_0/C_1\}$.

Theorem 1. For fixed i, if $L(i)$ is not empty then the optimal policy is to replace the failed unit at the k-th failure where $(k+1)$ is the minimal number of $L(i)$. If $L(i)$ is empty, the optimal policy is that no replacement will be done.

Proof. From lemma 1 and lemma 2, it is sufficient to consider the class of vectors such that $\pi(i) = (\frac{1}{n}, \dots, \frac{1}{n}, 0, \dots)$. Since there is one-one correspondence between $\pi(i)$ and $\Delta(i)$, we can see that $p_i = \dots = p_{i+n-1} = 1$ and $p_{i+n} = 0$ from (5), provided we define $\frac{0}{0} = 0$. This shows that the optimal policy is in the class of non-randomized policies. By using repeatedly lemma 3, we finally obtain the vector $\pi^*(i) = (\frac{1}{k}, \dots, \frac{1}{k}, 0, \dots)$ where k is the maximum of the numbers not belonging to $L(i)$. This shows that if $L(i)$ is not empty the optimal policy is to replace the unit at the k-th failure. In the case $L(i)$ is empty, it is easily seen that the optimal policy is to repair always.

Theorem 1 refers to an optimal policy in the case that we always replace the failed unit with a i-times failed unit. In practical cases, we may set $i = 0$ since we usually replace with a new one. Then (7) becomes

$$M_0(n) = \frac{m_0}{m_n} + (\frac{m_1}{m_n} - 1) + \dots + (\frac{m_{n-1}}{m_n} - 1) \quad (n = 0, 1, \dots).$$

Remark. Although we restricted ourselves to policies in $S(i)$ in section 2, the policy in which we repair always in theorem 1 is not contained in $S(i)$. However, since $S_{\Delta}(i) = \infty$ iff $\Delta(i) = (1, 1, \dots)$ in the class of non-randomized policies, if we adopt the policy $\Delta(i) = (1, 1, \ldots)$ whenever $s_\Delta(i) = \infty$, it is always true the optimal policy is in the class of non-randomized policies.

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