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A NOTE ON SUBMODULAR FUNCTIONS ON DISTRIBUTIVE LATTICES

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Abstract Let D be a distributive lattice formed by subsets of a finite set E with ϕ, E $\in$ D and let R be the set of reals. Also let f be a submodular function from **D** into R with $f(\phi) = 0$. We determine the set of extreme points of the base polyhedron

$$\mathbf{B}(f) = \{ x \mid x \in R^E, x(X) \leqq f(X)\ (X \in \mathbf{D}), x(E) = f(E) \}$$

and give upper and lower bounds of f which can be obtained in polynomial time in |E| under mild assumptions.

1. Introduction

Let E be a finite set, D be a distributive lattice formed by subsets of E with ϕ, E ε D and R be the set of reals. Also let f be a submodular function from D into R, i.e.,

$$(1.1) \qquad f(X) + f(Y) \geqq f(X \cup Y) + f(X \cap Y)$$

for any X, $Y \in$ D, and suppose $f(\phi) = 0$. Let us define a polyhedron B(f) by

$$(1.2) \qquad B(f) = \{x | x \in R^E,\ x(X) \leqq f(X)\ (X \in D),\ x(E) = f(E)\},$$

where for $X \in$ D and $x = (x(e)\colon e \in E) \in R^E$

$$(1.3) \qquad x(X) = \sum_{e \in X} x(e).$$

We call the pair (D,f) a *submodular system* and B(f) the *base polyhedron* associated with the submodular system (D,f).

We shall determine the set of extreme points of the base polyhedron B(f) and give upper and lower bounds of f which can be computed in polynomial time in |E|, the cardinality of E, under mild assumptions. Submodular functions play fundamental roles in many combinatorial optimization problems related to graphs, networks, matroids, polymatroids etc., and the present paper will contribute to further understanding of submodular functions.

2. Representation of Distributive Lattices

For a finite partially ordered set (poset) $\mathbf{P} = (P,\prec)$ a subset I of P is called an *ideal* of $\mathbf{P}$ if for every $a \in I$ and $b \in P - I$ we do not have $b \prec a$.

The following representation theorem for distributive lattices is classical and may be well known (see [1]).

Theorem 2.1: For any distributive lattice $\mathbf{D}$ formed by subsets of a finite set E with ϕ, $E \in \mathbf{D}$, there exists a unique poset $\mathbf{P} = (P,\prec)$ such that

(i) P is a partition $\{T_1, T_2, \ldots, T_k\}$ of E and

(2.1) (ii) $x \in \mathbf{D}$ if and only if

$$X = \cup\{T_i \mid T_i \in I\}$$

for some ideal I of $\mathbf{P}$.

Conversely, for any poset $\mathbf{P} = (P,\prec)$ with P being a partition $\{T_1, T_2, \ldots, T_k\}$ of E, the set $\mathbf{D}$ of all the subsets X of E which are expressed as (2.1) for ideals I of $\mathbf{P}$ is a distributive lattice with set union and intersection as the lattice operations and ϕ, $E \in \mathbf{D}$.

Given a distributive lattice $\mathbf{D}$, the poset $\mathbf{P} = (P,\prec)$ in Theorem 2.1 is determined as follows. For each $e \in E$, let $D(e)$ be the unique minimal element in $\mathbf{D}$ with $e \in D(e)$, i.e.,

$$D(e) = \cap\{X \mid e \in X \in \mathbf{D}\}. \tag{2.2}$$

Define a graph $G = (E,A^*)$ with the vertex set E and the arc set A^* by

$$A^* = \{(e_1,e_2) \mid e_1 \in E,\ e_2 \in D(e_1)\}. \tag{2.3}$$

The decomposition of G into strongly connected components yields a partition of the vertex set E and a partial order on the partition in a natural way which defines the required poset $\mathbf{P} = (P,\prec)$.

Without loss of generality we assume throughout the present paper that

(2.4) "each $T_i \in P$ of the poset $\mathbf{P} = (P,\prec)$ has cardinality one"

and we express $\mathbf{P}$ by $(E,\prec)$ instead of $(P,\prec)$ with $P = \{\{e\} \mid e \in E\}$. Note that without the above assumption the base polyhedron $B(f)$ does not have any extreme points. It should also be noted that because of this assumption both $D(e)$ and $D(e) - \{e\}$ belong to $\mathbf{D}$ for $D(e)$ $(e \in E)$ defined by (2.2) and that, for any integer i such that $0 \leqq i \leqq |E|$, there exists a set $X \in \mathbf{D}$ with $|X| = i$.

3. Extreme Points of the Base Polyhedron

First, we show the following lemma.

Lemma 3.1: Let

$$(3.1)\qquad (\phi =)\ S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_n\ (= E)$$

be a maximal chain in the distributive lattice $\mathbf{D}$. (Note that by the assumption (2.4) $|S_i - S_{i-1}| = 1$ $(1 \leqq i \leqq n)$ and $n = |E|$.) Also define $e_i \in E$ $(1 \leqq i \leqq n)$ by

$$(3.2)\qquad \{e_i\} = S_i - S_{i-1}.$$

Then for a vector $\hat{x} = (\hat{x}(e)\colon e \in E)$ defined by

$$(3.3)\qquad \hat{x}(e_i) = f(S_i) - f(S_{i-1}) \quad (1 \leqq i \leqq n)$$

we have

$$(3.4)\qquad \hat{x}(X) \leqq f(X) \quad (X \in \mathbf{D}).$$

$$(3.5)\qquad \hat{x}(E) = f(E),$$

i.e., $\hat{x} \in \mathbf{B}(f)$.

Proof: The inequality (3.4) with $X = \phi$ and equation (3.5) are trivial.

Suppose that (3.4) is valid for any $X \in \mathbf{D}$ with $|X| \leqq k$ for some k such that $0 \leqq k < n$ $(= |E|)$. For any $Y \in \mathbf{D}$ with $|Y| = k+1$ let e^* be an element of Y such that

$$(3.6)\qquad \{e^*\} = Y - S_{i^*-1}$$

and

$$(3.7)\qquad S_{i^*} \supseteqq Y$$

for some i^* $(1 \leqq i^* \leqq n)$. Then we have $Y - \{e^*\} \in \mathbf{D}$ and it follows from (3.6) and (3.7) and from the submodularity of f that

$$\begin{aligned} x(Y) &= x(e^*) + x(Y - \{e^*\}) \\ &\leqq x(e^*) + f(Y - \{e^*\}) \\ &= f(S_{i^*}) - f(S_{i^*-1}) + f(Y - \{e^*\}) \\ &\leqq f(Y). \end{aligned}$$

The lemma thus follows by induction. Q.E.D

From Lemma 3.1 we see that the base polyhedron $\mathbf{B}(f)$ is nonempty for any submodular function f, which may be well known.

For any weight vector $w \in R^E$ let us consider the problem:

$$\text{(3.8)} \qquad \Pi_w: \text{ Minimize } \sum_{e \in E} w(e)x(e) \quad \text{subject to } x \in B(f).$$

Suppose that the distinct values of $w(e)$ ($e \in E$) are given by

$$\text{(3.9)} \qquad w_1 < w_2 < \cdots < w_p$$

($p \leqq n$) and define

$$\text{(3.10)} \qquad A_i = \{e \mid e \in E,\ w(\varepsilon) \leqq w_i\} \quad (i=1, 2, \ldots, p).$$

Lemma 3.2: Problem Π_w has a finite optimal solution if and only if each set A_i ($i=1, 2, \ldots, p$) defined by (3.10) is an ideal of $P = (E,<)$ which represents the distributive lattice D.

Proof: The *"if" part*: By the assumption there exists a maximal chain

$$\text{(3.11)} \qquad (\phi =)\ S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_n\ (= E)$$

in the distributive lattice D such that A_i ($i=1, 2, \ldots, p$) are included in (3.11). (Note that such a maximal chain in D exists since A_i ($i=1, 2, \ldots, p$) form a chain in D.) Let $\hat{x} \in R^E$ be a vector defined by (3.11), (3.2) and (3.3). Then from Lemma 3.1 we have

$$\text{(3.12)} \qquad \hat{x} \in B(f).$$

Furthermore, for any vector $y \in B(f)$ we have from (3.3), (3.9) and (3.10)

$$\text{(3.13)} \qquad \begin{aligned} &\sum_{e \in E} w(e)y(e) - \sum_{e \in E} w(e)\hat{x}(e) \\ &= \sum_{i=1}^{p} \sum_{e \in A_i - A_{i-1}} w_i(y(e) - \hat{x}(e)) \\ &= \sum_{i=1}^{p} \left\{ \sum_{e \in A_i} w_i(y(e) - \hat{x}(e)) - \sum_{e \in A_{i-1}} w_i(y(e) - \hat{x}(e)) \right\} \\ &= \sum_{i=1}^{p-1} (w_{i+1} - w_i) \sum_{e \in A_i} (\hat{x}(e) - y(e)) + \sum_{e \in A_p} w_p(y(e) - \hat{x}(e)) \\ &= \sum_{i=1}^{p-1} (w_{i+1} - w_i)(f(A_i) - y(A_i)) \\ &\geqq 0, \end{aligned}$$

where $A_0 = \phi$ (and $A_p = E$).

Therefore, $\hat{x}$ is an optimal solution of Π_w.

The *"only if" part*: Let $\hat{x}$ be an optimal solution of Π_w. If for any A_k $(1 \leq k \leq p)$ A_k is not an ideal of $P = (E,<)$, then there is a pair (e_1,e_2) such that $e_1 < e_2$, $e_1 \in E - A_k$ and $e_2 \in A_k$. Since for every $X \in \mathbf{D}$ if $e_2 \in X$ then $e_1 \in X$, we have for any $d > 0$

$$(3.14) \qquad y \equiv \hat{x} + d\chi_{e_2} - d\chi_{e_1} \in \mathrm{B}(f),$$

where, for $e \in E$, $\chi_e \in R^E$ and

$$(3.15) \qquad \chi_e(e') = \begin{cases} 1 & (e' = e) \\ 0 & (e' \in E - \{e\}). \end{cases}$$

Consequently,

$$\sum_{e\in E} w(e)y(e) - \sum_{e\in E} w(e)\hat{x}(e)$$
$$= (w(e_2) - w(e_1))d$$
$$< 0.$$

This contradicts the optimality of $\hat{x}$. Therefore, A_i must be an ideal of $P = (E,<)$ for each $i = 1, 2, \ldots, p$. Q.E.D.

It should be noted that A_i $(i=1, 2, \ldots, p)$ in (3.10) are ideals of $P = (E,<)$ if and only if $w: E \rightarrow R$ is a monotone nondecreasing function from $P = (E,<)$ to $(R,\leq)$[7]. It should also be noted that Problem Π_w has a finite optimal solution if and only if the weight vector w belongs to the negative of the polar cone $C^*(f)$ of the recession cone

$$(3.16) \qquad C(f) = \{x \mid x \in R^E,\ x(X) \leq 0\ (X \in \mathbf{D}),\ x(E) = 0\}$$

of the base polyhedron $\mathrm{B}(f)$ (see, for example, [7], [5]). Therefore, Lemma 3.2 can be regarded as a characterization of the polar cone $C^*(f)$ of the recession cone $C(f)$.

The proof of the "if" part of Lemma 3.2 is a direct adaptation of a proof of the validity of the greedy algorithm for submodular functions on Boolean lattice 2^E given in [6].

In the proof of Lemma 3.2 we have already shown the following

Corollary 3.3: For any weight vector $w \in R^E$, if the problem Π_w has a finite optimal solution, i.e., A_i $(i=1, 2, \ldots, p)$ defined by (3.10) are ideals of $P = (E,<)$, then an optimal solution $\hat{x}$ is given by

(3.17) $\hat{x}(e_i) = f(S_i) - f(S_{i-1}) \quad (i=1, 2, \ldots, n),$

where

(3.18) $(\phi =)\ S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_n\ (= E)$

is a maximal chain in D with

(3.19) $\{e_i\} = S_i - S_{i-1} \quad (i=1, 2, \ldots, n)$

and

(3.20) $w(e_1) \leqq w(e_2) \leqq \cdots \leqq w(e_n).$

Corollary 3.3 provides an algorithm for solving Problem Π_w which is an extension of the so-called "greedy algorithm" for (poly-)matroids [3].

Theorem 3.4: The extreme points of B(f) are exactly those which are given by (3.17) - (3.19), each corresponds to a maximal chain (3.18) chosen from D.

Proof: Because of Corollary 3.3 we have only to show that for any vector $\hat{x}$ given by (3.17) - (3.19) there exists a weight vector $\hat{w} \in R^E$ such that $\hat{x}$ is a unique optimal solution of Problem $\Pi_{\hat{w}}$. For such a vector $\hat{x}$, let us choose a weight vector $\hat{w} \in R^E$ such that

(3.21) $\hat{w}(e_1) < \hat{w}(e_2) < \cdots < \hat{w}(e_n).$

Then $\hat{x}$ is an optimal solution of $\Pi_{\hat{w}}$ for such $\hat{w}$ due to Corollary 3.3. Moreover, $\hat{x}$ is a unique optimal solution because for any optimal solution y of Π_w we have, similarly as (3.13),

$$0 = \sum_{e \in E} \hat{w}(e)y(e) - \sum_{e \in E} \hat{w}(e)\hat{x}(e)$$

(3.22) $$= \sum_{i=1}^{n-1} (\hat{w}(e_{i+1}) - \hat{w}(e_i))(f(S_i) - y(S_i))$$

$$\geqq 0,$$

where $S_i = \{e_1, e_2, \ldots, e_i\}$ $(i=1, 2, \ldots, n)$. From (3.21) and (3.22),

$$\hat{x}(S_i) = f(S_i) = \hat{y}(S_i) \quad (i=1, 2, \ldots, n).$$

i.e.,

$$\hat{x}(e) = y(e) \quad (e \in E).$$

This concludes the proof of the theorem. Q.E.D.

Theorem 3.4 also easily follows from the fact that the rank of the coefficient matrix of $\hat{x}(S_i) = f(S_i)$ $(i=1, 2, \ldots, n)$ is equal to $n = |E|$.

Theorem 3.4 is a generalization of the extreme point theorem for (poly-)matroid polytopes by J. Edmonds [2].

4. Upper and Lower Bounds of Submodular Functions

It was shown in [4] that when f is an integer-valued submodular function the minimization of f can be performed by use of the so-called ellipsoid method in time polynomially bounded by $|E|$ and log B, with B being an integral upper bound of $|f(X)|$ ($X \in \mathbf{D}$), under the assumptions that the following operations (1) and (2) are carried out in unit time:

(1) to evaluate $f(X)$ for each $X \in \mathbf{D}$,

(2) to discern whether or not there is a set $X \in \mathbf{D}$ such that $e_1 \in X$ and $e_2 \notin X$ for each $e_1, e_2 \in E$

and that an integral upper bound B is previously known. We show that such an integral upper bound B can easily be computed.

We need some lemma to obtain an upper bound of f.

Lemma 4.1: For a vector $\bar{\alpha} = (\bar{\alpha}(e): e \in E)$ defined by

$$(4.1) \qquad \bar{\alpha}(e) = f(D(e)) - f(D(e) - \{e\})$$

we have

$$(4.2) \qquad \bar{\alpha}(X) \geqq f(X)$$

for any $X \in \mathbf{D}$, where $D(e)$ ($e \in E$) are defined by (2.2).

Proof: The inequality (4.2) is trivial for $X = \phi$.

Suppose that, for some integer k such that $0 \leqq k < |E|$, (4.2) is valid for any $X \in \mathbf{D}$ with $|X| \leqq k$. Then for any $Y \in \mathbf{D}$ with $|Y| = k+1$ let $\hat{e}$ be a maximal element of Y in $\mathbf{P} = (E, \prec)$. By the assumption and the submodularity of f we have

$$\begin{aligned}\bar{\alpha}(Y) &= \bar{\alpha}(\hat{e}) + \bar{\alpha}(Y - \{\hat{e}\}) \\ &\geqq \bar{\alpha}(\hat{e}) + f(Y - \{\hat{e}\}) \\ &= f(D(\hat{e})) - f(D(\hat{e}) - \{\hat{e}\}) + f(Y - \{\hat{e}\}) \\ &\geqq f(Y),\end{aligned}$$

where note that $Y - \{\hat{e}\} \in \mathbf{D}$ and $D(\hat{e}) \subseteqq Y$.

Therefore, the lemma follows by induction. Q.E.D.

From Lemma 4.1 we have an upper bound $\bar{B}$ of f given by

$$(4.3) \qquad \bar{B} = \Sigma\{\bar{\alpha}(e) \mid e \in E, \bar{\alpha}(e) > 0\}.$$

Furthermore, a lower bound of f is given as follows. Let $\hat{x}$ be an extreme point of B(f), which is obtained as in Corollary 3.3 Then

$$(4.4) \qquad \underline{B} = \Sigma\{\hat{x}(e) \mid e \in E,\ \hat{x}(e) < 0\}$$

is a lower bound of f since

$$(4.5) \qquad \underline{B} \leqq \hat{x}(X) \leqq f(X)$$

for any $X \in D$.

The upper and lower bounds $\bar{B}$ and $\underline{B}$ given by (4.3) and (4.4), respectively, can be obtained in polynomial time with respect to $|E|$ if we assume that the above-mentioned two operations (1) and (2) are carried out in unit time. It should be noted that the Hasse diagram for the poset $P = (E,<)$ can be obtained in polynomial time when Operation (2) is carried out in unit time (see Section 2).

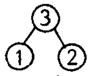
5. An Example

Consider a distributive lattice

$$(5.1) \qquad D = \{\phi, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\},$$

where $E = \{1,2,3\}$, and a submodular function f given by

$$(5.2) \qquad \begin{aligned} &f(\phi) = 0, \quad f(\{1\}) = -1, \quad f(\{2\}) = 3, \\ &f(\{1,2\}) = 1, \quad f(\{1,2,3\}) = 3. \end{aligned}$$

Observe that the distributive lattice D is the collection of ideals of a poset represented by the Hasse diagram  .

Now, an extreme point $\hat{x}$ of the base polyhedron B(f) is obtained by choosing a maximal chain in D given, for example, by

$$(5.3) \qquad \phi \subsetneqq \{1\} \subsetneqq \{1,2\} \subsetneqq \{1,2,3\}$$

and from (3.3)

$$(5.4) \qquad \begin{aligned} &\hat{x}(1) = f(\{1\}) - f(\phi) = -1, \\ &\hat{x}(2) = f(\{1,2\}) - f(\{1\}) = 2, \\ &\hat{x}(3) = f(\{1,2,3\}) - f(\{1,2\}) = 2. \end{aligned}$$

Furthermore, $D(e)$ $(e \in E)$ defined by (2.2) are given by

$$(5.5) \qquad D(1) = \{1\}, \quad D(2) = \{2\}, \quad D(3) = \{1,2,3\}.$$

Therefore, the vector $\bar{\alpha}$ in Lemma 4.1 becomes

$$\bar{\alpha}(1) = f(\{1\}) - f(\phi) = -1,$$

(5.6) $$\bar{\alpha}(2) = f(\{2\}) - f(\phi) = 3,$$

$$\bar{\alpha}(3) = f(\{1,2,3\}) - f(\{1,2\}) = 2.$$

From (4.3) an upper bound $\bar{B}$ of f is given by

(5.7) $$\bar{B} = \bar{\alpha}(2) + \bar{\alpha}(3) = 5.$$

Also from (4.4) and (5.4) we have an lower bound $\underline{B}$ of f as

(5.8) $$\underline{B} = \hat{x}(1) = -1.$$

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