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AN OPTIMAL ORDERING AND REPLACEMENT POLICY OF A MARKOVIAN DETERIORATION SYSTEM UNDER INCOMPLETE OBSERVATION

PART II

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Abstract We consider an optimal ordering and replacement policy of a discrete time Markovian deterioration system when the observation of the system is incomplete. The problem is to examine the structure of an optimal policy which minimizes the total expected discounted cost in an infinite time span. Formulating by a Markovian decision process, the optimality of a monotone policy is shown under some conditions on the deterioration system. Furthermore, some special cases are discussed.

1. Introduction

We treat a discrete time Markovian deterioration system, in which the degree of deterioration is quantized in many discrete states $0,1,\dots,N$, in the order of increasing deterioration. The state 0 is a good state, i.e., the system is like new, the states $1,\dots,N-1$ are deterioration states and the state N is a failed state. In a normal operation, these states are assumed to constitute a discrete time Markovian process with an absorbing state N. For such a system, optimal replacement problems have been studied by Derman[2], Koleasar[6], Rosenfield[10] and others. In their works, unlimited number of spare systems for immediate replacement are assumed to be available at any time. More generally, however, we should take account of the situation in which a spare system can be obtained only by ordering and it is delivered according to some arrival rate. Kaio and Osaki[4],[5], Nakagawa and Osaki[9] have considered optimal ordering policies for a system with two states, i.e., operating and failed. Mine and Kawai[7] showed the optimality of (S,T)-policy in which an order is placed at age S and preventive replacement is made for the system the age of which is more than or equal to T. An optimal ordering and replacement problem of a Markovian deterioration system has been discussed in Mine and Kawai[8] under the assumption that the observation is complete. In many practical cases, it often occurs that the state of the system can not be determined exactly, i.e., the observables need not coincide with the true state of the system. In this paper, we consider an optimal ordering and replacement problem of a Markovian deterioration system under incomplete observation, and investigate the structure of an optimal policy.

2. Model Description

We consider a system which takes any one of N+1 states $0,1,\ldots,N$. The transition of the system follows the one unit time transition probabilities p_{ij} ($i,j = 0,1,\ldots,N$). The system is observed at each time but we can not determine the true state exactly. To characterize this incompleteness of observation, we introduce the conditional probabilities $r_{i\theta}$ ($i,\theta = 0,1,\ldots,N$) that the observable is θ when the system is in state i. Here, for simplicity of discussion, the observables are assumed to have the same state space as the one of the system, i.e., we consider the case where we make correct or incorrect identification of the true state of the system with probability $r_{i\theta}$. Each ordered spare system is delivered with arrival rate a_k when the elapsed time from the placement of order is k. We assume that $a_0 = 0$ and $a_K = 1$ for some $K < \infty$.

The following costs are associated with the system.

$L = (L_0,L_1,\ldots,L_N)^T$: operating cost where L_i is the operating cost of the system in state i per unit time.

$C = (C_0,C_1,\ldots,C_N)^T$: replacement cost where C_i is the replacement cost of the system in state i. We assume that it takes one unit time to replace the system.

A : ordering cost of a spare system.

H : holding cost of a spare system per unit time.

For the above system, we consider an optimal ordering and replacement problem, that is, we have to determine when to order a spare system if no spare system has been ordered yet, and when to replace the system if a spare system is available in order to minimize the expected total discounted cost in an infinite time span. The problem can be regarded as a special case of the partially observable Markovian decision process considered by Astrom[1], Eckles[3], Smallwood and Sondik[11], Sondik[12] and others. In the sequel, we formulate our problem by a Markovian decision process and investigate the structure of an optimal policy.

3. Formulation by Markovian Decision Process

We make the following definitions as for the state of the system.

$x = (x_0, x_1, \ldots, x_N)$, $x_i \geqq 0$, $\sum_i x_i = 1$: the state probability vector of the system where x_i is the probability that the system is in state i.

$$y = \begin{cases} 0 & \text{if an order is not placed and we have no spare system,} \\ k & \text{if a spare system is on order and the elapsed time from the placement of order is } k,\ 1 \leqq k < \infty, \\ \infty & \text{if a spare system is available.} \end{cases}$$

Then, our problem can be formulated by a Markovian decision process with the state space $\{(x,y)\}$ (Eckles[3]). In the decision process the actions to be taken are as follows.

For $(x,0)$, we can consider the two actions,
action 0 : an order is not placed,
action 1 : an order is placed.
For (x,k), $1 \leqq k < \infty$, we have to wait the arrival of a spare system.
For (x,∞), we can consider the two actions.
action 2 : a replacement is not made,
action 3 : a replacement is made.

We let $v(x,y)$ denote the total β-discounted cost incurred when the initial information is (x,y) and an optimal policy is employed. Then, $v(x,y)$ obey the following equations.

$$(3.1)\qquad v(x,0) = \min\{\, v_0(x,0),\ v_1(x,0) \,\},$$

$$(3.2)\qquad v(x,k) = xL + \beta(1-a_{k+1})\sum_\theta q(\theta|x)v(T(x,\theta),k+1) + \beta a_{k+1}\sum_\theta q(\theta|x)v(T(x,\theta),\infty), \qquad 1 \leqq k < \infty,$$

$$(3.3)\qquad v(x,\infty) = \min\{\, v_2(x,\infty),\ v_3(x,\infty) \,\},$$

where

(3.4) $$v_0(x,0) = xL + \beta\sum_\theta q(\theta|x)v(T(x,\theta),0),$$

(3.5) $$v_1(x,0) = A + xL + \beta(1-a_1)\sum_\theta q(\theta|x)v(T(x,\theta),1) + \beta a_1\sum_\theta q(\theta|x)v(T(x,\theta),\infty),$$

(3.6) $$v_2(x,\infty) = H + xL + \beta\sum_\theta q(\theta|x)v(T(x,\theta),\infty),$$

(3.7) $$v_3(x,\infty) = xC + \beta v(e_0,0),$$

(3.8) $$q(\theta|x) = \sum_{i,j} x_i p_{ij} r_{j\theta},$$

(3.9) $$T(x,\theta) = \left(\sum_i x_i p_{i0} r_{0\theta}/q(\theta|x),\ \sum_i x_i p_{i1} r_{1\theta}/q(\theta|x),\ \ldots,\ \sum_i x_i p_{iN} r_{N\theta}/q(\theta|x) \right),$$

(3.10) $$e_i = (0_0, 0_1, \ldots, 0_{i-1}, 1_i, 0_{i+1}, \ldots, 0_N).$$

$v_0(x,0)$, $v_1(x,0)$, $v_2(x,\infty)$ and $v_3(x,\infty)$ correspond to the actions 0,1,2 and 3, respectively. Let $X(t)$, $x(t)$, $M(t)$ and $D(t)$ denote the state of the system, the state probability, the observable and the action at time t, respectively, then $q(\theta|x)$ and $T(x,\theta)$ imply that

(3.11) $$q(\theta|x) = \Pr\{ M(t+1) = \theta \mid x(t) = x,\ D(t) \neq 3 \},$$

(3.12) $$T(x,\theta)_j = \Pr\{ X(t+1) = j \mid x(t) = x,\ M(t+1) = \theta,\ D(t) \neq 3 \}.$$

4. Some Conditions on the Deterioration System

We give some conditions on the behavior of the deterioration system and derive some properties of the probabilities $q(\theta|x)$ and $T(x,\theta)$.
We assume that the following conditions hold.

C1. $p_{ij} = 0$ for $i > j$

C2. p_{ij} are Totally Positive of order $2(TP_2)$, in i,j, i.e.,

(4.1) $$p_{jk}p_{ih} - p_{jh}p_{ik} \geqq 0 \quad \text{for all } i < j,\ h < k.$$

C3. $r_{i\theta}$ are TP_2 in i, θ, i.e., (4.2) $r_{j\theta'}r_{i\theta} - r_{j\theta}r_{i\theta'} \geqq 0$ for all $i < j$, $\theta < \theta'$.

C4. a_k is nondecreasing in k.

C5. We introduce the following set of vectors,

$$F = \{ f \mid f = (f_0, f_1, \ldots, f_N)^T, f_i \leqq f_{i+1} \},$$

then $L, C \in F$.

C6. $L - C \in F$.

Condition 1 implies that the system can not recover its function without replacement. Condition 2 means that as the system deteriorates, it is more likely to make transition to higher states. Condition 3 reflects some tendency of observation, i.e., we can consider the case where we tend to guess the system to be in more deteriorating state as the system deteriorates. Condition 4 means that as the elapsed time after order is longer, the spare system is easier to arrive. Condition 5 shows that the operating cost and replacement cost are nondecreasing with the degree of deterioration. Condition 6 implies that the merit of replacement becomes bigger as the system deteriorates.

In the sequel, we discuss some properties of $q(\theta|x)$ and $T(x,\theta)$. For the purpose, we define the following relation on the set $\{ x = (x_0, x_1, \ldots, x_N),$ $x_i \geqq 0, \sum_i x_i = 1 \}$.

i) $x < x'$(TP) if and only if $x'_j x_i - x'_i x_j \geqq 0$ for all $i < j$.

ii) Let $w_k = (0_0, 0_1, \ldots, 0_{k-1}, 1_k, 1_{k+1}, \ldots, 1_N)^T$.

$x < x'$(ST) if and only if $xw_k \leqq x'w_k$ for all $k = 0, 1, \ldots, N$.

These relations are easily shown to be partial order.

Lemma 4.1. If $x < x'$(TP), then $x < x'$(ST).

The proof is easily done and is omitted.

Lemma 4.2. $x < x'$(ST) if and only if $xf \leqq x'f$ for all $f \in F$.

For proof, see Derman[2].

We let $p_i = (p_{i0}, p_{i1}, \ldots, p_{iN})$, $r_i = (r_{i0}, r_{i1}, \ldots, r_{iN})$, then C2 and C3 imply that $p_i < p_j$(TP) and $r_i < r_j$(TP) for $i < j$. As for $T(x,\theta)$ and $q(\theta|x)$, we have the following lemmas.

Lemma 4.3. Under C2, if $x < x'$(TP), then $T(x,\theta) < T(x',\theta)$(TP) for all $\theta = 0, 1, \ldots, N$.

Proof: If $x < x'$(TP) and $p_i < p_j$(TP) for $i < j$, then we have that

$$q(\theta|x)q(\theta|x')\{(T(x',\theta))_j(T(x,\theta))_i - (T(x',\theta))_i(T(x,\theta))_j\} \tag{4.3}$$

$$= \sum_k x'_k p_{kj} r_{j\theta} \sum_k x_k p_{ki} r_{i\theta} - \sum_k x'_k p_{ki} r_{i\theta} \sum_k x_k p_{kj} r_{j\theta}$$

$$= r_{i\theta} r_{j\theta} \sum_{h \leqq k} (x'_k x_h - x'_h x_k)(p_{kj} p_{hi} - p_{ki} p_{hj}) \geqq 0, \quad \text{for all } i < j,$$

which implies that $T(x,\theta) < T(x',\theta)$(TP) for all θ.

Lemma 4.4. Under C3, if $\theta < \theta'$, then $T(x,\theta) < T(x,\theta')$(TP) for all x.

Proof: If $\theta < \theta'$, we have that

$$q(\theta|x)q(\theta'|x)\{(T(x,\theta'))_j(T(x,\theta))_i - (T(x,\theta'))_i(T(x,\theta))_j\} \tag{4.4}$$

$$= \sum_k x_k p_{kj} r_{j\theta'} \sum_k x_k p_{ki} r_{i\theta} - \sum_k x_k p_{ki} r_{i\theta'} \sum_k x_k p_{kj} r_{j\theta}$$

$$= \sum_{h,k} x_h x_k p_{hi} p_{kj} (r_{j\theta'} r_{i\theta} - r_{i\theta'} r_{j\theta}) \geqq 0, \quad \text{for all } i < j,$$

which implies that $T(x,\theta) < T(x,\theta')$(TP).

Lemma 4.5. We let $q(x) = (\, q(0|x), q(1|x), \ldots, q(N|x)\,)$. Under C2 and C3, if $x < x'$(ST), then $q(x) < q(x')$(ST).

Proof:

$$q(x)w_k = \sum_i x_i \sum_j p_{ij} r_j w_k \tag{4.5}$$

Since $r_i < r_j$(ST) for $i < j$, we have that $(\, r_0 w_k, r_1 w_k, \ldots, r_N w_k\,)^T \in F$ for all k, which shows that $(\, \sum_j p_{0j} r_j w_k, \sum_j p_{1j} r_j w_k, \ldots, \sum_j p_{Nj} r_j w_k\,)^T \in F$ since $p_i < p_j$ (ST) for $i < j$. From lemma 4.2 and equation (4.5), we have that $q(x)w_k \leqq q(x')w_k$ for all k, which completes the proof.

5. Structure of an Optimal Policy

We examine the structure of an optimal ordering and replacement policy. For the purpose, we give the following lemma which plays an important role in our discussion.

Lemma 5.1. Let $g(x)$ be the function of the state probability such that if $x < x'$(TP), then $g(x) \leqq g(x')$. Under C2 and C3, if $x < x'$(TP), then it holds that

(5.1) $\sum_\theta q(\theta|x)g(T(x,\theta)) \leqq \sum_\theta q(\theta|x')g(T(x',\theta))$.

Proof: From lemmas 4.3 and 4.4, we have that

(5.2) $T(x,\theta) < T(x',\theta)(\mathrm{TP})$ for all θ,

and

(5.3) $T(x,\theta) < T(x,\theta')(\mathrm{TP})$ for all $\theta < \theta'$ and x.

Equations (5.2) and (5.3) show that

(5.4) $g(T(x,\theta)) \leqq g(T(x',\theta))$ for all θ,

and

(5.5) $(\, g(T(x,0)), g(T(x,1)), \ldots, g(T(x,N))\,)^{\mathrm{T}} \in \mathrm{F}$ for all x.

Froom lemma 4.5, we have that

(5.6) $q(x) < q(x')(\mathrm{ST})$

Using equations (5.4),(5.5),(5.6) and lemma 4.2, we have that

(5.7) $\sum_\theta q(\theta|x')g(T(x',\theta)) \geqq \sum_\theta q(\theta|x')g(T(x,\theta)) \geqq \sum_\theta q(\theta|x)g(T(x,\theta))$,

which completes the proof.

With respect to the optimal cost $v(x,k)$, $k = 0,1,\ldots,\infty$, we have the following lemma.

Lemma 5.2. Under C1 through C6, if $x < x'(\mathrm{TP})$, then it holds that

(5.8) $v(x,k) \leqq v(x',k)$ for $k = 0,1,\ldots,\infty$,

and

(5.9) $v(x,h) - v(x,k) \leqq v(x',h) - v(x',k)$, for $0 \leqq h < k \leqq \infty$.

Proof: The proof is done through induction. Let $v^n(x,k)$ denote the optimal n-period costs. Then

(5.10) $v^1(x,k) = xL,\ 0 \leqq k < \infty$,

(5.11) $v^1(x,\infty) = \min\{\, H + xL,\ xC\,\}$,

and for $n \geqq 2$,

(5.12) $v^n(x,0) = \min\{\, v_0^n(x,0),\ v_1^n(x,0)\,\}$,

(5.13) $v^n(x,k) = xL + \beta(1-a_{k+1})\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),k+1)$

$$+\beta a_{k+1}\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),\infty),\quad 1 \leqq k < \infty,$$

$$(5.14)\quad v^n(x,\infty) = \min\{ v_2^n(x,\infty),\ v_3^n(x,\infty) \},$$

where

$$(5.15)\quad v_0^n(x,0) = xL + \beta\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),0),$$

$$(5.16)\quad v_1^n(x,0) = A + xL + \beta(1-a_1)\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),1) + \beta a_1\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),\infty),$$

$$(5.17)\quad v_2^n(x,\infty) = H + xL + \beta\sum_\theta q(\theta|x)v^{n-1}(T(x,\theta),\infty),$$

$$(5.18)\quad v_3^n(x,\infty) = xC + \beta v^{n-1}(e_0,0).$$

From C5, C6, lemmas 4.1 and 4.2, it is easily seen that if $x < x'$(TP), then $v^1(x,k) \leqq v^1(x',k)$ for all k and $v^1(x,h) - v^1(x,k) \leqq v^1(x',h) - v^1(x',k)$ for $0 \leqq h < k \leqq \infty$. For simplicity of expression, we abbreviate

$$q = q(\theta|x),\ q' = q(\theta|x'),\ v(k) = v^{n-1}(T(x,\theta),k) \text{ and } v'(k) = v^{n-1}(T(x',\theta),k).$$

We assume that if $x < x'$(TP), then

$$(5.19)\quad v(k) \leqq v'(k),\quad k = 0,1,\ldots,\infty,$$

$$(5.20)\quad v(h) - v(k) \leqq v'(h) - v'(k),\quad \text{for } 0 \leqq h < k \leqq \infty.$$

From lemma 5.1 and equations (5.12)-(5.20), it is shown that if $x < x'$(TP), then $v^n(x,k) \leqq v^n(x',k)$ for all k. From now on, we show that if $x < x'$(TP), then $v^n(x,h) - v^n(x,k) \leqq v^n(x',h) - v^n(x',k)$ for $0 \leqq h < k \leqq \infty$. For the proof, it is sufficient to show that the following three equations hold if $x < x'$(TP).

$$(5.21)\quad v^n(x,0) - v^n(x,k) \leqq v^n(x',0) - v^n(x',k),\quad 1 \leqq k < \infty,$$

$$(5.22)\quad v^n(x,k) - v^n(x,\infty) \leqq v^n(x',k) - v^n(x',\infty),\quad 1 \leqq k < \infty,$$

$$(5.23)\quad v^n(x,h) - v^n(x,k) \leqq v^n(x',h) - v^n(x',k),\quad 1 \leqq h < k < \infty.$$

It should be noted that equations (5.21) and (5.22) imply that

$$(5.24)\quad v^n(x,0) - v^n(x,\infty) \leqq v^n(x',0) - v^n(x',\infty).$$

We let $D^n(x,k)$ denote the action which gives $v^n(x,k)$, e.g., if $D^n(x,\infty) = 3$, then we should replace the system by a spare system.

i) Proof of equation (5.21). We consider the two cases as for $D(x',0)$: (1) $D^n(x',0) = 0$, (2) $D^n(x',0) = 1$.

Case (1). From equations (5.12)-(5.16), lemma 5.1 and the assumption of induction, we have that

$$(5.25)\quad \{ v^n(x',0) - v^n(x',k) - (v^n(x,0) - v^n(x,k)) \}\beta^{-1}$$

$$\geqq \{ v_0^n(x',0) - v^n(x',k) - v_0^n(x,0) + v^n(x,k) \}\beta^{-1}$$

$$= \sum_\theta q'v'(0) - (1-a_{k+1})\sum_\theta q'v'(k+1) - a_{k+1}\sum_\theta q'v'(\infty)$$

$$- \sum_\theta qv(0) + (1-a_{k+1})\sum_\theta qv(k+1) + a_{k+1}\sum_\theta qv(\infty)$$

$$= \sum_\theta q'(v'(0) - v'(k+1)) - \sum_\theta q(v(0) - v(k+1)) + a_{k+1}\sum_\theta q'(v'(k+1) - v'(\infty))$$

$$- a_{k+1}\sum_\theta q(v(k+1) - v(\infty)) \geqq 0.$$

Case(2). From C4, we have that

$$(5.26)\quad \{ v^n(x',0) - v^n(x',k) - (v^n(x,0) - v^n(x,k)) \}\beta^{-1}$$

$$\geqq \{ v_1^n(x',0) - v^n(x',k) - v_1^n(x,0) + v^n(x,k) \}\beta^{-1}$$

$$= (1-a_1)\sum_\theta q'(v'(1) - v'(k+1)) - (1-a_1)\sum_\theta q(v(1) - v(k+1))$$

$$+ (a_{k+1}-a_1)\sum_\theta q'(v'(k+1) - v'(\infty)) - (a_{k+1}-a_1)\sum_\theta q(v(k+1) - v(\infty)) \geqq 0.$$

Hence, from equations (5.25) and (5.26), equation (5.21) holds.

ii) Proof of equation (5.22). We consider the two cases as for $D^n(x,\infty)$: (1) $D^n(x,\infty) = 2$, (2) $D^n(x,\infty) = 3$.

Case (1). From equations (5.13),(5.14),(5.17), lemma 5.1 and the assumption of induction, we have that

$$(5.27)\quad \{ v^n(x',k) - v^n(x',\infty) - (v^n(x,k) - v^n(x,\infty)) \}\beta^{-1}$$

$$\geqq \{ v^n(x',k) - v_2^n(x',\infty) - v^n(x,k) + v_2^n(x,\infty) \}\beta^{-1}$$

$$= (1-a_{k+1})\{ \sum_\theta q'(v'(k+1) - v'(\infty)) - \sum_\theta q(v(k+1) - v(\infty)) \} \geqq 0.$$

Case (2). From C6 and lemma 4.2, we have that

$$(5.28)\quad v^n(x',k) - v^n(x',\infty) - (v^n(x,k) - v^n(x,\infty))$$

$$\geqq v^n(x',k) - v_3^n(x',\infty) - v^n(x,k) + v_3^n(x,\infty)$$

$$= x'(L-C) - x(L-C) + \beta(1-a_{k+1})(\sum_\theta q'v'(k+1) - \sum_\theta qv(k+1))$$

$$+ \beta a_{k+1}(\sum_\theta q'v'(\infty) - \sum_\theta qv(\infty)) \geqq 0.$$

Hence, from equations (5.27) and (5.28), equation (5.22) holds.

iii) Proof of equation (5.23). From C4, equation (5.13), lemma 5.1 and the assumption of induction, we have that for $h < k$,

$$(5.29)\quad \{ v^n(x',h) - v^n(x',k) - (v^n(x,h) - v^n(x,k)) \}\beta^{-1}$$

$$= (1-a_{h+1})\{ \sum_\theta q'(v'(h+1) - v'(k+1)) - \sum_\theta q(v(h+1) - v(k+1)) \}$$

$$+ (a_{k+1}-a_{h+1})\{ \sum_\theta q'(v'(k+1) - v'(\infty)) - \sum_\theta q(v(k+1) - v(\infty)) \} \geqq 0.$$

Hence, equation (5.23) holds.

Therefore, $v^n(x,h) - v^n(x,k) \leqq v^n(x',h) - v^n(x',k)$ holds for all $0 \leqq h < k \leqq \infty$ and n. Since

$$(5.30)\quad \lim_{n\to\infty} v^n(x,k) = v(x,k)$$

(Eckles[3]), we have that if $x < x'$(TP), then $v(x,k) \leqq v(x',k)$ for all k, $v(x,h) - v(x,k) \leqq v(x',h) - v(x',k)$ for $0 \leqq h < k \leqq \infty$. This completes the proof.

Going back to equations (3.1)-(3.7) and subtracting $v_1(x,0)$ from $v_0(x,0)$, we have that

$$(5.31)\quad v_0(x,0) - v_1(x,0) = -A + \beta\sum_\theta q(\theta|x)\{ v(T(x,\theta),0) - v(T(x,\theta),1) \}$$

$$+ \beta a_1\sum_\theta q(\theta|x)\{ v(T(x,\theta),1) - v(T(x,\theta),\infty) \}.$$

Using lemmas 5.1 and 5.2, we can show that for $x < x'$(TP),

$$(5.32)\quad v_0(x,0) - v_1(x,0) \leqq v_0(x',0) - v_1(x',0).$$

Subtracting $v_3(x,\infty)$ from $v_2(x,\infty)$, we have that

$$(5.33)\quad v_2(x,\infty) - v_3(x,\infty) = H + x(L-C) + \beta\sum_\theta q(\theta|x)v(T(x,\theta),\infty) - \beta v(e_0,0).$$

Using lemmas 5.1 and 5.2, we can show that for $x < x'$(TP),

$$(5.34)\quad v_2(x,\infty) - v_3(x,\infty) \leqq v_2(x',\infty) - v_3(x',\infty).$$

Equations (5.32) and (5.34) gives us the following theorem as for an optimal ordering and replacement policy.

Theorem 5.1. We let $D(x,k)$ denote an optimal action for (x,k). Under C1 through C6, if $x < x'$(TP), then

(5.35) $D(x,0) \leqq D(x',0)$ and $D(x,\infty) \leqq D(x',\infty)$.

When $N = 2$, the state space is $\{ ((x_1,x_2),k) \mid x_1, x_2 \geqq 0, x_1 + x_2 \leqq 1 \}$, and is divided into two regions under an optimal ordering and replacement policy as shown in Fig.1.

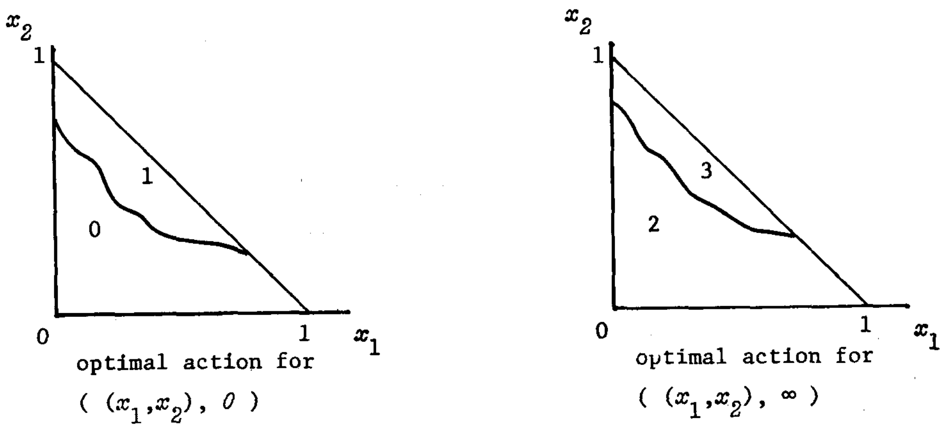

Fig.1. Optimal policy when $N = 2$.

6. Some Special Cases

We discuss the structure of an optimal policy in some special cases.

Case 1). No information case.

We consider the case where the observables gives us no information about the true state of the system. This implies that

(6.1) $r_{i\theta} = r_\theta$ for all i

We let $P_{ij}(t)$ denote the t-unit time transition probability from state i to state j associated with the state of the system, i.e., they are given by

(6.2)
$$P_{ij}(0) = \delta_{ij} \quad (\delta_{ij} \text{ is Kronecker's delta }),$$
$$P_{ij}(t+1) = \sum_k P_{ik}(t)p_{kj}.$$

We assume to start with the information ($e_0,0$) (it should be noted that immediately after replacement, we have always the information $(e_0, 0))$, then from equations (3.8),(3.9),(6.1) and (6.2), it holds that

(6.3) $T(e_0,\theta) = P_0(1)$,

(6.4) $T(P_0(t),\theta) = P_0(t+1)$,

where

(6.5) $P_0(t) = (P_{00}(t), P_{01}(t), \ldots, P_{0N}(t))$.

Equations (6.3) and (6.4) imply that the set of the state probability is given by

(6.6) $\{ x \} = \{ P_0(t),\ t = 0,1,\ldots \}$.

It is easily seen that $r_{i\theta}$ satisfy C2. Further, under C1 and C3, $P_{ij}(t)$ are shown to be TP_2 in j,t (Rosenfield[10]). Hence, we have that

(6.7) $P_0(t) < P_0(t+1)(TP)$,

which also implies that the relation <(TP) is total order on the set $\{ x \}$. From equations (6.6),(6.7) and lemma 5.1, we can conclude that there exists an optimal policy has the form of (S,T)-policy.

Case 2). Complete information case.

We consider the case where the observation is always correct, i.e.,

(6.8) $r_{i\theta} = \delta_{i\theta}$.

From equations (3.8),(3.9) and (6.8), we have that

(6.9) $T(x,i) = e_i,\ i = 0,1,\ldots,N$.

Hence, if we start with the information $(e_0, 0)$, then the set of the state probability is given by

(6.10) $\{ x \} = \{ e_i,\ i = 0,1,\ldots,N \}$

It is easily seen that $r_{i\theta}$ satisfy C3. Further, it is clear that $e_i < e_j$(TP) for $i < j$, which also implies that the relation <(TP) is total order on the set $\{ x \}$. Hence, from theorem 5.1, we can conclude that there exists an optimal policy which has the form of (m,M)-policy. This policy means that each order is placed if and only if the system enters the states $m,m+1,\ldots,N$ and each replacement is made if and only if the system enters the states $M,\ldots,N$ when a spare system is available.

Case 3). Two states case.

We consider the case where the system has only two states, i.e., a good state and a failed state. In this case, the behavior of the system and the accurancy of observation are characterized by

(6.11) $p_{00} + p_{01} = 1, \; p_{11} = 1, \; r_{00} + r_{01} = 1, \; r_{10} + r_{11} = 1.$

The state probability can be expressed by the failure probability g, i.e., $\{ x \} = \{ (1-g,g), \; 0 \leqq g \leqq 1 \}$. From equation (6.11), we have that

(6.12) $p_{11}p_{00} - p_{10}p_{01} = p_{11}p_{00} \geqq 0,$

(6.13) $r_{11}r_{00} - r_{10}r_{01} = r_{00} + r_{11} - 1.$

Hence, C2 holds and if $r_{00} \geqq 0.5$, $r_{11} \geqq 0.5$, then C3 holds. This condition for $r_{i\theta}$ reflects some real situation, i.e., it is natural to consider that the probability of correct observation is greater than the probability of incorrect observation. For the relation $<$(TP), it is clear that

(6.14) $x = (1-g,g) < x' = (1-g',g')(TP) \leftrightarrow g \leqq g' \leftrightarrow x < x'(ST).$

Therefore, from theorem 5.1, we can conculde that there exists an optimal policy which has the form of (g,G)-policy. This policy means that each order is placed if and only if the failure probability of the system is greater than or equal to g and each replacement is made if and only if the failure probability is greater than or equal to G when a spare system is available.

In case 1), consider the following weaker condition than C2,

(6.15) C2' $\quad p_i = (p_{i0},p_{i1},\ldots,p_{iN}) < p_j = (p_{j0},p_{j1},\ldots,p_{jN})(ST)$

for $i < j$, then it can be easily shown that

(6.16) $P_0(t) < P_0(t+1)(ST).$

By a similar discussion as in lemmas 4.4, 5.1, 5.2 and theorem 5.1, we can show the optimality of (S,T)-policy under the condition C2'.

In case 2), it is clear that $e_i < e_j(ST)$ for $i < j$. Hence, as in case 1), we can obtain the result in this case under the condition C2'.

7. Conclusion

In this paper, we have discussed an optimal ordering and replacement problem of a discrete time Markovian deterioration system under incomplete observation. The problem is formulated by a Markovian decision process and the optimality of a monotone policy has been shown under some reasonable conditions on physical and economic aspects of the deterioration system. The model discussed here contains the usual replacement model, ordering and replacement model under complete observation as special cases. In our model, it is assumed that only one spare system can be kept in inventory. In more general cases, however, this assumption should be relaxed, that is, any number of spares are allowed. It is also assumed that we can observe the system at any time without any cost. But in many practical situation, observation is costly. Hence, the control problem of the time interval between observations should be studied. These problems will be treated in future works.

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