Journal of the Operations Research Society of Japan
Vol. 26, No. 4, December 1983

AN OPTIMAL ORDERING AND REPLACEMENT POLICY OF A MARKOVIAN DEGRADATION SYSTEM UNDER COMPLETE OBSERVATION

PART I

Hajime Kawai
University of Osaka Prefecture

(Received May 27, 1982; Revised August 4, 1983)

Abstract This paper consideres an optimal ordering and replacement problem of a continuous time Markovian degradation system. An optimal policy minimizes the expected cost per unit time in an infinite time horizon. The problem is formulated by semi-Markov decision process and the optimality of an (n,N)-policy is shown. Further, the expected cost rate of the system operated under an (n,N)-policy is obtained.

1. Introduction

In this paper, we treat a Markovian degradation system, in which the level of degradation is quantized in many discrete states $0,1,\ldots,s,s+1$, in the order of increasing degradation. The state 0 is a good state, i.e., the system is like new, the states $1,2,\ldots,s$ are degradation states and the state $s+1$ is a failure state. In a normal operation, these states are assumed to constitute a continuous time Markov process with an absorbing state $s+1$. Such a system is called a Markovian degradation system. For the system, optimal replacement problems have been studied by Flehinger[2][3], Kawai[6][7], Mine and Kawai[9][10][12], Luss[8] and others. In the above studies, it is assumed that unlimited number of new spare systems for immediate replacement are always available. In some cases, this assumption does not hold. To overcome this, we should take account of the delivery time of a spare system. Kaio and Osaki[4][5], Nakagawa and Osaki[13] have determined optimal ordering policies for a system with two states, i.e., an operating state and a failure state. Mine and Kawai [11] showed an optimality of (S,T)-policy for ordering and replacement, under which a spare system is ordered at age S and preventive replacement is made when the age becomes larger than or equal to T.

In this paper, we consider an ordering and replacement problem for a continuous time Markovian degradation system. The problem is to determine an optimal policy which minimizes the expected cost per unit time in an infinite time span (cost rate). We formulate our problem by semi-Markov decision process and show the existence of an optimal (n,N)-policy. Under (n,N)-policy, each order of a spare system is placed when the system enters the states $n+1,...,s,s+1$ and a replacement is made when the system enters the states $N+1,...,s,s+1$. Furthermore, we derive the cost rate when each system is operated under an (n,N)-policy, from which we can also determine the optimal n^* and N^*.

2. Descriptin of the System

We consider our maintenance problem of the Markovian degradation system under the following situation.

i) The transition rates from one state to another are independent of time, i.e., are constant.

  β_{ij} : transition rate from state i to j.

ii) From a state, only transitions to higher states are possible, i.e.,

  $\beta_{ij} = 0$ for $i > j$.

iii) Operating cost are incurred at each state.

  a_i : operating cost of the system in state i per unit time.

iv) Replacement time is negligible and immediately after replacement the system is like new.

  c_i : replacement cost of the system in state i.

v) A spare system can be obtained only by ordering with random delivery time.

  c : cost for each order.

  $F(t)$: distribution function of the delivery time.

  T : expected delivery time.

vi) Only one spare system can be kept in inventory.

  h : holding cost of a spare system per unit time.

Our problem is to determine an optimal ordering and replacement policy which minimizes the expected cost per unit time in an infinite time span, that is, we have to determine the state at which a spare system is ordered when we have no spare system and the state at which the operating system is replaced when a spare system is available.

3. Notations and Assumptions

For the stochastic behavior of the system, we introduce the following notations.

λ_i : $\sum_j \beta_{ij}$,

p_{ij} : β_{ij}/λ_i,

$P_{ij}(t)$: t unit time transition probability from state i to state j.

In this paper, we let $\sum_k$ denote the summation over all the values that k can take. Transition probabilities $P_{ij}(t)$ are associated with the system with no replacement. For $P_{ij}(t)$, the following equations are easily obtained.

(3.1) $P_{ij}(t) = 0$ for $i > j$, $P_{s+1,s+1}(t) = 1$,

(3.2) $\mathrm{d}P_{ij}(t)/\mathrm{d}t = -\lambda_i P_{ij}(t) + \sum_k \beta_{ik} P_{kj}(t) = -\lambda_j P_{ij}(t) + \sum_k P_{ik}(t)\beta_{kj}$,

(3.3) $P_{i,s+1}(t) = \int_0^t \sum_{j=i}^{s} P_{ij}(x)\beta_{j,s+1}\mathrm{d}x$.

(3.4) $A_i(t)$: $\int_0^t \sum_j P_{ij}(x)a_j\mathrm{d}x$.

(3.5) A_i : $\int_0^\infty A_i(t)\mathrm{d}F(t)$.

(3.6) f_{ij} : $\int_0^\infty P_{ij}(t)\mathrm{d}F(t)$.

$A_i(t)$ is the expected operating cost during the time interval $(0,t)$ when the system starts from state i. A_i is the expected operating cost during a delivery when an order is placed at state i. f_{ij} is the probability that the system is in state j at the arrival time instant of a spare system when an order is placed at state i.

We make the following assumptions.

(A1) λ_i is nondecreasing in i.

(A2) $\sum_{j=k}^{s+1} p_{ij}$ is nondecreasing in i for any k.

This assumption is shown to be equavalent to the assumption (A3).

(A3) $\sum_j p_{ij}g_j$ is nondecreasing in i for any nondecreasing function g_i.

(A4) a_i/λ_i and c_i are nondecreasing in i.

(A5) $a_i/\lambda_i - c_i$ are nondecreasing in i, $i \leqq s$

(A6) $a_{s+1}/\lambda_s - c_{s+1} \geqq a_s/\lambda_s - c_s$.

(A1) implies that the expected sojourn time of each state becomes smaller as the system degrades. (A2) implies that the system is more likely to make transitions to higher states as the system degrades. (A4) implies that operating cost and replacement cost are nondecreasing with the degree of degradation. (A5) and (A6) imply that the merit of replacement becomes bigger as system degrades. These assumptions are reasonable to describe real degradation system from physical and economic point of view.

4. Transition Probability

As for the transition probabilities $P_{ij}(t)$, we have the following lemma.

Lemma 4.1.

$\sum_{j=k}^{s+1} P_{ij}(t)$ is nondecreasing in i for any k.

Proof: From equation (3.2), we have that

$$(4.1)\qquad P_{ij}(t) = \delta_{ij}e^{-\lambda_i t} + \int_0^t \lambda_i e^{-\lambda_i x} \sum_h p_{ih}P_{hj}(t-x)\mathrm{d}x,$$

where δ_{ij} is Kronecker's delta. We prove the lemma through induction. We let

$$(4.2)\qquad F_i(t) = \sum_{j=k}^{s+1} P_{ij}(t),$$

then we have that

$$(4.3)\qquad F_k(t) = F_{k+1}(t) = \ldots = F_{s+1}(t) = 1 \quad \text{for } t \geqq 0.$$

From equations (4.1)-(4.3), we have that

$$(4.4)\qquad F_i(t) = \int_0^t \lambda_i e^{-\lambda_i x}\sum_j p_{ij}F_j(t-x)\mathrm{d}x, \quad \text{for } i < k$$

Assume that

$$(4.5)\qquad F_{i+1}(t) \leqq F_{i+2}(t) \leqq \cdots \leqq F_{s+1}(t),$$

then from equation (4.4), (A1) and (A3), it holds that

$$(4.6)\qquad F_{i+1}(t) - F_i(t) \geqq \int_0^t (\lambda_{i+1}e^{-\lambda_{i+1}x} - \lambda_i e^{-\lambda_i x})\sum_j p_{ij}F_j(t-x)\mathrm{d}x$$

$$= \int_0^t (e^{-\lambda_i(t-x)} - e^{-\lambda_{i+1}(t-x)})\sum_j p_{ij}\mathrm{d}F_j(x) \geqq 0,$$

which implies that $F_{i+1}(t) \geqq F_i(t)$. This completes the proof.

The following lemma is easily shown to hold.

Lemma 4.2.

$\sum_j P_{ij}(t)g_j$ is nondecreasing in i for any nondecreasing function g_i.

5. Semi-Markov Decision Process Formulation

We formulate our problem as a semi-Markov decision process. First, we define the following time instants.

E_i : the time instant at which the system has just entered state i while a spare system has not been ordered,

F_i : the time instant at which the system has just entered state i while a spare system is available or a spare system has just been delivered while the system is in state i.

At each E_i ($i \leqq s$), we can take the two actions.

K : an order is not placed and the system continues to be operated,

O : an order is placed and the system continues to be operated.

At each F_i ($i \leqq s$), we can take the two actions.

L : the system is not replaced and it continues to be operated,

R : the system is replaced by a spare system.

At E_{s+1}, action O is taken and at F_{s+1}, action R is taken.

The problem is to select an optimal action at each E_i, F_i to minimize the expected cost per unit time. E_i and F_i are all the renewal points and our problem can be formulated by a semi-Markov decision process.

For an optimal policy, we define,

g : the cost rate,

v_i : the relative bias of starting from E_i,

w_i : the relative bias of starting from F_i.

Then g, v_i, w_i obey the following functional equations (Ross[15]).

(5.1) $v_i = \min \{ K_i, O_i \} \quad i = 0,1,\dots,s,$

(5.2) $v_{s+1} = O_{s+1},$

(5.3) $w_i = \min \{ L_i, c_i \} \quad i = 0,1,\dots,s,$

(5.4) $w_{s+1} = c_{s+1}$,

where

(5.5) $K_i = a_i/\lambda_i + \sum_j p_{ij}v_j - g/\lambda_i$,

(5.6) $O_i = c + A_i + \sum_j f_{ij}w_j - Tg$,

(5.7) $L_i = (a_i + h)/\lambda_i + \sum_j p_{ij}w_j - g/\lambda_i$.

K_i, O_i, L_i and c_i correspond to actions K,O,L and R, respectively.

6. Optimal Policy

We discuss the property of an optimal policy. For the purpose, we let

$D_v(i)$: an optimal action at E_i,
$D_w(i)$: an optimal action at F_i.

For the bias w_i, we have the following lemma.

Lemma 6.1. w_i is nondecreasing in i.

Proof : Proof is done through induction. From (A4), equations (5.3) and (5.4), it is clear that $w_s \leqq w_{s+1}$. Assume that

(6.1) $w_{i+1} \leqq w_{i+2} \leqq \cdots \leqq w_s \leqq w_{s+1}$,

then from (A3), it holds that

(6.2) $\sum_j p_{ij}w_j \leqq \sum_j p_{i+1,j}w_j$.

We consider two cases for the optimal action $D_w(i+1)$: 1) $D_w(i+1)$ = R, 2) $D_w(i+1)$ = L.

Case 1). In this case, from (A4), we have that

(6.3) $w_{i+1} - w_i \geqq c_{i+1} - c_i$,

which denotes that $w_i \leqq w_{i+1}$.

Case 2). In this case, from equations (5.3) and (6.1), it holds that

(6.4) $\lambda_{i+1}w_{i+1} = a_{i+1} + h + \sum_j \beta_{i+1,j}w_j - g \geqq a_{i+1} + \lambda_{i+1}w_{i+1} + h - g$,

which implies that

(6.5) $g - h \geqq 0$.

From (A1), (A5), equations (5.3), (6.2) and (6.4), it holds that

$$(6.6)\quad w_{i+1} - w_i \geqq L_{i+1} - L_i = a_{i+1}/\lambda_{i+1} - a_i/\lambda_i + \sum_j p_{i+1,j}w_j - \sum_j p_{ij}w_j$$
$$+ (1/\lambda_i - 1/\lambda_{i+1})(g - h) \geqq 0,$$

which denotes that $w_i \leqq w_{i+1}$.
In both cases, we have that $w_i \leqq w_{i+1}$. This completes the proof.

As for $D_w(i)$, we have the following theorem.

Theorem 6.1. There exists a limit number N such that

$$(6.7)\quad D_w(i) = \begin{cases} \text{L} & \text{for } i \leqq N, \\ \text{R} & \text{otherwise.} \end{cases}$$

Proof: If $D_w(i)$ = R for all i, then the theorem holds in a special form. If there exists i such that $D_w(i)$ = L, then in a similar way as in the proof of lemma 6.1, we have that $g - h \geqq 0$. Subtracting R_i from L_i, we have that

$$(6.8)\quad L_i - R_i = a_i/\lambda_i - c_i + \sum_j p_{ij}w_j - (g - h)/\lambda_i.$$

From (A3), (A5) and lemma 6.1, it is easily seen that $L_i - R_i$ is nondecreasing in i. This implies that the theorem holds.

From here, we discuss the property of $D_v(i)$. First, we give the following lemma as for the operating cost.

Lemma 6.2.

$$(6.9)\quad a_i + \sum_j \beta_{ij}A_j - \lambda_i A_i = \sum_j f_{ij}a_j, \quad i = 0,1,\ldots,s$$

Proof: From equations (3.1)-(3.3), it holds that

$$(6.10)\quad a_i + \sum_j \beta_{ij}A_j - \lambda_i A_i = \int_0^\infty \{ a_i + \sum_j \sum_k \beta_{ik}\int_0^x P_{kj}(y)a_j\mathrm{d}y$$
$$- \lambda_i\int_0^x \sum_j P_{ij}(y)a_j\mathrm{d}y \}\mathrm{d}F(x) = \int_0^\infty \{ a_i + \sum_j \int_0^x (\mathrm{d}P_{ij}(y)/\mathrm{d}y)a_j\mathrm{d}y \}\mathrm{d}F(x)$$
$$= \int_0^\infty \sum_j P_{ij}(x)a_j\mathrm{d}F(x) = \sum_j f_{ij}a_j.$$

This completes the proof.

We define

$$(6.11)\quad G_i = a_i + \sum_j \beta_{ij}O_j - g - \lambda_i O_i, \quad i = 0,1,\ldots,s.$$

From lemma 6.2 and equations (3.1)-(3.3), G_i is rewritten as follows.

(6.12) $G_i = \sum_j f_{ij} u_j - h,$

where

(6.13) $u_i = \begin{cases} a_i + h + \sum_j \beta_{ij} w_j - g - \lambda_i w_i, & \text{for } i \leq s \\ a_{s+1} + h - g, & \text{for } i = s+1. \end{cases}$

Concerning u_i, we have the following lemma.

Lemma 6.3. u_i is nondecreasing in i or $u_i \leq 0$ for all i.

Proof: From equations (5.3) and (5.7), $u_i \geq 0$ for $i \leq s$. From theorem 6.1, we have that

(6.14) $u_i = \begin{cases} 0, & \text{for } i \leq N \\ a_i - \lambda_i c_i + \sum_j \beta_{ij} c_j + h - g, & \text{for } N < i \leq s. \end{cases}$

From (A1), (A3), (A4) and (A5), it is clear that u_i is nondecreasing in i for $0 \leq i \leq s$. We consider two cases for $D_w(s)$: 1) $D_w(s)$ = R, 2) $D_w(s)$ = L.

Case 1). In this case, from (A6), we have that

(6.15) $u_{s+1} - u_s = a_{s+1} - a_s - \lambda_s c_{s+1} + \lambda_s c_s \geq 0,$

which implies that u_i is nondecreasing in i for $0 \leq i \leq s+1$.

Case 2). In this case, from theorem 6.1, we have that

(6.16) $D_w(i) = \text{L}$ for $i \leq s$,

hence

(6.17) $u_i = 0$ for $i \leq s$.

This completes the proof.

From the definition of f_{ij}, lemmas 4.2, 6.3 and equation (6.12), the following lemma is easily shown to hold.

Lemma 6.4. G_i is nondecreasing in i or $G_i \leq 0$ for all i.

As for the property of $D_v(i)$, we have the following theorem.

Theorem 6.2. There exists a limit number n such that

(6.18) $D_v(i) = \begin{cases} \text{K} & \text{for } i \leq n, \\ 0 & \text{otherwise.} \end{cases}$

Proof: For the proof, it is sufficient to show that if $D_v(i) = 0$, then $D_v(j) = 0$ for all $j > i$, or $D_v(i) = \text{K}$ for all $i \leq s$. From equations (5.1) and

(6.11), we have that

(6.19) $\lambda_i(K_i - O_i) \leqq a_i + \sum_j \beta_{ij}O_j - g - \lambda_i O_i = G_i.$

If $G_i \leqq 0$ for all i, then the theorem holds in a special form. From now on, we treat the case where G_i is nondecreasing. As for $D_v(s)$, we consider two cases: 1) $D_v(s)$ = K, 2) $D_v(s) = 0$.

Case 1). From equation (6.19), we have that

(6.20) $\lambda_i(K_i - O_i) \leqq G_i \leqq G_s = \lambda_s(K_s - O_s) \leqq 0.$

which implies that $D_v(i)$ = K for all $i \leqq s$.

Case 2). We assume that $D_v(i) = 0$ and $D_v(j+1) = D_v(j+2) = \ldots = D_v(s) = 0$, where $j > i$. Then, from equation (6.19), we have that

(6.21) $0 \leqq \lambda_i(K_i - O_i) \leqq G_i \leqq G_j = \lambda_j(K_j - O_j).$

Hence, $D_v(j) = 0$, which implies that $D_v(j) = 0$ for all $j > i$. This completes the proof.

From theorems 6.1 and 6.2, we have the following theorem concerning the structure of an optimal ordering and replacement policy.

Theorem 6.3. There exists an optimal (n,N)-policy.

7. Optimal (n,N)-policy

We give a procedure to obtain an optimal ordering and replacement policy. An optimal policy can be determined by a usual technique of Policy Iteration method as follows.

Policy Improvement Routine

For the current cost rate g, find the policy which satisfies,

(7.1) $v_i = \min \{ K_i, O_i \}, \quad i \leqq s, \quad v_{s+1} = O_{s+1},$

(7.2) $w_i = \min \{ L_i, c_i \}, \quad i \leqq s, \quad w_{s+1} = c_{s+1},$

where K_i, O_i and L_i are given in the same form as equations (5.5)-(5.7). The actions for E_i and F_i can be obtained by starting from F_s, E_s and working back. If the policy obtained in this routine is the same as the one in the previous routine, then an optimal policy (current policy) is determined. Otherwise, go to the value determination operation.

It should be noted that the policy obtained here has the form of (n,N)-

policy. The proof is done in the same way as in the previous section.

Value Determination Operation

Under the current policy ((n,N)-policy), solve the following set of equations with respect to v_i, w_i and g.

(7.3) $v_0 = 0,\ v_i = K_i,\ i = 1,\dots,n,\ v_i = O_i,\ i = n+1,\dots,s+1,$

(7.4) $w_i = L_i,\ i = 0,1,\dots,N,\ w_i = c_i,\ i = N+1,\dots,s+1.$

Then, go to policy improvement routine.

Solving the above equations, we have that

$$(7.5)\quad g(n,N) = [\ c + A_0 + \sum_i f_{0i}x_i + \sum_{i=0}^{n} Q_{0i}\{ -h\sum_{j=i}^{N} f_{ij} + \sum_{j=N+1}^{s} f_{ij}(\ a_j - \lambda_j c_j + \sum_k \beta_{jk}c_k\ ) + f_{i,s+1}a_{s+1}\ \}/\lambda_i\ ] \Big/ [\ T + \sum_i f_{0i}y_i + \sum_{i=0}^{n} Q_{0i}\sum_{j=N+1}^{s+1} f_{ij}/\lambda_i\ ],$$

where

$$(7.6)\quad x_i = \begin{cases} \sum_{j=i}^{N} Q_{ij}(\ a_j + h + \sum_{k=N+1}^{s+1} \beta_{jk}c_k\ )/\lambda_j, & i \leqq N \\ c_i, & i > N \end{cases}$$

$$(7.7)\quad y_i = \begin{cases} \sum_{j=i}^{N} Q_{ij}/\lambda_j, & i \leqq N \\ 0, & i > N \end{cases}$$

$$(7.8)\quad Q_{ii} = 1,\quad Q_{ij} = \sum_{k=i+1}^{j} p_{ik}Q_{kj} = \sum_{k=i}^{j-1} Q_{ik}p_{kj},\ \text{for } j > i.$$

It is easily shown that this policy iteration cycle gives us an optimal policy. It should be noted that $g(n,N)$ given by equation (7.5)-(7.8) is the cost rate of the system operated under a (n,N)-policy and the optimal n^* and N^* can be also determined directly from $g(n,N)$.

8. Numerical Example

The system has only two degradation states ($s = 2$) and the delivery time is constant T. As a numerical example, we let

$\beta_{02} = \beta_{03} = 0,\ \beta_{13} = 0,\ \beta_{01}(=\lambda_0) = \beta_{12}(=\lambda_1) = \beta_{23}(=\lambda_2) = 1,$

$a_0 = a_1 = a_2 = 0,\ a_3 = 20,\quad c_0 = c_1 = c_2 = 30,\quad c_3 = 70,\quad c = 10,$

and consider five cases for (h,T):{ (10,0.5), (10,1.0), (10,1.5) } and { (3,1.0), (10,1.0), (15,1.0) }. For each case, an optimal (n^*+1, N^*+1) and the optimal cost rate g^* are given below.

h	T	n^*+1	N^*+1	g^*
10	0.5	2	2	23.1
10	1.0	1	2	23.8
10	1.5	0	2	24.8
3	1.0	0	2	22.4
10	1.0	1	2	23.8
15	1.0	2	2	24.2

In this table, we can see that the optimal n^* becomes smaller as the delivery time becomes larger and that it becomes larger as the holding cost becomes larger. On the other hand, the optimal N^* is not sensitive to the delivery time and/or the holding cost. It is a natural result that the minimum cost g^* becomes larger as T and/or h becomes larger.

9. Conclusion

We have discussed an optimal ordering and replacement policy for a continuous time Markovian degradation system, which minimizes the expected cost per unit time in an infinite time span. The problem is formulated by a semi-Markov decision process and the optimality of (n,N)-policy has been shown under some reasonable conditions. Furthermore, the cost rate under (n,N) -policy has been obtained. In our problem, it is assumed that only one spare system can be kept in inventory. In more general cases, this assumption should be relaxed, that is, more than one spares should be allowed. Ordering and replacement problems in such a case will be treated in future work.

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Hajime KAWAI: Department of Business Administration, School of Economics, University of Osaka Prefecture, Sakai, Osaka, 591, Japan.