

NETWORK-FLOW ALGORITHMS FOR LOWER-TRUNCATED TRANSVERSAL POLYMATROIDS

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Abstract In this paper, we introduce *lower-truncated transversal polymatroids*, and develop *efficient algorithms of network-flow type* for those polymatroids. The lower-truncated transversal polymatroid contains, as special cases, a variety of useful matroids such as *cycle matroids of graphs*, *matroids in plane skeletal structures*, etc. We present simple and powerful theorems which enable us to solve various combinatorial optimization problems for those polymatroids by means of *network-flow algorithms*. Especially, we can solve *greedy-type optimization problems* concerning those polymatroids in a remarkably efficient manner. As greedy-type problems, we take up the problem of finding a maximum-weight independent vector, that of finding the principal partition and that of covering and packing, and give efficient solutions for them. Applying general algorithms for lower-truncated transversal polymatroids to cycle matroids of graphs and matroids in plane skeletal structures, we obtain various new results.

From the viewpoint of applications, lower-truncated transversal polymatroids are essentially related to *discrete systems with internal degrees of freedom* which arise in many fields of engineering, so that the algorithms for those polymatroids developed in this paper give efficient methods to analyze such systems in a unifying manner.

Introduction

In this decade, it has been pointed out that the (poly-)matroid theory is very useful to analyze discrete systems arising in operations research and other engineering systems, and various problems concerning such systems have been solved by means of (poly-)matroid theory newly, or, at least, in much simpler way than known methods (see [16], [17]). Here we introduce a class of polymatroids which contains various polymatroids arising in practical problems, and give network-flow algorithms for them. This enhances the efficiency of polymatroidal approach much.

It was Sugihara's recent work [26], [27], [29] concerning discrete systems with internal degrees of freedom that motivated this work. He has observed and shown that problems of checking inconsistency or redundancy of such discrete systems can

be formulated as problems of bipartite matchings in a generalized sense, and that the problems can be solved in a polynomial time by repeated applications of ordinary bipartite matching algorithms. In this paper, based upon Sugihara's work, we introduce *lower-truncated transversal polymatroids*, which give a unifying framework to treat discrete systems with internal degrees of freedom. Typical and useful examples of those polymatroids are cycle matroids of graphs and matroids in plane skeletal structures. We then give simple and powerful theorems which enable us to solve various combinatorial optimization problems concerning those polymatroids by means of network-flow algorithms. Specifically, we show that fundamental functions, such as the saturation function and the dependence function, of those polymatroids can be computed efficiently by solving simple maximum flow problems. It should be noted that, if one can compute fundamental functions of a polymatroid, one can solve the independent-flow problems [7], which is one of unifying problems in polymatroids optimization problems, concerning the polymatroid in practice.

Using the theorems, we can solve especially greedy-type optimization problems of lower-truncated transversal polymatroids in a very efficient way, so that we describe efficient algorithms for solving greedy-type problems in detail. In the paper, as greedy-type optimization problems, we take up the problem of finding a base or a maximum-weight independent vector and that of finding several kinds of so-called principal partitions of such polymatroid. (The principal partition is the decomposition theory useful to various problems concerning discrete systems in operations research and other fields of engineering science [15], [16], [17], [21], [31], [32].) The problem of covering and packing of lower-truncated transversal matroids is also considered.

By applying the theorems and the arguments for the above three optimization problems to cycle matroids of graphs and matroids in plane skeletal structures, we can obtain various new results. Let us specify the results concerning an undirected graph $G=(V, E)$ with vertex set V and edge set E . Let M_G be a cycle matroid of the graph G , and M_G^k the union of M_G with itself k times. (The matroid M_G^k is applied to fundamental problems in electric network.) We can find a base of M_G^k in $O(k^2|V|^2)$ time, and a maximum-weight independent set of M_G^k in $O(k^2|E||V|)$ time. We can find the principal partition of the graph G in the sense of Kishi and Kajitani [18] and Iri [14] in $O(|V|^2)$ time, and that in the sense of Tomizawa [31] in $O(|E|^3 \log|V|)$ time. On the problem of covering and packing, we can find the minimum number of forests whose union is G in $O(|E|^2)$ time, and the maximum number of disjoint spanning trees in $O(|E|^2 \log|V|)$ time. Specifying the results concerning a matroid in a plane skeletal structure associated with a graph $G=(V, E)$, which we denote by M_L , we can find a base of M_L in $O(|E||V|)$ time. The decomposition of a stable structure due to Nakamura and Sugihara [22] can be found in $O(|V|^2)$ time.

It should be noted that, though we do not discuss problems concerning respective systems, techniques developed in this paper can be applied to discrete systems with internal degrees of freedom [29] such as two typical examples mentioned above, line drawings of polyhedra [27] and other systems arising in various fields of engineering science.

1. Preliminaries

1.1. Basic concepts of polymatroids

For a finite set E , we denote by \mathbf{R}^E (resp. \mathbf{R}_+^E) the set of all functions (or vectors) from E to \mathbf{R} of reals (resp. \mathbf{R}_+ of nonnegative reals). For $x \in \mathbf{R}^E$ and $S \subseteq E$, we denote by $x|_S$ the restriction of x to S , and set $x(S) = \sum_{e \in S} x(e)$. We define a characteristic vector $\chi_S \in \mathbf{R}^E$ for $S \subseteq E$ by $\chi_S(e) = 1$ ($e \in S$) and $\chi_S(e) = 0$ ($e \notin S$). For $e \in E$, we denote $\chi_{\{e\}}$ simply by χ_e .

Polymatroids were introduced by Edmonds [3]. A function $\mu: 2^E \rightarrow \mathbf{R}$, where 2^E is the set of all subsets of E , is called a β_0 -function if

- (i) $\mu(X) \geq 0$ for each $X \in 2^E - \{\phi\}$
- (ii) nondecreasing: if $X \subseteq Y \subseteq E$, then $\mu(X) \leq \mu(Y)$
- (iii) submodular: for any $X, Y \subseteq E$, $\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y)$.

Further, μ is called a β -function if, also, $\mu(\phi) = 0$. For a β_0 -function $\mu: 2^E \rightarrow \mathbf{R}$, we define a polyhedron $\mathbf{P}(\mu)$ by

$$(1.1) \quad \mathbf{P}(\mu) = \{x \mid x \in \mathbf{R}_+^E, x(X) \leq \mu(X) \ (\phi \neq X \subseteq E)\}.$$

A *polymatroid* $P = (E, \mu)$ is a pair of a finite set E and a β -function $\mu: 2^E \rightarrow \mathbf{R}_+$ which is called the *rank function* of P . A vector $x \in \mathbf{P}(\mu)$ is called an *independent vector* of polymatroid P , and $\mathbf{P}(\mu)$ is called an *independence polyhedron* of P . An independent vector x of P such that $x(E) = \mu(E)$ is called a *base* of P .

A polymatroid can be obtained from a β_0 -function which is not necessarily a β -function. For a β_0 -function $\mu_0: 2^E \rightarrow \mathbf{R}$, $\mathbf{P}(\mu_0)$ becomes an independence polyhedron of a polymatroid whose rank function μ is given by

$$(1.2) \quad \mu(S) = \min \left\{ \sum_i \mu_0(X_i) \mid S = \cup_i X_i, X_i \neq \phi, X_i \cap X_j = \phi \ (i \neq j) \right\} \quad (S \subseteq E)$$

(see [1], [3]). Note that $\mathbf{P}(\mu_0) = \mathbf{P}(\mu)$ and $\mu(\phi) = 0$. μ is called the *lower-truncation* (or *Dilworth truncation*) of μ_0 .

For a vector $y \in \mathbf{R}_+^E$, a *reduction of polymatroid* (E, μ) with respect to y is a polymatroid (E, μ^y) where

$$(1.3) \quad \mu^y(S) = \min \{ \mu(X) + y(S - X) \mid X \subseteq S \} \quad (S \subseteq E).$$

Note that $\mathbf{P}(\mu^y) = \mathbf{P}(\mu) \cap \{x \mid x \in \mathbf{R}_+^E, x \leq y\}$.

A polymatroid (E, μ) is said to be *integral* if μ is integer-valued. Every extreme point of the independence polyhedron of an integral polymatroid is integer-valued. If (E, μ) is an integral polymatroid and $\mu(X) \leq |X|$ for any $X \subseteq E$, $M = (E, \mu)$ is a *matroid* and μ is its *rank function*. A subset X of E is *independent in M* iff $\mu(X) = |X|$. The *rank* of M is defined to be $\mu(E)$. The reduction of an integral polymatroid with respect to χ_E is a matroid. Let $M_1 = (E, \mu_1)$ and $M_2 = (E, \mu_2)$ be two matroids on the same set E . Then, $(E, \mu_1 + \mu_2)$ is a polymatroid but is not necessarily a matroid. However, the reduction of $(E, \mu_1 + \mu_2)$ with respect to χ_E is a matroid which is called the *union* of M_1 and M_2 . The collection of independent sets of the union matroid of M_1 and M_2 is given by

$$(1.4) \quad \{I \mid I = I_1 \cup I_2, I_i \subseteq E, I_i \text{ independent in } M_i (i=1,2)\}$$

For a polymatroid $P = (E, \mu)$, $x \in \mathbf{P}(\mu)$ and $u \in E$, a *saturation function* $\text{sat}(x)$ and a function $c^+(x, u)$, which may be called a *saturation capacity function*, are defined as follows.

$$(1.5) \quad \text{sat}(x) = \{v \mid v \in E, x + \delta \chi_v \notin \mathbf{P}(\mu) \text{ for any } \delta > 0\}$$

$$(1.6) \quad c^+(x, u) = \max\{\delta \mid \delta \geq 0, x + \delta \chi_u \in \mathbf{P}(\mu)\}$$

Note that $u \in \text{sat}(x)$ iff $c^+(x, u) = 0$. Further, define a *dependence function* $\text{dep}(x, u)$ by

$$(1.7) \quad \text{dep}(x, u) = \{v \mid v \in E, x + \delta \chi_u - \delta \chi_v \in \mathbf{P}(\mu) \text{ for some } \delta > 0\}$$

if $u \in \text{sat}(x)$, and $\text{dep}(x, u) = \emptyset$ if $u \notin \text{sat}(x)$. If $u \in \text{sat}(x)$, we define a function $c^\pm(x, u, v)$ for $v \in \text{dep}(x, u) - \{u\}$, which may be called an *exchange capacity function*, as follows.

$$(1.8) \quad c^\pm(x, u, v) = \max\{\delta \mid \delta > 0, x + \delta \chi_u - \delta \chi_v \in \mathbf{P}(\mu)\}$$

The above four functions, which, in the lump, we will refer to as *fundamental functions* of the polymatroid, were introduced by Fujishige [7]. If one can compute the fundamental functions of a polymatroid algorithmically, one can solve the independent-flow problems concerning the polymatroid in practice.

1.2. Hitchcock-type network flow problem

Let $B = (U, W; A)$ be a directed bipartite graph with left vertex set U , right vertex set W and arc set $A \subseteq U \times W$. Define functions $\Gamma: 2^U \rightarrow 2^W$ and $\gamma: 2^U \rightarrow \mathbf{R}_+$ by

$$(1.9) \quad \Gamma(X) = \{w \mid (u, w) \in A, u \in X\} \quad (X \subseteq U)$$

$$(1.10) \quad \gamma(X) = |\Gamma(X)| \quad (X \subseteq U).$$

As is well known, γ is a β -function. Let c_W be a vector in \mathbf{R}_+^W . We define a network $N(B, c_W)$ with vertex set \tilde{V} , arc set \tilde{A} and capacity function \tilde{c} as follows.

$$(1.11) \quad \tilde{V} = U \cup W \cup \{t\}$$

$$(1.12) \quad \tilde{A} = A \cup \{(w, t) \mid w \in W\}$$

$$(1.13) \quad \tilde{c}(a) = \begin{cases} \infty & a \in A \\ c_W(w) & a = (w, t) \quad (w \in W) \end{cases}$$

For a vector $f \in \mathbf{R}^{\tilde{A}}$, we define a vector $\partial f \in \mathbf{R}^{\tilde{V}}$ by

$$(1.14) \quad \partial f(v) = \sum_{a=(v,v') \in \tilde{A}} f(a) - \sum_{a=(v',v) \in \tilde{A}} f(a) \quad (v \in \tilde{V}).$$

A vector $f \in \mathbf{R}^{\tilde{A}}$ is a flow in $N(B, c_W)$ if f satisfies

$$(1.15) \quad \partial f(w) = 0 \quad (w \in W)$$

$$(1.16) \quad 0 \leq f(a) \leq \tilde{c}(a) \quad (a \in \tilde{A}).$$

A value of f is defined to be $\partial f(U)$. For a flow f , $\partial f|U$ is called a *supply vector*. For $c_U \in \mathbf{R}_+^U$, a flow f such that $\partial f|U \leq c_U$ is said to be *maximum with respect to c_U* if the value of f is maximum among those of all flows f' such that $\partial f'|U \leq c_U$. The following lemma is the famous maximum-flow minimum-cut theorem [6], [13].

Lemma 1.1. For $c_U \in \mathbf{R}_+^U$ and $c_W \in \mathbf{R}_+^W$, the value of a flow on $N(B, c_W)$ which is maximum with respect to c_U is equal to

$$(1.17) \quad \min\{c_W(\Gamma(X)) + c_U(U - X) \mid X \subseteq U\}. \quad \square$$

Let f be a flow on $N(B, c_W)$. Let $v, v' \in \tilde{V}$. An *augmenting path from v to v' on $N(B, c_W)$ with respect to flow f* is a sequence $v = v_0, a_1, v_1, a_2, v_2, \dots, v_{k-1}, a_k, v_k = v'$ ($k \geq 0$) such that $v_i \in \tilde{V} - \{t\}$ ($0 \leq i < k$), $v_i \neq v_j$ ($0 \leq i < j \leq k$), $a_i \in \tilde{A}$ ($1 \leq i \leq k$) and, for each $i = 1, \dots, k$, either $a_i = (v_{i-1}, v_i)$ and $f(a_i) < c(a_i)$ or $a_i = (v_i, v_{i-1})$ and $f(a_i) > 0$.

As is well known, the problem of finding a maximum flow with respect to χ_U in $N(B, c_W)$ is equivalent to the maximum matching problem of a bipartite graph B [6], [13]. This problem can be solved in $O(\sqrt{|U|}|A|)$ time [5], [9]. More generally, we obtain the following theorem, which can be shown almost similarly to the proof in [5], so that we omit the proof in this paper (in [11], a complete proof of this theorem is given).

Theorem 1.1. For an integer-valued vector $c_W \in \mathbf{R}_+^W$, a nonnegative integer d and $u \in U$, a maximum flow on $N(B, c_W)$ with respect to $\chi_U + d\chi_u$ can be found in $O(\sqrt{|U| + d}(|A| + d))$ time by using Dinic's algorithm [2]. \square

2. Lower-Truncated Transversal Polymatroids

2.1. Definitions

Let $B=(U, W; A)$ be a directed bipartite graph without isolated vertex. For $p, d \in \mathbf{R}_+$ ($p \neq 0$), consider a function $\gamma_{p,d}: 2^U \rightarrow \mathbf{R}$ defined by

$$(2.1) \quad \gamma_{p,d}(X) \equiv p\gamma(X) - d = p|\Gamma(X)| - d \quad (X \subseteq U)$$

(see (1.9) and (1.10)). Assume that the degree of each vertex in U is greater than or equal to d/p . Then, since γ is a β -function, we immediately have

Lemma 2.1. $\gamma_{p,d}$ is a β_0 -function. \square

Hence, $\mathbf{P}(\gamma_{p,d})$ is an independence polyhedron of some polymatroid on U . We denote this polymatroid by $P(p,d)$ whose rank function $\rho_{p,d}$ is the lower-truncation of $\gamma_{p,d}$ (see (1.2)). We will refer to a polymatroid which is the reduction of $P(p,d)$ with respect to some vector in \mathbf{R}_+^U as a *lower-truncated transversal polymatroid*. For $\alpha \geq 0$, we denote a polymatroid which is the reduction of $P(p,d)$ with respect to vector $\alpha\chi_U$ by $P(p,d;\alpha)$. Note that $P(p,d) = P(p,d;\gamma_{p,d}(U)) = P(p,d;n)$ where $n = \max\{\gamma_{p,d}(\{u\}) \mid u \in U\}$.

If p and d are integers, $P(p,d;1)$ is a matroid, which we will refer to as a (p,d) -*transversal matroid*, and denote by $M(p,d)$. $(1,0)$ -transversal matroid is just a transversal matroid in an ordinary sense [4]. Apparently, the union of $M(p,d)$ with itself k times is a matroid $M(kp, kd)$.

Let us show an example. Consider a lower-truncated transversal polymatroid $P(1,1)$ on U of bipartite graph $B=(U, W; A)$ ($U = \{u_1, u_2, u_3\}$) illustrated in Fig.2.1(a). Polyhedra $\mathbf{P}(\gamma)$ and $\mathbf{P}(\gamma_{1,1})$ are shown in Fig.2.1(b) and (c), respectively. In this case, the independence polyhedron $\mathbf{P}(\gamma_{1,1})$ of $P(1,1)$ corresponds to the polyhedron determined by a rank function of a graph depicted in Fig.2.1(d) (cf. §4.1).

In the so-called transversal theory, it is often considered whether the condition " $|X| \leq |\Gamma(X)| + k$ ($X \subseteq U$)" holds for bipartite graph B , where " k " is called a *defect* [24]. Hence, " d " in (2.1) may be considered as a *negative defect*. Sugihara [26] has observed that concerning plane skeletal structures a problem with a negative defect arises, and has shown that the problem can be solved by repeated applications of an ordinary bipartite-matching algorithm. He has shown that the similar types of problems arise in the discrete system with internal degrees of freedom, and propose a polynomial-time algorithm for checking structural inconsistency of such systems [29] (see also [27], [28]). Our approach is a straightforward generalization of his work. The lower-truncated transversal polymatroid introduced here gives a unifying framework to analyze those discrete systems. In the following, we develop useful techniques of network-flow type to treat lower-truncated transversal polymatroids algorithmically.

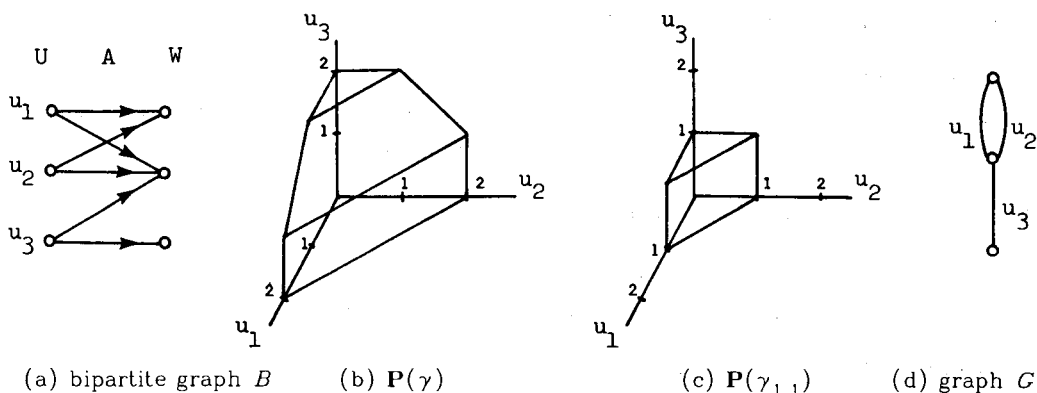


Fig.2.1. An example of a lower-truncated transversal polymatroid

2.2. Theorems

We begin with the following lemma, which is one of the so-called demand-supply theorems in network flow theory [6], [13]. Here, we give a proof based on the maximum-flow minimum-cut theorem.

Lemma 2.2. For $x \in \mathbf{R}_+^U$, network $N(B, p\chi_w)$ has a flow f such that $x = \partial f|U$ iff $x(X) \leq p\gamma(X)$ for any $X \subseteq U$.

Proof: $N(B, p\chi_w)$ has a flow f such that $x = \partial f|U$.

\leftrightarrow The value of a flow on $N(B, p\chi_w)$ which is maximum with respect to x is $x(U)$.

$\leftrightarrow p|\Gamma(X)| + x(U - X) \geq x(U)$ for any $X \subseteq U$ (due to Lemma 1.1).

$\leftrightarrow x(X) \leq p\gamma(X)$ for any $X \subseteq U$. \square

Using Lemma 2.2, we can obtain the following theorems, based on which we can compute the fundamental functions of polymatroid $P(p, d; \alpha)$ efficiently by a simple network flow algorithm. First, we give a necessary and sufficient condition so that a vector $x \in \mathbf{R}_+^U$ is contained in $\mathbf{P}(\gamma_{p,d})$.

Theorem 2.1. Let x be a vector in \mathbf{R}_+^U . Then, x is contained in $\mathbf{P}(\gamma_{p,d})$ iff, for any $u \in U$, $N(B, p\chi_w)$ has a flow f such that $x + d\chi_u = \partial f|U$.

Proof: For $x \in \mathbf{R}_+^U$ and $u \in U$, let $y = x + d\chi_u$.

If $x \in \mathbf{P}(\gamma_{p,d})$, we have $x(X) + d \leq p\gamma(X)$ for any X with $\phi \neq X \subseteq U$. Since $y(X) \leq x(X) + d$ for any $X \subseteq U$, and $y(\phi) = p\gamma(\phi) = 0$, we have $y(X) \leq p\gamma(X)$ for $X \subseteq U$, which, from Lemma 2.2, implies that $N(B, p\chi_w)$ has a flow f such that $\partial f|U = y (= x + d\chi_u)$.

If $N(B, p\chi_w)$ has a flow f such that $\partial f|U = x + d\chi_u (= y)$, then we have $y(X) \leq p\gamma(X)$ for $X \subseteq U$ from Lemma 2.2. For any Y with $u \in Y \subseteq U$, we have $y(Y) = x(Y) + d$, and hence $x(Y) \leq \gamma_{p,d}(Y)$. Since u can be taken arbitrarily, we

obtain the theorem. \square

Theorem 2.1 shows that, given a vector $x \in \mathbf{R}_+^U$, we can determine whether x is contained in $\mathbf{P}(\gamma_{p,d})$ by solving $|U|$ maximum flow problems, which can be solved quickly. Here, it should be noted that capacities of these $|U|$ maximum flow problems are almost similar to one another. If one makes use of this fact, the problem of testing independence of a given vector in $P(p,d)$ can be solved more efficiently. However, it is not generally true that one can compute the fundamental functions of $P(p,d;\alpha)$ quickly even though one can test independence of a given vector quickly. It is guaranteed by the following easy but useful theorem that the fundamental functions of $P(p,d;\alpha)$ can be computed efficiently.

Theorem 2.2. For a vector $x \in \mathbf{R}_+^U$, $u, v \in U$ and $\alpha, \delta \geq 0$, where $\alpha \leq x(v)$, let $y = x + \delta \chi_u - \alpha \chi_v$. Then, y is contained in $\mathbf{P}(\gamma_{p,d})$ iff network $N(B, p\chi_W)$ has a flow f such that $y + d\chi_u = \partial f|U$.

Proof: If $N(B, p\chi_W)$ has a flow f such that $y + d\chi_u = \partial f|U$, then, from Lemma 2.2, we have $y(X) + d \leq p\gamma(X)$ for any X with $u \in X \subseteq U$. For any X with $u \notin X \subseteq U$, $X \neq \emptyset$, we have $y(X) \leq x(X) \leq p\gamma(X) - d$ (note that $x \in \mathbf{P}(\gamma_{p,d})$). Hence, $y \in \mathbf{P}(\gamma_{p,d})$.

The inverse follows from Theorem 2.1. \square

From Theorem 2.2, we immediately obtain the following theorems for computing the fundamental functions of $P(p,d;\alpha)$ efficiently.

Theorem 2.3. For an independent vector x of $P(p,d;\alpha)$ and $u \in U$, let f_u be a flow on $N(B, p\chi_W)$ such that $x + d\chi_u = \partial f_u|U$. Then, we have

$u \in \text{sat}(x)$ iff $x(u) = \alpha$ or there is no augmenting path from u to t on $N(B, p\chi_W)$ with respect to flow f_u .

$$\text{dep}(x, u) = \begin{cases} \{v \mid v \in U, \text{ there is an augmenting path from } u \text{ to } v \text{ on } N(B, p\chi_W) \\ \text{with respect to flow } f_u\} & (\text{ if } u \in \text{sat}(x), x(u) < \alpha) \\ \{u\} & (\text{ if } u \in \text{sat}(x), x(u) = \alpha) \end{cases}$$

$$c^+(x, u) = \min\{\alpha - x(u), \delta - x(u) - d\} \quad (u \in U),$$

where $\delta = \max\{\partial f(u) \mid f: \text{flow on } N(B, p\chi_W), \partial f(v) = x(v) \quad (v \in U - \{u\})\}$.

$$c^\pm(x, u, v) = \min\{\alpha - x(u), \epsilon - x(u) - d\} \quad (u \in \text{sat}(x), v \in \text{dep}(x, u) - \{u\}),$$

where $\epsilon = \max\{\partial f(u') \mid f: \text{flow on } N(B, p\chi_W), \partial f(u') = x(u') \quad (u' \in U - \{u, v\})\}$.

Proof: Concerning the saturation function, if $x(u) = \alpha$, we obviously have $u \in \text{sat}(x)$. Suppose that $x(u) < \alpha$. Let \tilde{V} be a set of vertices v on $N(B, p\chi_W)$ such that there is an augmenting path from u to v with respect to flow f_u . In the case of $t \in \tilde{V}$, there exists a flow f' such that $x + (d + \delta)\chi_u = \partial f'|U$ for some $\delta > 0$. Then, from

Theorem 2.2, we see that $x + \delta\chi_u$ is an independent vector of $P(p, d; \alpha)$, and hence $u \notin \text{sat}(x)$. In the case of $t \notin \tilde{V}$, let \tilde{U} be defined by $\tilde{U} = U \cap \tilde{V}$. Then, from the definition of \tilde{V} and the assumption of $t \notin \tilde{V}$, we have $u \in \tilde{U}$ and $x(\tilde{U}) + d = p\gamma(\tilde{U})$, and hence $u \in \text{sat}(x)$.

Concerning the dependence function, the case of $u \in \text{sat}(x)$ and $x(u) = \alpha$ is trivial. Suppose that $u \in \text{sat}(x)$ and $x(u) < \alpha$. Let \tilde{V} be defined as above, where from the assumption we have $t \notin \tilde{V}$. For $v \in U - \{u\}$, if $v \in \tilde{V}$, there exists a flow \tilde{f} such that $x + (d + \delta)\chi_u - \delta\chi_v = \partial\tilde{f}|U$ for some $\delta > 0$. Then, from Theorem 2.2, we see that $x + \delta\chi_u - \delta\chi_v$ is an independent vector of $P(p, d; \alpha)$, and hence $v \in \text{dep}(x, u)$. In the case of $v \notin \tilde{V}$, let \tilde{U} be defined by $\tilde{U} = U \cap \tilde{V}$. Then, we see that $x(\tilde{U}) + d = p\gamma(\tilde{U})$, $u \in \tilde{U}$ and $v \notin \tilde{U}$, and hence $v \notin \text{dep}(x, u)$.

The remaining parts can be shown almost analogously. \square

Using Theorem 2.3, we can generally construct efficient algorithms for optimization problems concerning polymatroid $P(p, d; \alpha)$.

In the case of $d \leq p$, we can also compute the fundamental functions of $P(p, d; \alpha)$ in a slightly different way. The theorems below enable us to compute those fundamental functions for a subset of U simultaneously.

Theorem 2.4. In the case of $d \leq p$, concerning a vector x in \mathbf{R}_+^U , x is contained in $P(\gamma_{p,d})$ iff, for any $w \in W$, $N(B, p\chi_w - d\chi_w)$ has a flow f such that $x = \partial f|U$.

Proof: For u and w such that $u \in U$, $w \in W$ and $(u, w) \in A$, it is obvious that $N(B, p\chi_w)$ has a flow f' such that $x + d\chi_u = \partial f'$ iff $N(B, p\chi_w - d\chi_w)$ has a flow f such that $x = \partial f|U$. Then, the theorem follows from Theorem 2.1. \square

Theorem 2.5. In the case of $d \leq p$, for an independent vector x of $P(p, d; \alpha)$ and $u \in U$, let w be such that $w \in \Gamma(\{u\})$, and f_w a flow on $N(B, p\chi_w - d\chi_w)$ such that $x = \partial f_w|U$. Then, we have

$u \in \text{sat}(x)$ iff $x(u) = \alpha$ or there is no augmenting path from u to t on $N(B, p\chi_w - d\chi_w)$ with respect to flow f_w .

$$\text{dep}(x, u) = \begin{cases} \{v \mid v \in U, \text{ there is an augmenting path from } u \text{ to } v \text{ on } N(B, p\chi_w - d\chi_w) \\ \text{with respect to flow } f_w\} & (\text{ if } u \in \text{sat}(x), x(u) < \alpha) \\ \{u\} & (\text{ if } u \in \text{sat}(x), x(u) = \alpha) \end{cases}$$

$$c^+(x, u) = \min\{\alpha - x(u), \delta' - x(u)\},$$

where $\delta' = \max\{\partial f(u) \mid f: \text{flow on } N(B, p\chi_w - d\chi_w), \partial f(v) = x(v) \ (v \in U - \{u\})\}$.

$$c^\pm(x, u, v) = \min\{\alpha - x(u), \epsilon' - x(u)\} \ (u \in \text{sat}(x), v \in \text{dep}(x, u) - \{u\}),$$

where $\epsilon' = \max\{\partial f(u) \mid f: \text{flow on } N(B, p\chi_w - d\chi_w), \partial f(u') = x(u') \ (u' \in U - \{u, v\})\}$.

Proof: One can easily prove this theorem by applying the similar arguments in the proof of Theorem 2.4 to Theorem 2.3. \square

Let us show some examples. First, consider a lower-truncated transversal polymatroid $P(2,3)$ on U of bipartite graph $B=(U, W; A)$ ($U=\{u_i \mid i=1, \dots, 8\}$) in Fig.2.2(a). Setting $U_k=\{u_i \mid i=1, \dots, k\}$ ($k=1, \dots, 8$), we show a flow f on $N(B, 2\chi_W)$ such that $\chi_{U_6}=\partial f|U$ in Fig.2.2(b). In (b), it can be easily checked that the flow can be augmented at least by 3 from each u_i to t , so that χ_{U_6} is an independent vector of $P(2,3)$ (Theorem 2.1). Especially, the flow can be augmented by 4 from u_7 to t . Hence, we see that $c^+(\chi_{U_6}, u_7)=1$ (Theorem 2.3) and χ_{U_7} is an independent vector of $P(2,3)$. In Fig.2.2(c), we show a flow f' on the same network such that $\chi_{U_7}=\partial f'|U$. In (c), we can easily see that $u_8 \in \text{sat}(\chi_{U_7})$ and $\text{dep}(\chi_{U_7}, u_8)=\{u_3, u_4, u_5, u_6, u_7, u_8\}$ (Theorem 2.3).

Next, as an example of the case of $d \leq p$, consider $P(2,2)$ and $M(2,2)(=P(2,2;1))$ on U of bipartite graph $B=(U, W; A)$ ($U=\{u_i \mid i=1, \dots, 8\}$, $W=\{w_j \mid j=1, \dots, 5\}$) in Fig.2.3(a). It can be readily seen that $T \equiv U - \{u_4\}$ is a base of matroid $M(2,2)$. In Fig.2.3(b), we show a flow f on $N(B, 2\chi_W - 2\chi_{w_2})$ such that $\chi_T = \partial f|U$, from which we see the following concerning $P(2,2)$ (Theorem 2.5):

$$\begin{aligned} u_1 &\notin \text{sat}(\chi_T), \\ \text{dep}(\chi_T, u_2) &= \text{dep}(\chi_T, u_3) = \{u_2, u_3\}, & \text{dep}(\chi_T, u_4) &= \{u_2, u_3, u_4\}, \\ \text{dep}(\chi_T, u_6) &= \{u_2, u_3, u_5, u_6\}, & \text{dep}(\chi_T, u_7) &= \{u_2, u_3, u_5, u_6, u_7, u_8\}. \end{aligned}$$

Note that $\{u_i \mid u_i \in U, (u_i, w_2) \in A\} = \{u_1, u_2, u_3, u_4, u_6, u_7\}$.

We conclude this section with the following theorems which give several results of Theorem 2.1~5 from the standpoint of the algorithmic complexity. In the discussions below, we suppose p, d and n are integers with $p, n > 0$ and $d \geq 0$. We further consider the following condition (2.2):

(2.2) For an integer-valued independent vector x of $P(p, d; n)$, an integral flow f on $N(B, p\chi_W)$ such that $x = \partial f|U$ is given.

Theorem 2.6. Suppose the condition (2.2) holds. For $u \in U$, we can determine whether u is contained in $\text{sat}(x)$, and compute $\text{dep}(x, u)$ in $O((1+d)|A|)$ time. We can compute $c^+(x, u)$ and $c^\pm(x, u, v)$ in $O((n+d)|A|)$ time where we suppose that $v \in \text{dep}(x, u) - \{u\} \neq \phi$.

Proof: Since p, d and n are integers, given an integral flow f on $N(B, p\chi_W)$ such that $x = \partial f|U$, we can find a flow f_u on the same network such that $x + d\chi_u = \partial f_u|U$ in $O((1+d)|A|)$ time by solving a simple integral maximum flow problem. Given the flow f_u , we can determine whether u is contained in $\text{sat}(x)$ and compute $\text{dep}(x, u)$ in $O(|A|)$ time. Given the flow f_u , we can also compute $c^+(x, u)$ and $c^\pm(x, u, v)$ in $O(n|A|)$ time, since $c^+(x, u), c^\pm(x, u, v) \leq n$. Hence the theorem holds. \square

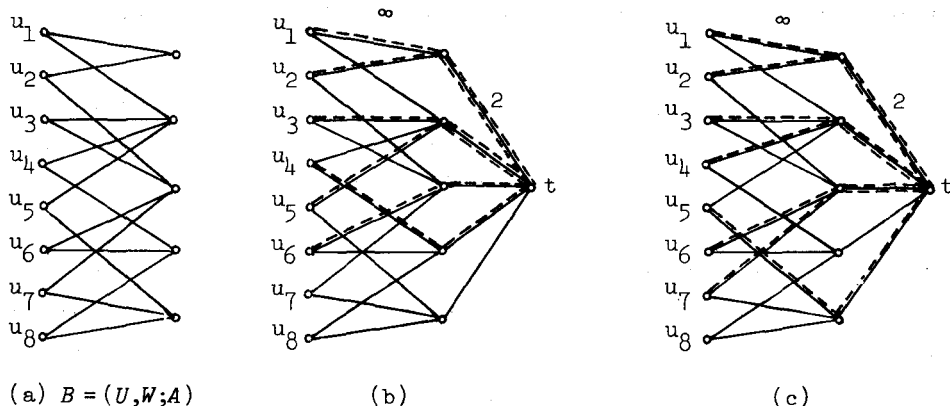


Fig.2.2. A lower-truncated transversal polymatroid $P(2,3)$
 ("-----" denotes the flow of value 1.)

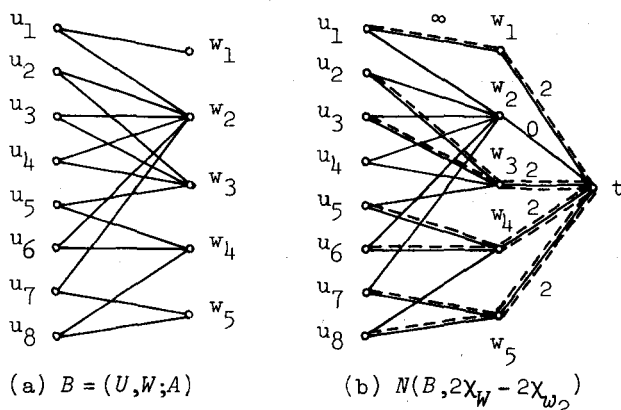


Fig.2.3. Lower-truncated transversal polymatroids $P(2,2)$ and $M(2,2)$
 ("-----" denotes the flow of value 1.)

Theorem 2.7. Suppose the condition (2.2) holds and $d \leq p$. For $w \in W$, let \tilde{U} be a nonempty subset of U such that $(u, w) \in A$ for each $u \in \tilde{U}$.

(i) We can find $S \equiv \tilde{U} \cap \text{sat}(x)$ and $D \equiv \cup \{\text{dep}(x, u) \mid u \in S\}$ in $O((1+d)|A|)$ time.

(ii) For a sequence u_1, u_2, \dots, u_k ($k = |\tilde{U}|$) of all the elements of \tilde{U} , consider vectors x_i ($i = 1, \dots, k$) defined by

$$(2.3) \quad x_0 := x; \quad x_i := x_{i-1} + c^+(x_{i-1}, u_i) \chi_{u_i} \quad (i = 1, \dots, k),$$

(Note that each x_i is an independent vector of $P(p, d; n)$.) Then, the vector x_k can

be found in $O((1+d+x_k(U)-x(U))|A|)$ time.

Proof: Given a flow f on $N(B, p\chi_w)$ such that $x=\partial f|U$, we can find a flow \hat{f} on $N(B, p\chi_w-d\chi_w)$ such that $x=\partial \hat{f}|U$ in $O((1+d)|A|)$ time.

(i) Let X be a set of all vertices v in U such that there is an augmenting path from v to t on $N(B, p\chi_w-d\chi_w)$ with respect to flow \hat{f} . The set X can be found in $O(|A|)$ time. From Theorem 2.5, we have $S=\{u \in \tilde{U} \mid u \notin X \text{ or } x(u)=\alpha\}$, which can be found in $O(|A|)$ time provided that the flow \hat{f} is given. Hence the set S can be found in $O((1+d)|A|)$ time in total.

Let S' be defined by $S'=\{u \mid u \in S, x(u) < n\}$. Let Y be a set of all vertices v in U such that there is an augmenting path from some vertex in S' to v on $N(B, p\chi_w-d\chi_w)$ with respect to flow \hat{f} . Given the flow \hat{f} and the set S' of vertices, we can find Y in $O(|A|)$ time. From Theorem 2.5, we have $D=Y \cup S$, which can be found in $O((1+d)|A|)$ time in total.

(ii) Starting with flow $f_0 \equiv \hat{f}$ on $N(B, p\chi_w-d\chi_w)$, we iterate to augment the flow from u_1 to t until either there comes to be no augmenting path from u_1 to t or the amount of flow going out of u_1 comes up to n . Let f_1 be the flow thus obtained at the end of this procedure. Then, from Theorem 2.5, we have $c^+(x_0, u_1) = \partial f_1(u_1) - \partial f_0(u_1)$ and $x_1 = \partial f_1|U$. Next, as in the case of u_1 , starting with flow f_1 , we iterate to augment the flow from u_2 to t until a flow f_2 such that $x_2 = \partial f_2|U$ is obtained, and so on. Applying this procedure for each $i=1, \dots, k$, we can find the flow f_i from the flow f_{i-1} in $O((1+\partial f_i(U) - \partial f_{i-1}(U))|A|)$ time. Hence we can find the flow f_k (i.e., $x_k = \partial f_k|U$) in $O((1+\partial f_k(U) - \partial \hat{f}(U))|A| + |A||\tilde{U}|)$ time, provided that the flow \hat{f} is given.

In the above-mentioned complexity for finding the flow f_k , the term of $|A||\tilde{U}|$ can be dispensed with in the following way. Suppose that an ordinary labeling procedure [6], [13] is employed for finding the flow f_i from the flow f_{i-1} . Consider the case when, in the labeling procedure, it comes to be found that there is no augmenting path from u_i to t with respect to the current flow (this flow is f_i). Here, if we delete all the current labels and then start to augment the flow from u_{i+1} to t , the term of $|A||\tilde{U}|$ is surely needed. However, we can re-use all the labels as they are and start to augment the flow from u_{i+1} to t with those labels, since it has been found that there is no augmenting path from each currently-labeled vertex to t . If we take this technique, the term of $|A||\tilde{U}|$ is done without. Hence, given the flow \hat{f} , we can find f_k in $O((1+\partial f_k(U) - \partial \hat{f}(U))|A|)$ time.

Thus, we can find x_k in $O((1+d+x_k(U)-x(U))|A|)$ time in total. \square

Note that the fundamental functions of polymatroid $P(p, d; n/m)$ such that p, d, m and n are integers can be computed efficiently by applying the theorems above to polymatroid $P(mp, md; n)$.

3. Greedy-Type Optimization Problems of Lower-Truncated Transversal Polymatroids

As is noted repeatedly, if one can compute the fundamental functions of a polymatroid, one can solve the independent-flow problems concerning the polymatroid. Since in the last section we have shown that the fundamental functions of lower-truncated transversal polymatroids can be computed efficiently by means of network-flow algorithms, we can generally solve various optimization problems of these polymatroids. In this section, we take up greedy-type optimization problems of the polymatroids especially. We show that such problems can be solved in a remarkably efficient manner by means of network-flow algorithms. Here, by greedy-type optimization problems of a polymatroid, we mean problems to which the only polymatroid is essentially related (e.g., the polymatroid intersection problem is not of greedy type). So far as a class of lower-truncated transversal polymatroids is concerned, the union of a matroid in this class with itself is again a matroid in this class, so that problems related to union of a matroid in this class with itself can be regarded as greedy-type ones.

3.1. The greedy algorithm

Let z be a weight vector in \mathbf{R}_+^U . Then, a weight of an independent vector x of a polymatroid P on U is defined to be $\sum_{u \in U} z(u)x(u)$. The greedy algorithm finds a maximum-weight independent vector of polymatroid P . If $z = \chi_U$, this algorithm finds a base of P . The greedy algorithm works as follows.

- (step 1) let σ be an ordering of U such that $z(\sigma(1)) \geq z(\sigma(2)) \geq \dots \geq z(\sigma(|U|))$;
 $x := 0$;
 (step 2) **for** $i=1$ **to** $|U|$ **do** $x := x + c^+(x, \sigma(i))\chi_{\sigma(i)}$;

Then, the vector obtained by this algorithm is a maximum-weight independent vector of P with respect to z [3].

Since we can compute c^+ for $P(p, d; \alpha)$ efficiently, we can apply the greedy algorithm to polymatroid $P(p, d; \alpha)$ in practice. Evaluating the complexity of this algorithm, we have

Theorem 3.1. A maximum-weight independent vector of $P(p, d; n/m)$ with respect to a weight vector, where p, d, m and n are integers ($m, n > 0$), can be found in $O((n + dm)|A||U|)$ time. A maximum-weight independent set of $M(p, d)$ can be found in $O(\min\{1 + d, \sqrt{|U|\}\}|A||U|)$ time.

Proof: It suffices to consider the problem for $P(mp, md; n)$. Though we obtain the theorem from Theorems 2.3 and 2.6 almost immediately, we describe the algorithm in order to make sure. We can find a maximum-weight independent vector of

$P(mp, md; n)$ by starting with zero flow f_0 on $N(B, mp\chi_W)$ and finding flow f_i from flow f_{i-1} for each $i=1, \dots, |U|$ as follows, where σ is the ordering of U as above:

- (i) starting with the flow f_{i-1} , augment the flow from $u_{\sigma(i)}$ to t as much as possible on $N(B, mp\chi_W)$ where the amount of flow going out of $u_{\sigma(i)}$ should not exceed n (let \tilde{f}_{i-1} be the currently-obtained flow);
- (ii) find a flow f_i on $N(B, mp\chi_W)$ such that $\partial f_i|U = \partial \tilde{f}_{i-1}|U - md\chi_{u_{\sigma(i)}}$.

For the flow $f|_{|U|}$ obtained by this algorithm, $\partial f|_{|U|}|U$ is a maximum-weight independent vector of $P(mp, md; n)$, where the validity of this procedure follows from the proposition concerning c^+ in Theorem 2.3.

Evaluating the complexity of each part of this procedure, we see that (i) can be done in $O((n+dm)|A|)$ time for each i (Theorem 2.6), and (ii) can be done in $O(1+|\delta^+u_{\sigma(i)}|)$ time, where $|\delta^+u_{\sigma(i)}|$ denotes the number of arcs going out of $u_{\sigma(i)}$ on the bipartite graph B . Hence, a maximum-weight independent vector can be found in $O((n+dm)|A||U|)$ time in total.

The proposition concerning $M(p, d)$ can be obtained by employing Dinic's algorithm in augmenting the flow from $u_{\sigma(i)}$ to t in the procedure (Theorem 1.1). \square

A base of $P(p, d; \alpha)$ can be found by executing step 2 of the greedy algorithm for any ordering σ of U , from which, in the case of $d \leq p$, we obtain the following theorem.

Theorem 3.2. For a polymatroid $P(p, d; n/m)$ such that p, d, m and n are integers ($m, n > 0$) and $d \leq p$, its base can be found in $O((p+d)m|A||W|)$ time. For a matroid $M(p, d)$ such that $d \leq p$, its base can be found in $O(\min\{(p+d), \sqrt{|U|}\}|A||W|)$ time.

Proof: A base of $P(mp, md; n)$ can be found by the following algorithm:

- (step 1) $S := U$; let f be zero flow on $N(B, mp\chi_W)$;
- (step 2) (i) while $S \neq \emptyset$, take $w \in \Gamma(S)$ and execute (ii);
- (ii) $\tilde{U} := \{u \mid u \in S, (u, w) \in A\}$; $S := S - \tilde{U}$;
- making use of the flow f on $N(B, mp\chi_W)$, execute the procedure of Theorem 2.7(ii) (cf. its proof) for $x = \partial f|U$ and \tilde{U} ; update f by the currently-obtained flow on $N(B, mp\chi_W - md\chi_w)$ (the updated f is also the flow on $N(B, mp\chi_W)$);

In the above algorithm, for the flow f obtained at the end of it, $x^* \equiv \partial f|U$ is a base of $P(mp, md; n)$. Since (i), (ii) of step 2 are executed at most $|W|$ times, the complexity of this algorithm is $O((x^*(U) + md|W|)|A|)$ (Theorem 2.7(ii)). Since $x^*(U) \leq mp|W|$, this complexity is $O((p+d)m|A||W|)$.

The proposition concerning $M(p, d)$ can be obtained by employing Dinic's algorithm. Here it should be remarked that, by Dinic's algorithm, we cannot increment

the flow from u_i to t for each $i=1, \dots, k$ sequentially in the procedure of Theorem 2.7(ii). However, for the purpose of finding a base merely, we have only to be able to find an independent vector x_k of $P(mp, md; n)$ such that $x_k|(U-\tilde{U})=x|(U-\tilde{U})$ and $\tilde{U} \subseteq \text{sat}(x_k)$ in the case of Theorem 2.7(ii). For this purpose, Dinic's algorithm suffices. \square

3.2. The principal partition

The theory of the principal partition has been developed on fine structures of polymatroid intersections, and has been applied to many kinds of systems problems in engineering science [15], [17], [21], [32]. Here, we consider the principal partition of a polymatroid with respect to a modular function, which was considered by Tomizawa [31] and was refined in terms of polymatroids by Fujishige [8].

We first review the decomposition theory of a Boolean sublattice briefly from a rather algorithmic point of view. Given a family L of subsets of U which forms a Boolean sublattice (i.e., $X, Y \in L \rightarrow X \cup Y, X \cap Y \in L$), we can obtain a partition of U with a partial order as follows. For L , let $U^+(L)$ be the complement of the maximum of L , $U^-(L)$ be the minimum of L and $U^*(L) = U - U^+(L) - U^-(L)$. For each $u \in U^*(L) \cup U^-(L)$, define $D(L, u)$ by

$$(3.1) \quad D(L, u) = \bigcap \{X \mid u \in X \in L\}.$$

Note that $D(L, u) \in L$, and, if $v \in D(L, u)$, then $D(L, v) \subseteq D(L, u)$. Consider a directed graph $G^* = (U^*(L), A^*)$ with vertex set $U^*(L)$ and arc set A^* defined by

$$(3.2) \quad A^* = \{(u, v) \mid u, v \in U^*(L), v \in D(L, u)\}.$$

Let $U_i^*(L)$ ($i=1, \dots, k$) be a partition of $U^*(L)$ obtained from the decomposition of G^* into strongly connected components with which a partial order is naturally associated. Then, we obtain a partition of U into blocks $U^+(L), U_1^*(L), U_2^*(L), \dots, U_k^*(L), U^-(L)$ with the partial order.

Let us consider the principal partition of $P(p, d) = (U, \rho_{p,d})$ with respect to $\alpha \chi_U$ ($\alpha > 0$), where we suppose $\rho_{p,d}(\{u\}) > 0$ for $u \in U$ for simplicity. Let $L(\alpha)$ be defined by

$$(3.3) \quad L(\alpha) = \{S \mid S \subseteq U, \rho_{p,d}(S) + \alpha|U - S| = \rho^*\},$$

$$\text{where } \rho^* = \min\{\rho_{p,d}(X) + \alpha|U - X| \mid X \subseteq U\}.$$

As is readily deduced from the submodularity of $\rho_{p,d}$, $L(\alpha)$ forms a Boolean sublattice. We call the partition of U with the partial order associated with $L(\alpha)$ the *principal partition of $P(p, d)$ with respect to $\alpha \chi_U$* . Let x_α be a base of $P(p, d; \alpha)$. Then, we have

$$(3.4) \quad U^+(L(\alpha)) = \{u \mid u \in U, u \notin \text{sat}(x_\alpha) \text{ in } P(p, d)\}$$

$$(3.5) \quad U^-(L(\alpha)) = \cup \{\text{dep}(x_\alpha, u) \text{ in } P(p, d) \mid u \in U, x_\alpha(u) < \alpha\}$$

$$(3.6) \quad D(L(\alpha), u) = \text{dep}(x_\alpha, u) \cup U^-(L(\alpha)) \quad (u \in U^*(L(\alpha))),$$

where in (3.6) the dependence function is for $P(p, d)$. Since we can find the base x_α , and compute sat and dep efficiently as is shown in the section 2, we can find this principal partition in practice. Evaluating the complexity for finding the partition, we obtain

Theorem 3.3. The principal partition of $P(p, d)$ with respect to $\alpha \chi_U$, where p and d are integers and $\alpha = n/m$ for positive integers m and n , can be found in $O((n + dm)|A||U|)$ time. In the case of $d \leq p$, this principal partition can be found in $O((p + d)m|A||W|)$ time.

Proof: The proposition concerning general p and d are immediately obtained from Theorems 2.6 and 3.1.

In the case of $d \leq p$, we can find an integer-valued base x of $P(mp, md; n)$ in $O((p + d)m|A||W|)$ time. For this base x , we can find $U - U^+ = \text{sat}(x)$ in $P(mp, md)$ and $U^- = \cup \{\text{dep}(x, u) \text{ in } P(mp, md) \mid u \in U, x(u) < n\}$ in $O((1 + md)|A||W|)$ time (Theorem 2.7(i)). For $U^* = U - U^- - U^+$, we have $|U^*| \leq mp|W|/n$, and therefore we can list $\text{dep}(x, u)$ in $P(mp, md)$ for all $u \in U^*$ in $O((1 + md)|A||W| + |A||U^*|) = O((p + d)m|A||W|)$ time. Thus, in the case of $d \leq p$, the principal partition can be found in $O((p + d)m|A||W|)$ time. \square

By varying α as a parameter, we can obtain a finer partition of U as follows. If $0 \leq \alpha < \alpha'$, then we have $U^-(L(\alpha)) \subseteq U^-(L(\alpha'))$. Hence, there exist at most $|U|$ numbers α^* such that $U^-(L(\alpha)) \subset U^-(L(\alpha^*))$ for any α with $0 \leq \alpha < \alpha^*$. We call those numbers the critical values. Let the critical values be given by α_j ($j = 1, \dots, l$) so that $0 = \alpha_0 < \alpha_1 < \dots < \alpha_l < \alpha_{l+1} = \infty$. Then, we have

$$(3.7) \quad \phi = U^-(L(\alpha_1)) \subset \dots \subset U^-(L(\alpha_l)) \subset U^-(L(\alpha_{l+1})) = U$$

$$(3.8) \quad U^+(L(\alpha_j)) \cup U^-(L(\alpha_{j+1})) = U, \quad U^+(L(\alpha_j)) \cap U^-(L(\alpha_{j+1})) = \phi$$

for $j = 0, 1, \dots, l$. Thus, we can obtain a fairly fine partition of U into blocks $U^*_i(L(\alpha_j))$ ($i = 1, \dots, k; j = 1, \dots, l$) with a partial order, which we will refer to as the *refined principal partition of $P(p, d)$ with respect to χ_U* . It is known that, if p and d are integers, this refined principal partition can be found, with the technique of bisection, by solving $O(|U|)$ times the problem of finding the principal partition of $P(p, d)$ with respect to $\alpha \chi_U$ where α is such that $\alpha = n/m$ for integers m and n with $1 \leq m \leq |U|$ and $1 \leq n \leq \rho_{p,d}(U)$. Then, from Theorems 2.6 and 3.1, we can find the refined principal partition in $O(|A||U|^2(|U| + |W|))$ time, where we consider p and d are

constants

Remark 4.1. Theoretically, we can easily reduce the above time bound to $O(|A||U|(|U|+|W|)\log(|U|+|W|))$ by some technique, which may be called balanced bisection, in the case

(3.9) concerning $P(p, d)$ on U of the bipartite graph $B=(U, W; A)$, the contraction of $P(p, d)$ to $\tilde{U}(\subseteq U)$ is a polymatroid $P(p, d)$ on \tilde{U} of the bipartite graph $\tilde{B}=(\tilde{U}, \tilde{W}; \tilde{A})$ which can be obtained from B by shortcircuiting all the arcs incident to vertices in $U-\tilde{U}$.

See [11] for detailed discussions. \square

Next, consider a bipartite graph $B=(U, W; A)$ such that an integer-valued base x of $P(p, d)$ on U satisfies $x(U)=\gamma_{p,d}(U)$ where p and d are positive integers ($d \neq 0$). For the base x of $P(p, d)$ and $u^* \in U$, consider $\tilde{L}(x, u^*)$ defined by

$$(3.10) \quad \tilde{L}(x, u^*) = \{X \mid u^* \in X \subseteq U, x(X) = \gamma_{p,d}(X)\} \cup \{\phi\}.$$

Then, $\tilde{L}(x, u^*)$ forms a Boolean sublattice. We call the partition with the partial order obtained from $\tilde{L}(x, u^*)$ the *principal partition associated with $\tilde{L}(x, u^*)$* . In this case, we have

$$(3.11) \quad U^+(\tilde{L}(x, u^*)) = U^-(\tilde{L}(x, u^*)) = \phi$$

$$(3.12) \quad D(\tilde{L}(x, u^*), u) = \{v \mid v \in U, \text{ there is an augmenting path from } u \text{ to } v$$

$$\text{on } N(B, p\chi_W) \text{ with respect to flow } f \text{ such that } x + d\chi_{u^*} = \partial f|U\} \quad (u \in U).$$

Note that u^* is surely contained in the set \tilde{D} given in the right-hand side of (3.12) for each $u \in U$ (if $u^* \notin \tilde{D}$, then we have $x(\tilde{D}) = p\gamma(\tilde{D})$, i.e., $x(\tilde{D}) > \gamma_{p,d}(\tilde{D})$, which contradicts the fact that x is an independent vector of $P(p, d)$). From (3.12), the principal partition associated with $\tilde{L}(x, u^*)$ can be obtained by decomposing a bipartite graph $\tilde{B}=(U, W; \tilde{A})$ into strongly connected components where

$$(3.13) \quad \tilde{A} = A \cup \{a' \mid a \in A, f(a) > 0 \text{ for a flow } f \text{ on } N(B, p\chi_W) \text{ such that}$$

$$x + d\chi_{u^*} = \partial f|U, \quad a': \text{reorientation of } a\}.$$

Given an integral flow f' on $N(B, p\chi_W)$ such that $x = \partial f'|U$, we can find the principal partition associated with $\tilde{L}(x, u^*)$ in $O((1+d)|A|)$ time.

3.3. Covering and packing

By the *covering number* of a matroid M on a set E , we mean the minimum number of independent sets of M whose union is E , and, by the *packing number* of M , we mean the maximum number of disjoint bases of M . In this section, we consider a problem of computing the covering and packing numbers of a (p, d) -

transversal matroid $M(p, d)$ on U of a bipartite graph $B=(U, W; A)$.

Let us consider the problem of finding the covering number of $M(p, d)$ on U . The covering number of $M(p, d)$ is at most $|U|$, and is equal to the minimum integer k such that U is independent in $M(kp, kd)$. In the case of $d=0$, from Theorems 1.1 and 2.1, we can determine whether U is independent in $M(kp, 0)$ in $O(\sqrt{|U|}|A|)$ time, and therefore we can find the covering number in $O(\sqrt{|U|}|A| \log|U|)$ time by employing a binary search technique. Similarly, in the case of $d>0$, from Theorem 3.1, we can find the covering number in $O(|A||U|^{3/2} \log|U|)$ time.

In the case of $0 < d \leq p$, we can find the covering number of $M(p, d)$ more quickly by the following algorithm.

- (step 1) find the covering number k_0 of $M(p, 0)$, and a flow f on $N(B, k_0 p \chi_W)$ such that $\chi_U = \partial f|U$; $k := k_0$; $Y := W$; go to step 2;
- (step 2) if $Y = \emptyset$ then halt $\{k$ is the covering number of $M(p, d)\}$; otherwise take $w \in Y$ and $Y := Y - \{w\}$; go to step 3;
- (step 3) (i) if $f(w, t) \leq kp - kd$, then return to step 2; otherwise $f' := f$ and go to (ii);
 - (ii) find a maximum flow f^* in $N(B, kp \chi_W - kd \chi_w)$ with respect to χ_U by making use of the flow f' ; go to (iii)
 - (iii) if $\partial f^*|U = \chi_U$, then return to step 2; otherwise, $k := k + 1$; $f' := f^*$ and return to (ii);

The validity of the above algorithm follows from Theorem 2.4. Let us estimate its time complexity. Step 1 can be executed in $O(\sqrt{|U|}|A| \log|U|)$ time. Each execution of step 3 takes $O((f(w, t) + k_i + 1)|A|)$ time where k_i is the increment of k during its step 3. Since steps 2 and 3 are executed for each $w \in W$, and the total increment of k through the computation is at most $|U|$, and $\sum_{w \in W} f(w, t) = |U|$, the total time complexity of the above algorithm is $O(|A|(|U| + |W|))$.

Thus, we obtain the following theorem.

Theorem 3.4. The covering number of a (p, d) -transversal matroid $M(p, d)$ can be found

- (i) in $O(\sqrt{|U|}|A| \log|U|)$ time if $d=0$,
- (ii) in $O(|A|(|U| + |W|))$ time if $0 < d \leq p$ and
- (iii) in $O(|A||U|^{3/2} \log|U|)$ time if $d > p$. \square

Let us consider the problem of finding the packing number of $M(p, d)$ on U . Let r be the rank of $M(p, d)$. The packing number of $M(p, d)$ is equal to the maximum integer k such that the rank of $M(kp, kd)$ is kr . Then, as in the case of the problem of finding the covering number, we have the following theorem.

Theorem 3.5. The packing number of a (p, d) -transversal matroid $M(p, d)$ whose

rank is τ can be found

- (i) in $O(\sqrt{|U|}|A| \log|U|)$ time if $d=0$,
- (ii) in $O(\min\{|U|/\tau, \sqrt{|U|}\}|A||W| \log|U|)$ time if $0 < d \leq p$ and
- (iii) in $O(\min\{|U|/\tau, \sqrt{|U|}\}|A||U| \log|U|)$ time if $d > p$.

Proof: Immediate from Theorems 1.1, 3.1 and 3.2 and the fact that the packing number of $M(p, d)$ is at most $|U|/\tau$. \square

4. Cycle Matroids of Graphs and Matroids in Plane Skeletal Structures

In this section, we take up cycle matroids of graphs and matroids in plane skeletal structures as typical and useful examples of lower-truncated transversal polymatroids. We apply the arguments discussed in the above sections to problems concerning these two useful matroids, by which we obtain various new results, and demonstrate the practical usefulness of the network-flow approaches proposed in this paper.

4.1. Cycle matroids of graphs

For an undirected graph $G=(V, E)$ with vertex set V and edge set E , we consider a directed bipartite graph $\hat{B}(G)=(E, V; \hat{A})$ with left vertex set E , right vertex set V and arc set $\hat{A} \subseteq E \times V$ defined by

$$(4.1) \quad \hat{A} = \{(e, v_i), (e, v_j) \mid e = \{v_i, v_j\} \in E \text{ in } G\}.$$

Let M_G be a cycle matroid on E of a graph G , and M_G^k be the union of M_G with itself k times. For $X \subseteq E$, let $V(X)$ be the set of all vertices incident to some edge in X . Note that, for $X \subseteq E$, $V(X)$ coincides with $\Gamma(X)$ on the bipartite graph $\hat{B}(G)$. The following is the famous theorem due to Nash-Williams [23].

Theorem 4.1. For $S \subseteq E$, S is independent in M_G^k iff $k|V(X)| - k \geq |X|$ for any X with $\emptyset \neq X \subseteq S$. \square

Thus, a (k, k) -transversal matroid $M(k, k)$ on E of $\hat{B}(G)$ is the matroid M_G^k , and therefore we can apply the algorithms presented in the above sections to problems concerning M_G^k . (Note that the degree of each vertex in E on $\hat{B}(G)$ is $2 \geq 1 = k/k$.)

Let us show an example. Consider a graph $G=(V, E)$ ($E = \{u_1, \dots, u_8\}$, $V = \{w_1, \dots, w_5\}$) depicted in Fig.4.1. For this graph G , the bipartite graph $\hat{B}(G)$ is that in Fig.2.3(a). Hence, from the discussions concerning the example of Fig.2.3, we see that $T = E - \{u_4\}$ is a base of M_G^2 , and, for $P(2, 2)$ on E of $\hat{B}(G)$, $u_1 \notin \text{sat}(\chi_T)$, $\text{dep}(\chi_T, u_4) = \{u_2, u_3, u_4\}$, etc.

To return, let us consider greedy-type optimization problems of M_G^k . First, consider the problem of finding a base. From Theorem 3.2, we can find a base of M_G^k in $O(k|E||V|)$ time in general (note that $|\hat{A}| = O(|E|)$). We can easily reduce the

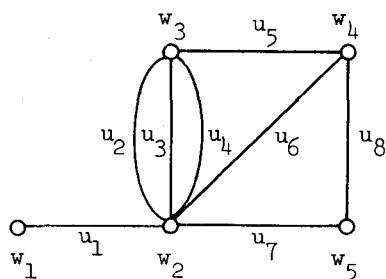


Fig.4.1. A graph G

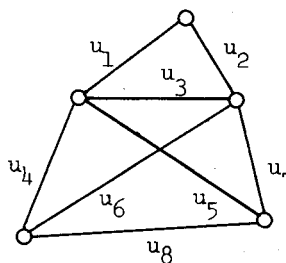


Fig.4.2. A graph G

above-mentioned complexity to $O(k^2|V|^2)$ by utilizing the following two facts:

- (i) The degree of each vertex of E on $\hat{B}(G)$ is 2.
- (ii) The value of any flow considered in the course of the algorithm is at most $k|V|$.

Furthermore, from the algorithmic point of view, in finding a base by solving an integral maximum flow problem with the labeling procedure, we employ the following two techniques:

- (i) When we remove labels on the network after the flow augmentation, we do not manipulate all the vertices but only a set of labeled vertices, which can be done by having a list of labeled vertices at all times.
- (ii) At any moment of the algorithm, concerning the current flow f on the network, we record a set of arcs $a=(u,v) \in \hat{A}$ in $\hat{B}(G)$ such that $f(a) > 0$ for each $v \in V$, and, in the case of searching the vertex v , we scan the arc (v,t) and arcs in that set for v .

Then, one can easily show that the complexity of the algorithm modified as above is $O(k^2|V|^2)$. In a similar way, we can find a maximum-weight independent set of M_G^k in $O(k^2|E||V|)$ time (Theorem 3.1).

Next, consider the problem of finding the covering and packing numbers of M_G . In [30], Tarjan announced that both numbers can be found in $O(|E|^2)$ time. Picard and Queyranne [25] and Ichimori [10] presented $O(|V|^4)$ and $O(\min\{|V|^2, |E| \log |V|\} |V|^2)$ algorithms, respectively, for computing the covering number of M_G . Applying the general algorithms in section 3.3, we can find the covering number in $O(|E|^2)$ time, and the packing number in $O(|E|^2 \log |V|)$ time.

Consider the problem of finding the principal partition of a graph G . The principal partition in the sense of Kishi and Kajitani [18] and Iri [14], which is the principal partition of $P(2,2)$ on E of $\hat{B}(G)$ with respect to χ_V , can be found in $O(|E||V|)$ time from Theorem 3.3 in general. This complexity can be reduced to $O(|V|^2)$ as follows. First, from the above discussion, we can find an integer-valued base x^* of $P(2,2;1)$ in $O(|V|^2)$ time. We can then find $U^- \equiv \cup \{\text{dep}(x^*, u) \text{ in } P(2,2) \mid u \in E, x^*(u) < 1\}$ in

$O(|E|+|V|^2)=O(|V|^2)$ time (Theorem 2.7(i)). Since $|E-U^-|$ is $O(|V|)$, we can easily find the whole principal partition in $O(|V|^2)$ time. The principal partition of a graph G in the sense of Tomizawa [31], which is the refined principal partition of $P(1,1)$ on E of $\hat{B}(G)$ with respect to χ_V , can be found in $O(|E|^3 \log |V|)$ time, since, for this $P(1,1)$, the condition (3.9) holds (see Remark 4.1).

4.2. Matroids in plane skeletal structures

Plane skeletal structures are two-dimensional frameworks composed of rigid rods mutually connected at terminal vertices with rotatable joints. A plane skeletal structure S can be considered to consist of a graph $G=(V,E)$ and a map from V to \mathbf{R}^2 by regarding rods as edges and joints as vertices. If the vertices of structure S are in general position, the rigidity depends upon the structure of the graph G only. Then, G (or, edge set E) is called generic independent if S has no redundant rod, and G is called stiff if S is rigid and has no redundant rod. The collection of generic independent subsets of E forms a collection of independent sets of some matroid, which we denote by M_L . The following theorem on rigidity was first given by Laman [19] (see also [20], [26]).

Theorem 4.2. A graph $G=(V,E)$ is generic independent iff $2|V(X)|-3 \geq |X|$ for any X with $\phi \neq X \subseteq E$. G is generic independent and stiff iff $2|V|-3=|E|$ and $2|V(X)|-3 \geq |X|$ for any X with $\phi \neq X \subseteq E$. \square

Thus, $M(2,3)=(P(2,3))$ on E of $\hat{B}(G)$ is the matroid M_L .

Let us show an example. Consider a graph $G=(V,E)$ ($E=\{u_1, \dots, u_8\}$) in Fig.4.2. For this graph G , the bipartite graph $\hat{B}(G)$ is that depicted in Fig.2.2(a). From the discussions concerning Fig.2.2, we see that G is not generic independent (i.e., G contains a redundant edge), and that $L=\{u_1, \dots, u_7\}$ is a maximal generic independent and stiff subgraphs of G . Since $\text{dep}(\chi_L, u_8)=\{u_3, \dots, u_8\}$ for $P(2,3)$ on E of $\hat{B}(G)$, we see that $(L \cup \{u_8\}) - \{u\}$ is generic independent in the case of $u \in \{u_3, \dots, u_8\}$, but is not generic independent in the case of $u \in \{u_1, u_2\}$.

Applying the algorithms developed in this paper, we can determine whether a graph G is generic independent, and find a base of M_L in $O(|E||V|)$ time. Note that it has been only known that the independence testing problem can be solved in $O(|E|^{3.5})$ time [26] (see also [20]).

The principal partition of M_L tells us how the redundancy is distributed on G . However, if a graph G is generic independent and stiff, the principal partition gives no meaningful partition of E , that is, it gives a partition of E into respective edges with no partial order among them, since a structure consisting of an edge is generic independent and stiff. Nakamura and Sugihara [22] gave how to obtain a meaningful partition of E on such a graph G . For $e \in E$, define $D(e)$ by

$$(4.2) \quad D(e) = \bigcap \{X \mid e \in X \subseteq E, 2|V(X)| - 3 = |X| \geq 2\},$$

and consider a directed graph $G^* = (E, A^*)$ with vertex set E and arc set A^* defined by

$$(4.3) \quad A^* = \{(e_i, e_j) \mid e_i, e_j \in E, e_j \in D(e_i)\}.$$

Then, the partition of E with a partial order obtained by decomposing G^* into strongly connected components completely describes the internal structure of generic independent and stiff subgraph $G' = (V', E')$ with $|E'| \geq 2$ of G .

In obtaining the partition due to Nakamura and Sugihara, the main problem is to compute $D(e)$ for $e \in E$. By using the principal partition associated with $\tilde{L}(\chi_E, e)$ for the bipartite graph $\hat{B}(G) = (E, V; \hat{A})$ and $e \in E$, where $\tilde{L}(\chi_E, e) = \{X \mid e \in X \subseteq E, 2|V(X)| - 3 = |X|\}$, we can find $D(e)$ for $e \in E$ in $O(|E|)$ time. Hence, we can obtain this partition in $O(|E|^2) = O(|V|^2)$ time.

Concluding Remarks

We have given the theorems for computing the fundamental functions of lower-truncated transversal polymatroids, based on which we can solve various combinatorial optimization problems for those polymatroids efficiently from a unifying point of view by means of network-flow algorithms. We have also discussed in detail three greedy-type problems among those optimization problems, and have shown that the problems for cycle matroids of graphs and matroids in plane skeletal structures can be solved very efficiently by means of network-flow algorithms proposed in this paper.

From the standpoint of applications, though we have touched upon, as examples, cycle matroids of graphs and matroids in plane skeletal structures only, the lower-truncated transversal polymatroid is strongly connected with the discrete system with internal degrees of freedom, which has been investigated by Sugihara [29] (e.g., line drawings of polyhedra [27]), and therefore the techniques of network-flow type developed in this paper are widely applicable to problems of analyzing such systems.

From the viewpoint of combinatorial optimization, in this paper, we have taken up only the greedy-type optimization problem among various ones, and we have shown that those problems can be solved very efficiently (in many cases, more quickly compared with known methods). The method proposed in this paper can be applied to general optimization problems as a tool for computing the fundamental functions in constructing so-called auxiliary networks in the independent-flow problems.

From the rather theoretical point of view, the following result has been obtained concerning the lower-truncation of β_0 -functions. Consider two β_0 -functions μ_1 and μ_2 on a finite set E . Let λ_1 and λ_2 be lower-truncations of μ_1 and μ_2 , respectively (cf. (1.2)). Although we have $\mathbf{P}(\lambda_i) = \mathbf{P}(\mu_i)$ ($i=1,2$), and obviously $\mathbf{P}(\lambda_1 + \lambda_2) \subseteq$

$\mathbf{P}(\mu_1 + \mu_2)$, it is not necessarily true that $\mathbf{P}(\lambda_1 + \lambda_2) = \mathbf{P}(\mu_1 + \mu_2)$ holds. In [12], a necessary and sufficient condition so that $\mathbf{P}(\lambda_1 + \lambda_2) = \mathbf{P}(\mu_1 + \mu_2)$ holds is given. Though this type of approach is rather hopeless from the practical point of view, we can apply this arguments to the sum of rank functions of two graphs. Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be graphs with the same edge set E . Let γ^i be $\gamma_{1,1}$ on E of bipartite graph $\tilde{B}(G_i)$ and ρ^i the lower-truncation of γ^i ($i=1,2$). Note that ρ^i is the rank function of graph G_i . Then, we see that $\gamma^1 + \gamma^2$ corresponds to $\gamma_{1,2}$ on E of bipartite graph $\tilde{B} = (E, V_1 + V_2; \tilde{A})$ defined by

$$\tilde{A} = \{(e, u_1), (e, v_1), (e, u_2), (e, v_2) \mid \{u_1, v_1\} = e \in E \text{ in } G_1, \{u_2, v_2\} = e \in E \text{ in } G_2\}.$$

Thus, $\gamma^1 + \gamma^2$ determines a lower-truncated transversal polymatroid $P(1,2)$ on E of \tilde{B} , which can be algorithmically treated in an efficient manner as shown in the paper. However, as stated above, $\mathbf{P}(\gamma^1 + \gamma^2)$ does not necessarily correspond to $\mathbf{P}(\rho^1 + \rho^2)$; we only have $\mathbf{P}(\rho^1 + \rho^2) \subseteq \mathbf{P}(\gamma^1 + \gamma^2)$ and we can not apply the techniques of network-flow type to the problem concerning the sum of rank functions of two graphs G_1 and G_2 straightforwardly. For conditions so that $\mathbf{P}(\rho^1 + \rho^2) = \mathbf{P}(\gamma^1 + \gamma^2)$ holds, see [12].

Finally, in this paper, we consider the theorems for bipartite graphs only. It is straightforward to generalize the theorems to those for ordinary graphs. Applying the generalized theorems for graphs, we can efficiently solve the problem related to orientations of graphs.

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