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LIFO ALLOCATION PROBLEM FOR PERISHABLE COMMODITIES

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Abstract This paper discusses an allocation problem of perishable commodities based on an LIFO issuing policy. Under the rotation allocation policy, commodities are distributed from a regional center to n locations in the region. Costs are charged at each location for every unit short, outdated or transported. The purposes of this paper are to clarify the existence of the optimal single period allocation policy and to propose an algorithm of obtaining its solution. Furthermore, the influences of shortage, transportation and outdating costs upon the optimal policy are investigated.

1. Introduction

We consider perishable commodities [1] [2] [3] [4] which become obsolete and cannot be used after a certain period of time. Optimal allocation policies for perishable commodities were analyzed first by Prastacos [5] [6] [7]. These models were discussed under charging cost only for shortage and outdating. But, if we discuss the allocation problem, transportation cost should be also an important factor.

In this paper, we discuss a single period allocation problem of perishable commodities based on an LIFO issuing and a rotation allocation policies considering shortage, outdating and transportation costs. Blood, photographic film and fresh commodities are typical examples of our model. Section 2 gives some assumptions and notations used throughout this paper.
Section 3 constructs the problem formulation and clarifies the existence of the optimal allocation policy.
Section 4 presents an algorithm of obtaining its optimal solution. In addition, we give an example in section 5 in order to examine the influences of

shortage, outdating and transportation costs upon the optimal allocation policy.

2. Preliminaries

A periodic review inventory model is considered for one planning horizon and single item. That is, ordering takes place at the start of a period and costs are incurred during a period. The period length is arbitrary but fixed. And the following assumptions are permitted to construct the model.

(1) Maximum lifetime of the perishable commodities is fixed and equal to L periods.

(2) Inventory is depleted by demand at the start of each period according to a LIFO issuing policy.

(3) The stock remaining at the end of each period ages one period monotonously and is returned from each location to regional center. (rotation policy)

(4) If the commodities have not been depleted by demand until the period it reaches age L, then it outdates and must be discarded.

(5) Costs are charged at each location for every unit short, outdated or transported.

(6) Demands at each location are independent random variables with known distribution $F_k(\cdot)$ and density $f_k(\cdot)$.

(7) The following notations are used throughout this paper.

s_k: shortage cost per unit of demand unfilled at location k

w_k: outdating cost per unit perished at location k

u_k: transportation cost per unit shipped from regional center to location k or returned from location k to regional center

N_k: allocated quantity not subject to outdating at location k (age 0 to age L-2)

B_k: allocated quantity subject to outdating at location k (age L-1)

$\mathbf{N} = (N_1, N_2, \ldots, N_n)$

$\mathbf{B} = (B_1, B_2, \ldots, B_n)$

$$N = \sum_{k=1}^{n} N_k$$

$$B = \sum_{k=1}^{n} B_k$$

3. LIFO Allocation Model

When fresher units N and older units B are prepared, the total expected cost $C(N,B)$ is given by

$$(3.1)\qquad C(N,B) = \sum_{k=1}^{n} [s_k \int_{N_k+B_k}^{\infty} (x-N_k-B_k)dF_k(x) + w_k \int_{N_k}^{N_k+B_k} (N_k+B_k-x)dF_k(x)$$

$$+ w_k B_k F_k(N_k) + (N_k+B_k)u_k + u_k \int_0^{N_k} (N_k-x)dF_k(x)]$$

$$(3.2)\qquad \frac{\partial C(N,B)}{\partial B_k} = (s_k+w_k)F_k(N_k+B_k)-s_k+u_k .$$

$$(3.3)\qquad \frac{\partial^2 C(N,B)}{\partial B_k^2} = (s_k+w_k)f_k(N_k+B_k) \geqq 0$$

$$\lim_{B_k\to\infty} \frac{\partial C(N,B)}{\partial B_k} = w_k+u_k > 0 .$$

$$(3.4)\qquad \frac{\partial C(N,B)}{\partial N_k} = (s_k+w_k)F_k(N_k+B_k)-s_k+u_k+ (u_k-w_k)F_k(N_k) .$$

$$(3.5)\qquad \frac{\partial^2 C(N,B)}{\partial N_k^2} = (s_k+w_k)f_k(N_k+B_k) + (u_k-w_k)f_k(N_k) \geqq 0$$

where, $u_k > w_k$.

$$\lim_{N_k\to\infty} \frac{\partial C(N,B)}{\partial N_k} = 2u_k > 0$$

$$(3.6)\qquad \begin{vmatrix} \frac{\partial^2 C(N,B)}{\partial N_k^2} & \frac{\partial^2 C(N,B)}{\partial N_k \partial B_k} \\ \frac{\partial^2 C(N,B)}{\partial B_k \partial N_k} & \frac{\partial^2 C(N,B)}{\partial B_k^2} \end{vmatrix} = (u_k - w_k)(s_k+w_k)f_k(N_k)\, f_k(N_k+B_k) \geqq 0$$

Together the above inequalities (3.3), (3.5), (3.6) and the property of convex combination, the following proposition is obtained.

Proposition 1.

Total expected cost function $C(\mathrm{N},\mathrm{B})$ is convex with respect to N_k and B_k.

4. Algorithm for Optimal Allocation Policy

Introducing Lagrange multipliers λ and μ into equation (3.1), the following Lagrange function $\overline{C(\mathrm{N},\mathrm{B})}$ is given by

$$(4.1) \qquad \overline{C(\mathrm{N},\mathrm{B})} = C(N,B) - \lambda\left(\sum_{k=1}^{n} B_k - B\right) - \mu\left(\sum_{k=1}^{n} N_k - N\right) .$$

$$(4.2) \qquad \frac{\partial \overline{C(\mathrm{N},\mathrm{B})}}{\partial B_k} = (s_k+w_k)F_k(N_k+B_k) - s_k + u_k - \lambda = 0 .$$

$$(4.3) \qquad \frac{\partial C(\mathrm{N},\mathrm{B})}{\partial N_k} = (s_k+w_k)F_k(N_k+B_k) - s_k + u_k - \mu + (u_k-w_k)F_k(N_k) = 0 .$$

An algorithm is developed to solve the following Kuhn-Tucker conditions with respect to N_k and B_k, $k=1, \ldots, n$.

(i) $\sum_{k=1}^{n} B_k = B, \quad \sum_{k=1}^{n} N_k = N$

(ii) $\frac{\partial \overline{C(\mathrm{N},\mathrm{B})}}{\partial B_k} \geqq 0, \quad \frac{\partial \overline{C(\mathrm{N},\mathrm{B})}}{\partial N_k} \geqq 0$

(iii) $\frac{\partial \overline{C(\mathrm{N},\mathrm{B})}}{\partial B_k} \cdot B_k = 0, \quad \frac{\partial \overline{C(\mathrm{N},\mathrm{B})}}{\partial N_k} \cdot N_k = 0$

Algorithm

First, n locations are arranged and numbered so as to be the following order,

$$u_1-s_1 \leqq u_2 - s_2 \leqq \ \ldots, \ \leqq u_n - s_n \ ,$$

$$I_i = (u_i-s_i \ , \ u_i-s_i, \ u_{i+1}-s_{i+1}), \ i=1, \ \ldots, \ n$$

$$I_0 \triangleq (-\infty, \ u_1-s_1), \ u_{n+1} - s_{n+1} \triangleq + \infty$$

(i) When B=0, go to (ii) and when $B \neq 0$, go to (v).

(ii) Set $i \leftarrow 0$. Then, n locations are arranged and numbered so as to be the above order,

(iii) Set $\mu \varepsilon I_i$. And the followings are obtained.

- $N_l=0 \ : \ l=i+1, \ \ldots, \ n.$
- $N_l=F_l^{-1}(\frac{s_l-u_l+\mu}{s_l+u_l}) \ : \ l=1, \ \ldots, \ i.$

If $\Sigma_l F_l^{-1} \ (\frac{s_l-u_l+u_i-s_i}{s_l+u_l}) \leqq N$ and $\Sigma_l F_l^{-1} \ (\frac{s_l-u_l+u_{i+1}-s_{i+1}}{s_l+u_l}) > N$, there exists μ_0 within I_i so that $\Sigma_{l=1}^{n} N_l=N$ and terminates the algorithm with $N_l(l=1, \ \ldots, n)$. Otherwise, go to (iv).

(iv) Set $i \leftarrow i+1$ and go to (iii).

(v) Set $i \leftarrow 1$.

(vi) Set $\lambda \varepsilon I_i$, $\mu \varepsilon I_i$. Then,

$B_i = 0 \ ; \ l=i+1, \ \ldots, \ n$ is obtained.

From the equation (4.2),

$N_p+B_p=F_p^{-1}(\frac{s_p-u_p+\lambda_0}{s_p+w_p})$; $p=1,2, \ldots, i$ is obtained.

Introducing $B_p=0$ into the equation (4.3),

$N_p=F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})$; $p=1,2, \ldots, i$ is gained.

Two kinds of sets, J and K, are defined:

$J \triangleq \{\text{location } p \mid p=1, \ldots, i, B_p \neq 0\}$

$K \triangleq \{\text{location } p \mid p=1, \ldots, i, B_p = 0\}$.

Put each location p (=1, ..., i) into either type of sets K or J respectively and if both λo and

μ_o exist within I_i so that

$$\sum_{p=1}^{i}(N_p+B_p)=\sum_{p\varepsilon J}F_p^{-1}(\frac{s_p-u_p+\lambda_0}{s_p+w_p})+\sum_{p\varepsilon K}F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})=N+B$$

$$\sum_{p=1}^{i}N_p=\sum_{p\varepsilon J}F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})+\sum_{p\varepsilon K}F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})=N,$$

then go to (ix).

And the otherwise, set $q \leftarrow i+1$ and go to (vii).

(vii) Put each location p(=1, ..., i) into either type of sets K or J respectively and if λ_0 and μ_0 exist within I_i and I_q respectively so that

$$\cdot\sum_{p=1}^{q}N_p=\sum_{p\varepsilon J}F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})+\sum_{p\varepsilon K}^{i}F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})+\sum_{p=i+1}^{q}F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})=N$$

$$\cdot\sum_{p=1}^{i}(N_p+B_p)+\sum_{p=i+1}^{q}N_p=\sum_{p\varepsilon J}F_p^{-1}(\frac{s_p-u_p+\lambda_0}{s_p+w_p})+\sum_{p\varepsilon K}F_p^{-1}(\frac{w_p-u_p+\mu_0}{s_p+u_p})$$

$$+\sum_{p=i+1}^{q}F_p^{-1}(\frac{s_p-u_p+\mu_o}{s_p+u_p})=N+B\ ,$$

and if $\frac{s_p-u_p+\lambda_0}{s_p+w_p} \geqq \frac{\mu_0-\lambda_0}{u_p-w_p}$ $(p=1,2, \ldots i)$,

then the optimal solution is obtained as follows:

- $B_p=F_p^{-1}(\frac{s_p-u_p+\lambda_0}{s_p+w_p})-F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})$: $p=1, \ldots, i$ $(p\varepsilon J)$

- $B_p=0$: $p=1, \ldots, i$ $(p\varepsilon K)$

- $N_p=F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})$: $p=i+1, \ldots, q$

- $N_p=F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})$: $p=1, \ldots, i$ $(p\varepsilon J)$ $N_p=F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})$:

$p=1, \ldots, i$ $(p\varepsilon K)$

And the otherwise, go to (viii).

(viii) Set $q \leftarrow q+1$. If $q=n+1$, go to (x). Otherwise, go to (vii).

(ix) If

$$\frac{s_p-u_p+\lambda_0}{s_p+w_p} \geqq \frac{\mu_0-\lambda_0}{u_p-w_p} ,$$

then the optimal solution is obtained as follows:

- $N_p=F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})$: $p=1, \ldots, i$ $(p\varepsilon J)$, $N_p=F_p^{-1}(\frac{s_p-u_p+\mu_0}{s_p+u_p})$:

- $B_p=F_p^{-1}(\frac{s_p-u_p+\lambda_0}{s_p+w_p}) -F_p^{-1}(\frac{\mu_0-\lambda_0}{u_p-w_p})$: $p=1, \ldots, i$ $(p\varepsilon J)$

$p=1, \ldots, i$ $(p\varepsilon K)$

- $B_p=0$: $p=1, \ldots, i$ $(p \in K)$

- $N_p=0$: $p=i+1, \ldots, n$

- $B_p=0$: $p=i+1, \ldots, n$.

The otherwise, set $q \leftarrow i+1$ and go to (vii).

(x) If $i < n+1$, then set $i \leftarrow i+1$ and go to (vi).

Proposition 2.

The increase of outdating cost w_k makes N_k+B_k decrease.

(Proof)

From the equation (4.2),

$$N_k + B_k = F_k^{-1}\left(\frac{s_k-u_k+\lambda}{s_k+w_k}\right)$$

is obtained. With respect to the above equation, when w_k increases, $N_k + B_k$ decreases. If $N_k + B_k$ remains constant, we should increase λ. While λ increases, other locations' $N_l + B_l$ $(l \neq k)$ increase. This causes $\sum_{i=1}^{n} (N_i + B_i) > N+B$. So, we must decrease $N_k + B_k$ instead of increasing λ.

Proposition 3.

The increase of shortage cost s_k makes $N_k + B_k$ increase.

(Proof)

From the equation (4.2), this proposition could be proved by the similar way of proving the proposition 2.

5. Numerical Example

This section provides an example in order to illustrate the results of Section 3 by the use of the proposed algorithm described in Section 4. The demand distribution functions at each location are the identical uniform distribution from 0 to 10. In this example, we consider n=3 locations and N=6, B=2.

The cost parameters (s_k, w_k, u_k) at location k are given in Table 1. Table 2 shows the optimal allocation policy and its total expected cost. Table 3 shows the influence of shortage cost s_1 upon the optimal allocation policy at location *1* in order to illustrate the proposition 3. In this table, cost parameters are fixed except s_1 in Table 1. Table 4 shows the influence of transportation cost u_1 upon the optimal allocation policy at location *1*. In this table, cost parameters are fixed except u_1 in Table 1. Table 5 shows the influence of outdating cost w_1 upon the optimal allocation policy at location *1* in order to illustrate the proposition 2. In this table, cost parameters are fixed except w_1 in Table 1.

Table 1. Cost parameters

Cost / k	s_k	U_k	W_k
1	5	10	5
2	10	15	5
3	15	20	5

Table 2. Optimal allocation policy and its total expected cost

location k	N	B	Cost
1	2.28169	2.0	
2	2.16901	0	210.549
3	1.54930	0	

Table 3. Influence of shortage cost upon the optimal allocation policy

s_l	$N_l + B_l$
2	2.82132
3	3.08157
4	3.32361
5	4.28169
6	4.46866
7	4.64380
8	4.80818
9	4.96278

Table 4. Influence of transportation cost upon the optimal allocation policy

u_l	$N_l + B_l$
6	6.20195
7	5.66771
8	5.17221
9	4.71137
10	4.28169
11	3.10627
12	2.69129
13	2.30179

Table 5. Influence of outdating cost upon the optimal allocation policy

w_l	$N_l + B_l$
2	4.48451
3	4.41690
4	4.34930
5	4.28169
6	4.21408
7	4.14648
8	4.07887
9	4.01162

6. Conclusion

In this paper, the model for a single period allocation problem of perishable commodities based on an LIFO issuing policy has been formulated under the rotation allocation policy and the existence of the optimal allocation policy has been proved. An algorithm of obtaining its optimal allocation policy has been proved. An algorithm of obtaining its optimal allocation policy has been proposed. Furthermore, a numerical example has given and investigated the influences of shortage, transportation and outdating costs upon the optimal allocation policy.

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