Journal of the Operations Research
Society of Japan
Vol. 26, No. 2, June 1983

BOUNDS AND AN APPROXIMATION FOR SINGLE SERVER QUEUES

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(Received January 6, 1982; Final October 27, 1982)

Abstract In this note, we derive bounds for the waiting time in a single server queueing system with general independent arrival and general service times, using a regenerative process representation of this queue. We present easily computable new upper bounds for the variance and new upper and lower bounds for the second and higher moments of the waiting time. Also we present a simple approximation for the variance of waiting time.

Introduction

Mori [10] presented an elaborate study of bounds for the single (GI/G/1) and multiple (GI/G/m) server queueing systems with general independent arrival and service times. He has derived easily computable lower and upper bounds for the mean waiting time and a lower bound for the variance of waiting time in a GI/G/1 queue. Similar results for the GI/G/m queue are also given in [10]. Upper and lower bounds for the mean and variance of waiting times for special cases of GI/G/1 are also given. However, no upper bound for the variance of waiting time is given for the general GI/G/1 queue (see page 168 of [10]). In this note, we derive an easily computable upper bound and approximations for the variance of waiting time in a GI/G/1 queue.

Considerable interest has been shown in developing bounds (see [1,2,4-10,12,15,16]) and several approximations (see [14] for a review and comparison) for the GI/G/1 queue. In this paper, using a regenerative process representation of the GI/G/1 queue, we will develop easily computable upper and lower bounds for the mean, variance, second, and higher moments for the waiting time. In this process we develop easily computable bounds for the moments of the idle time.

In section 2, we discuss the GI/G/1 queueing model and derive equations relating the moments of the waiting time to the moments of idle time. We also present upper and lower bounds for the moments of the idle periods. Bounds of mean waiting time is briefly discussed in section 3, and bounds for variance of waiting time is presented in section 4. A recursive scheme to compute the second and higher moments of the waiting time is discussed in section 5. A simple approximation for the variance of waiting time is given in section 6. Numerical results for the bounds and approximation are given in section 7.

2. The Single Server Queue

We consider a single server queueing system to which the arrivals form a renewal process: the n-th customer arrival occurs at time τ_n and requires a service of length B_n. The interarrival times $A_n = \tau_{n+1} - \tau_n$, $n \geq 1$ and the service times B_n, $n \geq 1$, are assumed to form two independent sequences of independently and identically distributed random variables. Define $P\{A_n \leq x\} = A(x)$, $P\{B_n \leq x\} = B(x)$, and $P\{B_n - A_n \triangleq X_n \leq x\} = X(x)$.

Let W_n be the waiting time of the n-th customer, $W_1 = 0$. Note that $\{W_n\}_1^\infty$ forms a discrete time regenerative process with regeneration epoch corresponding to the customer arrivals to an empty system. Let N be the number of customers served during the first regeneration cycle formed by the busy cycle of GI/G/1. Then,

(1) $\quad W_1 = 0$

(2) $\quad W_{n+1} = W_n + B_n - A_n,\ n = 1,2,\ldots,N-1$

(3) $\quad W_{N+1} - I = W_N + B_N - A_N$

where $W_{N+1} = 0$ w.p. 1, and I is the server idle time during the first regeneration cycle. Taking the (r+1)-th power of equations (1), (2), and (3), summing up and taking the expectations on both sides we get,

$$E\{\sum_{i=1}^{N} W_i^{r+1}\} + (-1)^{r+1} E(I^{r+1}) = \sum_{j=0}^{r+1} E\{\sum_{i=1}^{N} W_i^j X_i^{r+1-j}\}\binom{r+1}{j}$$

where E is the expectation operator. Now assuming that $\rho \triangleq E(B)/E(A) < 1$ and $E(|X|^j) < \infty$, $j = 1,2,\ldots,r+1$, from the theory of regenerative processes (see e.g., Crane and Lemoine [3]) we can show that

$$E\{\sum_{i=0}^{N} W_i^j \, X_i^{r+1-j}\}/E(N) = \lim_{n\to\infty} E(W_n^j \, X_n^{r+1-j})$$

$$= E(W^j)E(X^{r+1-j}),$$

since W_n and X_n are independent and $W \underline{\underline{D}} \lim_{n\to\infty} W_n$. Now dividing the previous equation by E(N) and using the above result one has

$$(4) \qquad E(W^r) = \frac{(-1)^{r+1}}{(r+1)} \frac{E(I^{r+1})}{E(N)E(X)} - \frac{1}{(r+1)} \sum_{j=0}^{r-1} \binom{r+1}{j} E(W^j) \frac{E(X^{r+1-j})}{E(X)}$$

$$r \geq 1,$$

Similarly, summing equations (1), (2), and (3), and taking expectations on both sides, we get

$$(5) \qquad E(I) = -E(N)E(X).$$

Then, substituting (5) for E(I) in (4), we have

$$(6) \qquad E(W^r) = \frac{(-1)^r}{(r+1)} \frac{E(I^{r+1})}{E(I)} - \frac{1}{(r+1)E(X)} \sum_{j=0}^{r-1} \binom{r+1}{j} E(W^j)E(X^{r+1-j}),$$

$$r \geq 1,$$

a result obtained by Lemoine [7] using an alternate approach. If we have upper and lower bounds for $E(W^j)$, $j = 1,2,\ldots,r-1$, and $E(I^{r+1})/E(I)$, we can develop upper and lower bounds for $E(W^r)$. That is, (6) gives a recursive relation useful in developing bounds for the r-th moment of the waiting time in GI/G/1 queue. Next we will develop bounds for $E(I^r)/E(I)$, $r \geq 2$.

Now define $Y_n = -X_n = A_n - B_n$, $n = 1,2,\ldots,N$. Multiply equation (1) by Y_0^r, (2) by Y_n^r, and (3) by Y_N^r, adding them and taking expectation, we get

$$E\{\sum_{i=1}^{N} W_i \, Y_{i-1}^r\} - E(IY_N^r)$$

$$= E\{\sum_{i=1}^{N} W_i \, Y_i^r\} - E\{\sum_{i=1}^{N} Y_i^{r+1}\},$$

where $Y_0 \underline{\underline{D}} Y_n$, $n = 1,2,\ldots$.
Now noting that

$$E\{\sum_{i=1}^{N} W_i \, Y_{i-1}^r\}/E(N) = \lim_{n\to\infty} E\{W_{n+1} \, Y_n^r\}$$

and

$$E\{\sum_{i=1}^{N} W_i\, Y_i^r\}/E(N) = \lim_{n\to\infty} E\{W_n\, Y_n^r\} = E(W)E(Y^r)$$

and dividing the above equation by E(N) one obtains,

$$(7)\qquad Cov(W,Y^r) - \frac{E(IY_N^r)}{E(N)} = -\,E(Y^{r+1}),\ r \geq 1,$$

where

$$Cov(W,Y^r) = \lim_{n\to\infty} Cov(W_{n+1},Y_n^r).$$

$Cov(W,Y^r) \leq 0$, for r = 1,3,5,..., since as Y_n increases, Y_n^r increases and W_{n+1} will not increase. Now noting that $Y_N \geq I$, from (7) we get

$$(8)\qquad E(I^{r+1}) \leq E(N)E(X^{r+1}),\ r = 1,3,5,\ldots\ .$$

It should be noted that the upper bound for $E(I^{r+1})$ derived by Lemoine [7] (equation 2.11) is better than that given by equation (8). This bound (equation 2.11 of [7]) can be derived by a procedure similar to the above except using $Y_n^+ = \max(Y_n,0)$ instead of Y_n and noting that $(Y_n)(Y_n^+)^r = (Y_n^+)^{r+1}$ and $Y_n^+ = Y_N$. To compute this bound given in [7], however, we need to know the distribution functions A(•) and B(•). Thus, for quick calculations with partial information of only the moments of A(•) and B(•), equation (8) may be used and it should prove to be a useful bound.

Now dividing (8) by (5), we get

$$(9)\qquad \frac{E(I^{r+1})}{E(I)} \leq -\frac{E(X^{r+1})}{E(X)},\quad r = 1,3,5,7,\ldots\ .$$

A similar derivation using A_n instead of Y_n gives

$$(10)\qquad \frac{E(I^{r+1})}{E(I)} \leq \frac{E(A^{r+1}) - E(B)E(A^r)}{E(A) - E(B)},\quad r \geq 1,$$

Equations (9) and (10) give upper bounds for $E(I^r)/E(I)$. Next we need to develop lower bounds for these quantities. Obviously

$$(11)\qquad \frac{E(I^{r+1})}{E(I)} \geq 0.$$

Stronger lower bounds can be developed using the properties of positive random variables and Jensen's inequality to convex functions. It is

(12) $$\frac{E(I^r)}{E(I)} \geq (E(I))^{r-1}, \quad r = 1,2,\ldots .$$

An alternate equation for $E(W^r)$ can be derived from (6) by considering the ratio of $E(I^{r+1})/E(I^r)$. It is

(13) $$E(W^r) = -\frac{1}{(r+1)E(X)} \sum_{j=0}^{r-1} \{\binom{r+1}{j} E(X^{r+1-j}) + \frac{E(I^{r+1})}{E(I^r)} \binom{r}{j} \cdot E(X^{r-j})\} E(W^j), \quad r \geq 1$$

Now to use equation (13) to derive bounds for $E(W^r)$ we need upper and lower bounds for $E(I^{r+1})/E(I^r)$. Lower bounds for this can be derived as follows. Let I_{k+1} be the residual life of the renewal process formed by random variables with the same cumulative distribution function $I_k(\cdot)$ as that of I_k. That is,

(14) $$I_{k+1}(x) = \frac{\int_0^x (1-I_k(y))dy}{E(I_k)}, \quad k \geq 0,$$

where we set $I_0 = I$ and assume that $E(I_k) < \infty$. Then

(15) $$E(I_{k+1}^j) = \frac{E(I_k^{j+1})}{(j+1)E(I_k)}, \quad k \geq 0, \; j \geq 1,$$

Using (15) recursively for $k = r-1, r-2, \ldots, 0$, we can show that

(16) $$E(I_r) = \frac{E(I^{r+1})}{(r+1)E(I^r)}, \quad r \geq 0,$$

and

(17) $$E(I_r^2) = \frac{2}{(r+2)(r+1)} \frac{E(I^{r+2})}{E(I^r)}, \quad r \geq 0.$$

Then, using the property that $Var(I_r) = E(I_r^2) - (E(I_r))^2 \geq 0$, we get

(18) $$\frac{2}{(r+1)(r+2)} \frac{E(I^{r+2})}{E(I^r)} \geq \left(\frac{E(I^{r+1})}{(r+1)E(I^r)}\right)^2, \quad r \geq 0$$

Rearranging, we get

(19) $$\frac{E(I^{r+2})}{E(I^{r+1})} \geq \frac{E(I^{r+1})}{E(I^{r})} \cdot \frac{(r+2)}{2(r+1)}, \quad r \geq 0.$$

Repeated use of (19) leads to

(20) $$\frac{E(I^{r+2})}{E(I^{r+1})} \geq \frac{E(I^{2})}{E(I)} \cdot \frac{(r+2)}{2^{r+1}}, \quad r \geq 0.$$

Now we need lower bounds for $E(I)$ or $E(I^2)/E(I)$ to use (12) and (20).

The following bounds can be derived from the results available in the literature, or using equation (5).

(21) $$\frac{E(I^{2})}{E(I)} \geq E(I) \geq (1-\rho)E(A).$$

The above lower bound can be derived using (5) and noting that $E(N) \geq 1$ (see Kingman [5] and Marshall [9]).

(22) $$\frac{E(I^{2})}{E(I)} \geq E(I) \geq (1-\rho)E(A)/X(0).$$

The above lower bound can be derived using (5) and noting that $E(N) \geq 1/X(0)$ (see Mori [10] and Lemoine [7]).

Using Daley's upper bound for $E(W)$ reported by Calo and Schwartz [1], we can show that

(23) $$\frac{E(I^{2})}{E(I)} \geq (1-\rho)E(A)(1+C_a^2),$$

where C_a^2 is the squared coefficient of variation of interarrival times. Note that (23) is a better lower bound than (21). Now, using the results given in Mori [10] and (23), we get

(24) $$\frac{E(I^{3})}{3E(I)} \geq \left(\frac{E(I^{2})}{2E(I)}\right)^2 \geq \frac{(1-\rho)^2(E(A))^2(1+C_a^2)}{4}$$

3. Bounds on Mean Waiting Time

In this section we indicate how the bounds available for $E(W)$ in the literature can be derived from equations (6), (9), (10), (21), and (23). From (6) we get, for $r = 1$,

$$(25)\qquad E(W) = -\frac{E(X^2)}{2E(X)} - \frac{E(I^2)}{2E(I)}.$$

Now using (25) and (9), we get a trivial lower bound $E(W) \geq 0$. However, using (25) and (10) gives the lower bound

$$(26)\qquad E(W) \geq \frac{Var(B)}{2E(A)(1-\rho)} - \frac{E(B)}{2}$$

derived by Stoyan and Stoyan [15] and Mori [10]. Note that when the squared coefficient of variation C_b^2, of the service time is such that, $C_b^2 < (1-\rho)/\rho$, the lower bound (26) is negative. See Table 1 for other lower bounds available in the literature. Now usine (25) and (21), we get the upper bound of Kingman [4] and Marshall [9]. Using (25) and (22), we get the upper bound of Mori [10] and Lemoine [7]. For other upper bounds, see Table 1.

TABLE 1. Upper and Lower Bounds for the Mean Waiting Time E(W).

E(W)

LOWER BOUND		UPPER BOUND	
$\frac{E(X^+)^2}{2(1-\rho)E(A)}$	, Kingman [5]	$\frac{E(A)(C_a^2+\rho^2 C_b^2)}{2(1-\rho)}$	, Kingman [4] Marshall [9]
ω_2, the unique solution of $\omega = \int_{-\omega}^{\infty}(1-X(x))dx$	, Marshall [9]	$\frac{E(A)(C_a^2+\rho^2 C_b^2)}{2(1-\rho)} - \frac{(1-\rho)C_a^2 E(A)}{2}$	, Daley Reported in [1]
$\frac{Var(B)}{2(1-\rho)E(A)} - \frac{E(B)}{2}$	, Stoyan & Stoyan [15] Mori [10]	$\frac{E(A)(C_a^2+\rho^2 C_b^2)}{2(1-\rho)} - \frac{(1-\rho)(1-X(0))E(A)}{2X(0)}$	, Stoyan [15] Lemoine [7]
$E(X^+ + X)^+$	, Calo & Schwartz [1]	ω_1, the unique positive root of $2(1-\rho)E(A)\omega = E(X^2) - E((\omega+X)^-)^2$	Calo & Schwartz [1]
$\frac{E(X^2)-E((X^+ + X)^-)^2}{2(1-\rho)E(A)}$	, Calo & Schwartz [1]		

4. Bounds on Variance of Waiting Time

In this section we derive new upper bounds for the variance of waiting time Var(W). From (6) and (25), we can show that

$$(27)\qquad \mathrm{Var}(W) = -\frac{E(X^3)}{3E(X)} + \left(\frac{E(X^2)}{2E(X)}\right)^2 + \frac{E(I^3)}{3E(I)} - \left(\frac{E(I^2)}{2E(I)}\right)^2,$$

a result given by Marshall [9]. Mori [10] has shown that $(E(I^3)/3E(I)) - (E(I^2)/2E(I))^2 \geq 0$ (see equation (18) for r = 1), and that

$$(28)\qquad \mathrm{Var}(W) \geq -\frac{E(X^3)}{3E(X)} + \left(\frac{E(X^2)}{2E(X)}\right)^2.$$

Now using (10) for r = 2 and (23) in (27), we get

$$(29)\qquad \mathrm{Var}(W) \leq \left(\frac{E(X^2)}{2E(X)}\right)^2 - \left(\frac{(1-\rho)E(A)(1+C_a^2)}{2}\right)^2 - \Delta,$$

where

$$\Delta = \left(\frac{3E(B^2)E(A) - 2E(B)E(A^2) - E(B^3)}{3E(A)(1-\rho)}\right).$$

Alternate upper bound can be developed using (22) instead of (23). It is

$$(30)\qquad \mathrm{Var}(W) \leq \left(\frac{E(X^2)}{2E(X)}\right)^2 - \left(\frac{(1-\rho)E(A)}{2X(0)}\right)^2 - \Delta.$$

TABLE 2. Upper and Lower Bounds for Variance of Waiting Time Var(W).

Var(W)

LOWER BOUND	UPPER BOUND
$-\frac{E(X^3)}{3E(X)} + \left(\frac{E(X^2)}{2E(X)}\right)^2$, Mori [10]	$-\Delta + \left(\frac{E(X^2)}{2E(X)}\right)^2 - \left(\frac{(1-\rho)E(A)(1+C_a^2)}{2}\right)^2$ *
	$-\Delta + \left(\frac{E(X^2)}{2E(X)}\right)^2 - \left(\frac{(1-\rho)E(A)}{2X(0)}\right)^2$ *
	$\min_s \{\frac{4}{e^2 s^2} \log(\frac{1}{1-X(s)})\}$, Kingman [5]

*developed in this article $\Delta = \frac{3E(B^2)E(A) - 2E(B)E(A^2) - E(B^3)}{3E(A)(1-\rho)}$

5. Bounds for Second and Higher Moments of Waiting Time

In this section we will formally present the recursive relation to compute bounds for $E(W^r)$, $r \geq 2$. Let U_r be the upper and L_r be the lower bounds of $E(W^r)$ for $r \geq 1$. Set U_1 equal to the minimum of the upper bounds for $E(W)$ presented in Table 1 and L_1 equal to the maximum of the lower bounds for $E(W)$ presented in Table 1. Define u_{r+1} equal to the minimum of the right hand side of (9) and (10). Let ℓ_1 equal the maximum of the right hand side of (22) and (23). Let ℓ_{r+2} be equal to the right hand side of (12) with $E(I)$ replaced by ℓ_1. Then from (6) we get, for $r = 1,2,\ldots,$

$$(31)\qquad U_{2r} = u_{2r+1} - \frac{1}{(2r+1)E(X)} \sum_{j=0}^{r-1} \binom{2r+1}{2j} L_{2j} E(X^{2t+1-2j}) - \frac{1}{(2r+1)E(X)} \sum_{j=0}^{r-1} \binom{2r+1}{2j+1} U_{2j+1} E(X^{2r-2j})$$

$$L_{2r} = \ell_{2r+1} - \frac{1}{(2r+1)E(X)} \sum_{j=0}^{r-1} \binom{2r+1}{2j} U_{2j} E(X^{2r+1-2j}) - \frac{1}{(2r+1)E(X)} \sum_{j=0}^{r-1} \binom{2r+1}{2j+1} L_{2j+1} E(X^{2r-2j})$$

$$(32)\qquad U_{2r+1} = -\ell_{2r+2} - \frac{1}{(2r+2)E(X)} \sum_{j=0}^{r} \binom{2r+2}{2j} U_{2j} E(X^{2r+2-2j}) - \frac{1}{(2r+2)E(X)} \sum_{j=0}^{r-1} \binom{2r+2}{2j+1} L_{2j+1} E(X^{2r+1-2j})$$

$$L_{2r+1} = -u_{2r+2} - \frac{1}{(2r+2)E(X)} \sum_{j=0}^{r} \binom{2r+2}{2j} L_{2j} E(X^{2r+2-2j}) - \frac{1}{(2r+2)E(X)} \sum_{j=0}^{r-1} \binom{2r+2}{2j+1} U_{2j+1} E(X^{2r+1-2j}).$$

Now equations (31) and (32) can be alternatively evaluated for $r = 1,2,\ldots,$ until the bounds for the desired moment of the queueing time is obtained.

Similarly, using (13), (20), (21), and the upper bound

$$\frac{E(I^{r+2})}{E(I^{r+1})} \le \frac{E(I^{r+2})}{(E(I))^{r+1}}$$

along with (9) and (10) can be used to compute bounds for $E(W^r)$, $r \ge 1$.

6. Approximation for the Variance Var(W)

In this section we derive an approximation for the variance Var(W) of waiting time in the GI/G/1 queue. We know of no other approximations available for Var(W). Here we propose two approximations for Var(W), one for each of two different cases of GI/G/1. The two cases are $1 \ge C_s^2 \ge C_a^2 \ge 0$ and $1 \ge C_a^2 > C_s^2 \ge 0$, chosen purely based on numerical results of these approximations (carried out in [13]), where C_a^2 and C_s^2 are the squared coefficient of variation of the interarrival and service times, respectively.

The approximations are chosen following the approach similar to Page's [11] and Takahashi [17]. However, weights are chosen to be C_a^3 and C_s^4 instead of C_a^2 and C_s^2 because they gave better approximations. The proposed approximations are:

(i) for $1 \ge C_s^2 \ge C_a^2 \ge 0$.

$$\text{Var}(W) \simeq C_s^4 \,\text{Var}\{W(G/M)\} + C_a^3(1-C_s^4)\,\text{Var}\{W(M/D)\}, \tag{33}$$

where Var{W(G/M)} and Var{W(M/D)} are respectively the variance of time spent queueing in equivalent GI/M/1 and M/D/1 queues. The equivalent GI/M/1 queue is chosen so that its interarrival process and server utilizations are identical to the given GI/G/1 queue. Similarly, the equivalent M/D/1 queue is chosen so that its mean arrival and service rates are identical to the given GI/G/1 queue.

Unfortunately there is no closed form solution for Var{W(G/M)}. A very good approximation can be obtained, however. If E[W(B/M)] is the mean waiting time in GI/M/1 queue with mean service time E(B), it can be shown that (from the distribution function for W (G/M) [6]),

$$\text{Var}\{W(G/M)\} = E[W(G/M)](E[W(G/M)] + 2E(B)) .$$

We have already given a very good approximation (L(PAG)) for the mean number in the GI/M/1 system in [14]. Then, from equation (8) of [14] with the substitution $C_s^2 = 1$, we get

$$E[W(G/M)] \simeq k,$$

where

$$k = \frac{\rho E(B)}{2(1-\rho)} \{2C_a^2 + (1-C_a^2) \exp\{\frac{-2(1-\rho)}{3\rho}\}\}.$$

Then

$$(34) \qquad Var\{W(G/M)\} \simeq k(k + 2E(B)).$$

When the arrival distribution is Erlang, a closed form for Var{W(G/M)} is available [6]. Thus we can check the approximation (33) against the exact value in the Erlang case.

A comparison of this approximation to the exact values are given in Table 6. The exact value for Var{W(M/D)} is:

$$Var\{W(M/D)\} = \frac{\rho(4-\rho)(E(B))^2}{12(1-\rho)^2} .$$

Now substituting equation (34) and the above into (33), we get

$$(35) \qquad Var(W) \simeq k(k+2E(B))C_s^4 + \frac{\rho(4-\rho)(E(B))^2}{12(1-\rho)^2} \; C_a^3(1-C_s^4),$$

$$1 \geq C_s^2 \geq C_a^2 \geq 0.$$

Similarly: (ii) for $1 \geq C^{a}_{2} > C_s^2 \geq 0$ we propose

$$(36) \qquad Var(W) \simeq C_a^4 \; Var\{W(M/G)\} + C_s^3(1-C_a^4)Var\{W(D/M)\},$$

where Var{W(M/G)} and Var{W(D/M)} are respectively the variance of time spent queueing in equivalent M/G/1 and D/M/1 queues. The exact value for Var{W(M/G)} depends on the third moment of the service time distribution. We will approximate Var{W(M/G)} by an equivalent $M/E_k/1$ queue so that

(37) $$\mathrm{Var}\{W(M/G)\} \simeq \frac{\rho(1+C_s^2)\{4-\rho+(8-5\rho)C_s^2\}(E(B))^2}{12(1-\rho)^2}.$$

We have already given an approximation for Var{W(G/M)} by equation (34). Substituting $C_a^2 = 0$ in (34), we get

$$\mathrm{Var}\{W(D/M)\} \simeq k_1(k_1+2E(B)),$$

where

(38) $$k_1 = \frac{\rho E(B)}{2(1-\rho)} \exp\{-\frac{2(1-\rho)}{3\rho}\}.$$

Substituting equations (37) and (38) in (36), we get

(39) $$\mathrm{Var}(W) \simeq \frac{\rho(1+C_s^2)\{4-\rho+(8-5\rho)C_s^2\}(E(B))^2 C_a^4}{12(1-\rho)^2} + k_1(k_1+2E(B))C_s^3(1-C_a^4)$$

7. Some Numerical Comparisons

In this section, we will look at the goodness of the bounds and the approximation for the variance of waiting time in $E_k/E_\ell/1$ queueing systems. For $M/E_\ell/1$ and $E_k/M/1$ exact results are obtained and compared to the bounds (see Tables 3 and 4) and to the approximations (see Table 6). For other cases, simulation is used to obtain estimates of the variance of waiting time. These results, with the bounds are given in Table 5 and with the approximations are given in Tables 7 and 8.

We observed that the bounds are good in general only when the server utilization is higher than .6. It is also observed that the lower bound is closer to the exact result than the upper bound. However, when the server utilization is lower than .6, most of the time the lower bound is negative and the upper bound is very much higher than the exact value. Of course, the negative values of the lower bound (27) should be read as zero. Therefore, it is strongly suggested that the bounds be used only for cases with high server utilizations.

TABLE 3. Exact, lower and upper bounds for the variance of waiting time of an arbitrary customer in an $M/E_\ell/1$ queue for various values of the Erlang parameter ℓ of E_ℓ and server utilization ρ.

Variance of Waiting Time Var(W)

ρ↓	ℓ→	1	2	3	5	10
.6	LB	2.4722	-0.0122	-0.6667	-1.1278	-1.4372
	EX	5.2500	2.7656	2.1111	1.6500	1.3406
	UB	10.3611	7.8769	7.2222	6.7611	6.4517
.7	LB	8.0703	3.3500	2.1073	1.2259	0.6328
	EX	10.1111	5.3958	4.1481	3.2667	2.6736
	UB	15.1429	10.4276	9.1799	8.2984	7.7054
.8	LB	22.4375	11.4375	8.5116	6.4375	5.0375
	EX	24.0000	13.0000	10.0741	8.0000	6.6000
	UB	29.6667	18.6667	15.7407	13.6667	12.2667
.9	LB	97.7654	53.3279	41.4321	32.9654	27.2279
	EX	99.0000	54.5625	42.6667	34.2000	28.4625
	UB	107.6296	63.1921	51.2963	42.8296	37.0921

LB - Lower Bound (equation (28))
EX - Exact
UB - Upper Bound (equation (29))

TABLE 4. Exact, lower and upper bounds for the variance of waiting time of an arbitrary customer in an $E_k/M/1$ queue for various values of the Erlang parameter k of E_k and server utilization ρ.

Variance of Waiting Time Var(W)

ρ↓	k→	1	2	5
.6	LB	2.472	1.821	1.3611
	EX	5.250	2.934	1.817
	UB	10.3611	5.043	2.8678
.8	LB	22.438	13.160	8.6875
	EX	24.000	13.774	8.938
	UB	29.667	15.729	9.7067

LB - Lower bound (equation (28))
EX - Exact
UB - Upper bound (equation (29))

TABLE 5. Simulated results, lower and upper bounds for the variance of waiting time of an arbitrary customer in an $E_k/E_\ell/1$ queue for various values of the Erlang parameters k and ℓ of E_k and E_ℓ and server utilization ρ = 0.6.

Variance of Waiting Time Var(W)

k↓	ℓ→	2	5
2	LB	0.118	-0.529
	SIM	1.248	0.472
	UB	3.340	2.693
5	LB	0.127	-0.239
	SIM	0.517	0.204
	UB	1.633	1.268

LB - Lower bound (equation (28))
SIM - Simulation results
UB - Upper bound (equation (29))

TABLE 6. Variance of Waiting Time in the $E_k/M/1$ Queue

ρ	C_a^2	EXACT	APPROXIMATE
0.8	1.0	24.000	24.000
	0.5	13.774	13.795
	0.2	8.938	8.950
	0.1	7.541	7.548
	0.0	6.250	6.252
0.6	1.0	5.250	5.250
	0.5	2.934	2.962
	0.2	1.817	1.838
	0.1	1.491	1.505
	0.0	1.190	1.193

TABLE 7. Variance of the Waiting Time in the $E_k/E_2/1$ Queue

ρ	C_a^2	SIMULATION	APPROXIMATE
0.8	1.0	13.000*	13.000
	0.5	5.389	4.863
	0.2	2.291	2.595
0.6	1.0	2.7656	2.7656
	0.5	1.248	1.022
	0.2	0.517	0.531

*Exact values

TABLE 8. Variance of the Waiting Time in the $E_k/E_5/1$ Queue

ρ	C_a^2	SIMULATION	APPROXIMATE
0.8	1.0	8.000*	8.000
	0.5	2.489	2.419
	0.2	0.679	0.816
0.6	1.0	1.650*	1.650
	0.5	0.472	0.493
	0.2	0.204	0.165

*Exact values

The proposed approximation given exact values for $M/E_k/1$ systems, very good approximations for $E_k/M/1$ systems, and reasonably good approximations for $E_k/E_\ell/1$ systems with $\ell,k = 2,5$.

Acknowledgements

The author would like to thank the referees for their helpful comments.

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