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AN OPTIMUM INSPECTION POLICY FOR A ONE-UNIT SYSTEM WITH PREVENTIVE MAINTENANCE

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Abstract This paper deals with a one-unit system that the system's failure can be detected only by inspection. This inspection takes a non-negligible random time. Consequently the system is down during the inspection whether it is operable or not. When the system's failure is detected by i-th inspection ($i = 1, 2, \ldots, n+1$), the system is repaired. When the system is operable at the time of the (n+1)-st inspection, preventive maintenance is performed. It is assumed that a system is as good as new after repair or preventive maintenance is performed and is put in operation immediately.

Under this inspection policy, the Laplace transform of the point-wise availability and the stationary availability of the system are derived by using the method of supplementary variables.

We discuss the optimum inspection policy maximizing the stationary availability. It is to determine an optimal number of times of inspection and optimal inspection periods. It is shown that there exists an optimum inspection policy under some conditions on the failure distribution and the mean maintenance time.

1. Introduction

We deal with stochastically failing system, in which failure can be detected only by actual inspection (required to take non-negligible time). Barlow, Hunter and Proschan [1,2] studied the optimum checking procedures minimizing the total expected value of the cost of the elapsed time between system failure and its detection and the cost of checking. It may be quite difficult to find such an optimal procedure. To avoid this difficulty, Munford and Shahani [7,8] and Tadikamalla [11] suggest a nearly optimal checking interval. On the other hand, Keller [5] and Osaki [9] proposed a smooth density $n(t)$ where $n(t)$ denotes the number of checks per unit time and obtained a sequence of approximate inspection times by the usual methods of the calculus of variations.

Further, Bosch and Jensen [3], Luss and Kander [6], Sengupta [10], Wattanapanom and Shaw [12] and Zuckerman [14] considered models which are related to Barlow, Hunter and Proschan [1].

We are interested in the availability of system. For example, a machine produces units continuously. The quality of the output is checked at various time to determine whether the machine is functioning satisfactorily. Upon detection of malfunction, repair is made, production resumes and inspection continues. Coleman and Abrams [4] and Weiss [13] dealed with such models.

In such situation we further incorporate a policy that preventive maintenance is performed if the system is operable at the time of the (n+1)-st inspection. Under this inspection policy, the Laplace transform of the pointwise availability and the stationary availability of the system are derived by using the supplementary variable methods. And it is shown that there exists an optimum inspection policy which maximizes the stationary availability under suitable conditions. Finally, we shall present numerical example.

2. Definition of Model and Availability

We define a model as follows:

(i) The system's life time has an arbitrary distribution $F(x)$ with a finite mean λ, the differentiable density function $f(x)$ and the failure rate $\lambda(x) = f(x)/\bar{F}(x)$, where, in general, for a function $K(x)$, $\bar{K}(x) = 1 - K(x)$.

(ii) The system's failure can be detected only by inspection and the probability of its detection equals one.

(iii) The system is shut down while being inspected. So the system is down during inspection whether it is operable or not.

(iv) Inspection is scheduled to begin after X_I units of time from the instant at which the system is renewed by repair or preventive maintenance. (See Fig. 1)

(v) The subsequent inspections are scheduled to X_i units of time after the conclusion of the $(i-1)$-st inspection if no repair has taken place ($i=2,\ldots,$ n+1). Where X_i ($i=1,2,\ldots,$n+1) is a random variable having a distribution function $A_i(x)$ with a finite mean and the density function $a_i(x)$. We call this interval the inspection period. (See Fig. 1)

(vi) Each inspection takes X_I units of time, where X_I is a random variable having a distribution function $G(x)$ with a finite mean μ and the density function $g(x)$.

(vii) When a failure of the system is detected by inspection, the system undergoes repair at once. The repair time has a distribution function $H(x)$ with a finite mean r and the density function $h(x)$.

(viii) Preventive maintenance on the system is performed when it is operable at the conclusion of the (n+1)-st inspection. The preventive maintenance time has a distribution function $H_s(x)$ with a finite mean r_s and the density function $h_s(x)$.

(ix) The system is as good as new after repair or preventive maintenance is performed and is put in operation immediately.

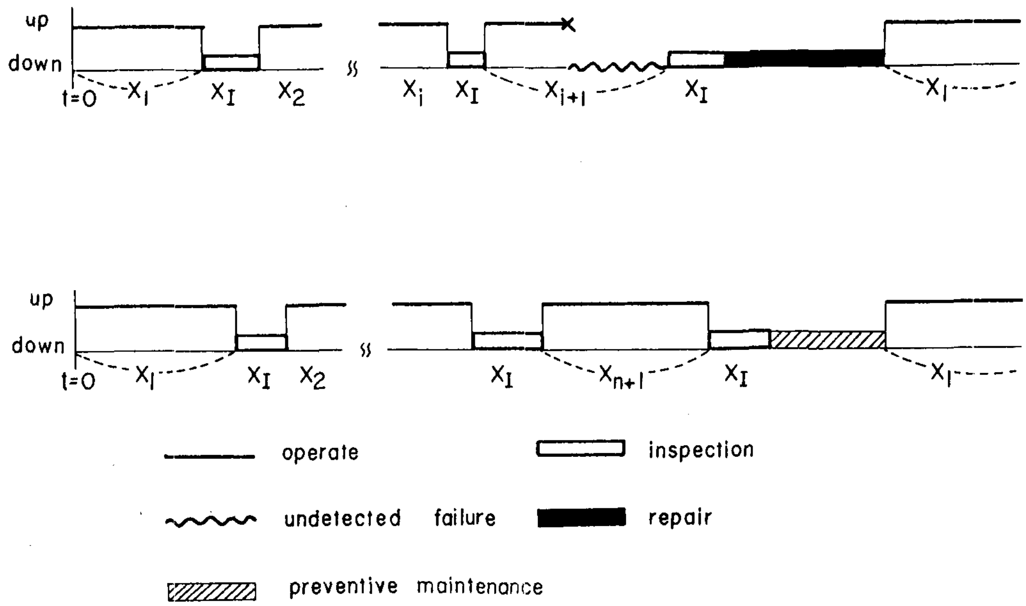

Fig. 1 A configulation of the inspection policy.

The optimization problem we consider is to maximize the stationary availability of the system. Then, we do not wish to check too often from the point of view of the availability since each inspection takes a time (i.e., it is a down time.). On the other hand, there is a down time between the occurrence of a system's failure and its detection. So, our problem is to determine a number of inspection until preventive maintenance is performed and a sequence of inspection periods.

Letting $p_A(t)$ be the point-wise availability of the system at time t, the Laplace transform $p_A^*(s)$ of $p_A(t)$ is explicitly given as follows (See Appendix):

$$(1) \quad p_A^*(s) = \sum_{i=0}^{n} g^*(s)^i \psi_i^*(s) / [1-g^*(s)^{n+1} h_s^*(s) \phi_{n+1}^*(s)$$

$$-\sum_{i=0}^{n} g^*(s)^{i+1} h^*(s)\{a^*_{i+1}(s)\phi^*_i(s)-\phi^*_{i+1}(s)\}],$$

where

$$\phi^*_i(s) = \int_0^\infty \exp[-sx]\bar{F}(x)a^{(i)}(x)dx \qquad (i=1,\dots,n+1),$$

$$\phi^*_0(s) = 1,$$

$$a^{(i)}(x) = \int_0^x a^{(i-1)}(x-y)a_i(y)dy \qquad (i=2,\dots,n+1),$$

$$a^{(1)}(x) = a_1(x),$$

$$\psi^*_i(s) = \int_0^\infty \exp[-sx]\bar{F}(x)\bar{A}^{(i)}(x)dx \qquad (i=0,1,\dots,n),$$

$$\bar{A}^{(i)}(x) = \int_0^x a^{(i)}(x-y)\bar{A}_{i+1}(y)dy \qquad (i=1,\dots,n),$$

$$\bar{A}^{(0)}(x) = \bar{A}_1(x)$$

and * denotes the Laplace transform.

Consequently, we obtain the stationary availability $p_A(\infty)$ of the system as follows:

$$(2) \quad p_A(\infty) = \sum_{i=0}^{n} \psi^*_i(0)/[(n+1)\mu\phi^*_{n+1}(0)+r_s\phi^*_{n+1}(0)+\sum_{i=0}^{n}\{(r+(i+1)\mu) \times(\phi^*_i(0)-\phi^*_{i+1}(0))+\phi^*_i(0)\cdot\int_0^\infty \bar{A}_{i+1}(x)dx\}].$$

3. Optimum Inspection Policy

In the preceding section, we have obtained the stationary availability under a random inspection period, in which X_i is distributed $A_i(x)$. Hereafter, we consider only a regular inspection period which could be applied in practical fields. So, we assume that

$$(3) \quad A_i(x) = \begin{cases} 0 & \text{for } 0 \leqq x < T_i \\ 1 & \text{for } T_i \leqq x \end{cases} \qquad (i=1,\dots,n+1)$$

Then $p_A(\infty)$ is a function of (n+1) variables $T_1, \dots, T_{n+1}$. Denoting it by $p_A(n,T)$,

$$(4) \quad p_A(n,T) = \int_0^{S_{n+1}} \bar{F}(x)dx/[\sum_{i=0}^{n} (\mu+T_{i+1})\bar{F}(S_i)+r-(r-r_s)\bar{F}(S_{n+1})]$$

where

$$(5) \quad S_i = T_0 + T_1 + \dots + T_i\,, \quad T_0 = 0 \qquad (i=0,1,\dots,n+1).$$

We shall discuss the optimum inspection period maximizing eq.(4) i.e., the problem determining the number of inspection n* until preventive maintenance is performed and the inspection period $\hat{T}_1, \hat{T}_2, \dots, \hat{T}_{n^*+1}$.

If $T_i = 0$ for all $i = 1,\dots,n+1$ or $T_1 \to \infty$, then eq.(4) equals zero. Eq.(4) is a continuous function in $T_1,\dots,T_{n+1}$ for fixed n. Consequently, we remark that there is at least one set $(T_1,T_2,\dots,T_{n+1})$ maximizing eq.(4).

Let us suppose that $T_i = T$ for all $i=1,\ldots,n+1$ (periodic inspection). Only for the case of exponential failure, periodic inspection procedure is optimum over all inspection procedures. This is clear by the memoryless property of exponential distribution. Then eq.(4) is

$$(6)\quad p_A(n,T) = \lambda\{1-\exp[-(n+1)T/\lambda]\}/[(T+\mu)\{1-\exp[-(n+1)T/\lambda]\}/\{1-\exp[-T/\lambda]\} +r-(r-r_s)\exp[-(n+1)T/\lambda]].$$

Eq.(6) is non-decreasing in n for all $T > 0$ i.e.,

$$\lim_{n\to\infty} p_A(n,T) \equiv p_A(\infty,T) \geqq p_A(n,T) \quad \text{for all } T > 0.$$

Consequently, to perform preventive maintenance is meaningless. From eq.(6), it is

$$(7)\quad p_A(\infty,T) = [r/\lambda+(T+\mu)/\lambda(1-\exp[-T/\lambda])]^{-1}.$$

To maximizing $p_A(\infty,T)$ equals to minimizing $D(T) = (T+\mu)/(1-\exp[-T/\lambda])$. Taking the derivative and setting it equal to zero, we obtain

$$(8)\quad \exp[-T/\lambda] - T/\lambda = 1 + \mu/\lambda$$

which has a unique solution in T. This coincides with Weiss [13] and Barlow, Hunter and Proschan [1].

Next it is of interest to consider the optimum sequential inspection procedure maximizing eq.(4) for any fixed n. A necessary condition that a set $(T_1,\ldots,T_{n+1})$ is optimum inspection procedure is that

$$(9)\quad \partial p_A(n,T)/\partial T_i = 0 \qquad \text{for all } i=1,\ldots,n+1.$$

Hence, using eq.(4) we obtain

$$(10)\quad T_{i+1} = [\bar{F}(S_{i-1})-\bar{F}(S_i)]/f(S_i)-\mu \qquad (i=1,\ldots,n),$$

$$(11)\quad [(r-r_s)\bar{F}(S_{n+1})+\bar{F}(S_n)/\bar{F}(S_{n+1})]\int_0^{S_{n+1}} \bar{F}(x)dx = r-(r-r_s)\bar{F}(S_{n+1}) + \sum_{i=0}^{n}(\mu+T_{i+1})\bar{F}(S_i).$$

From eq.(10), T_{i+1} $(i=1,\ldots,n)$ is evidently expressed as a function of T_1 only. Let $\xi_{i+1}(T_1)$ denote the right-hand side of eq.(10) for $i = 1,\ldots,n$ and $\xi_1(T_1) = T_1$. Accordingly $T_2,\ldots,T_{n+1}$ are determined recursively once we determine T_1 by eq.(11). To show that this set is unique, we need the following lemmas.

Lemma 1. If the failure density $f(x)$ is PF_2 and $f(x) > 0$ for all $x > 0$, then $T_{i+1} = \xi_{i+1}(T_1)$ is non-decreasing in T_1 $(i=1,\ldots,n)$.

Proof: We will prove by the induction. From eq.(10), $T_2 = F(T_1)/f(T_1) - \mu$. By Cor. 3.1 of Barlow, Hunter and Proschan [1], T_2 is non-decreasing in T_1. Suppose that T_{j+1} is non-decreasing T_1 for all $j=1,\ldots,i$. By Theorem 3 of Barlow, Hunter and Proschan [1], the following inequality holds.

(12) $[F(x)-F(x-\Delta)]/f(x) \geqq [F(x-\varepsilon)-F(x-\Delta-\varepsilon)]/f(x-\varepsilon)$ for all $\Delta,\varepsilon \geqq 0$.

Also by the assumption there exists $\Delta'_j \geqq 0$ such that

(13) $\xi_j(T_1+\Delta) = \xi_j(T_1) + \Delta'_j \ (\geqq \xi_j(T_1))$

for all $\Delta \geqq 0$ and $j=1,\ldots,i+1$. Hence we have for all $\Delta \geqq 0$,

$$\begin{aligned}
&\xi_{i+2}(T_1+\Delta) + \mu \\
&= [F(\sum_{j=0}^{i}\xi_{j+1}(T_1+\Delta))-F(\sum_{j=0}^{i-1}\xi_{j+1}(T_1+\Delta))]/f(\sum_{j=0}^{i}\xi_{j+1}(T_1+\Delta)) \\
&= [F(\sum_{j=0}^{i}\{\xi_{j+1}(T_1)+\Delta'_{j+1}\})-F(\sum_{j=0}^{i-1}\{\xi_{j+1}(T_1)+\Delta'_{j+1}\})]/f(\sum_{j=0}^{i}\{\xi_{j+1}(T_1)+\Delta'_{j+1}\}) \\
&\geqq [F(\sum_{j=0}^{i}\xi_{j+1}(T_1))-F(\sum_{j=0}^{i-1}\xi_{j+1}(T_1)-\Delta'_{i+1})]/f(\sum_{j=0}^{i}\xi_{j+1}(T_1)) \quad \text{(by eq.(12))} \\
&\geqq [F(\sum_{j=0}^{i}\xi_{j+1}(T_1))-F(\sum_{j=0}^{i-1}\xi_{j+1}(T_1))]/f(\sum_{j=0}^{i}\xi_{j+1}(T_1)) \\
&= \xi_{i+2}(T_1) + \mu. \quad ||
\end{aligned}$$

Hence by the assumption that $f(x)$ is differentiable, we have

(14) $d\xi_i(T_1)/dT_1 \geqq 0$ for all $i = 1,\ldots,n+1$.

Lemma 2. $\lim_{T_1\to 0} \xi_2(T_1) < 0$.

Proof: Since $f(x)$ is PF_2, it is unimodal. Let m be the mode of $f(x)$. Letting $m > 0$, then $f(x)$ is non-decreasing for all $x \leqq m$. Since $T_1 \to 0$, suppose that T_1 is sufficiently small.

$$T_2 = \xi_2(T_1) = F(T_1)/f(T_1)-\mu \leqq T_1f(T_1)/f(T_1)-\mu < T_1 \to 0 \quad \text{as } T_1 \to 0.$$

When $m \leqq 0$, $f(x)$ is non-increasing and $f(0)$ is positive. Hence it is trivial. ||

When T_2 is negative, it is meaningless. So we define $T_2 = \max(\xi_2(T_1),0)$. We rewrite it $\xi_2(T_1)$. Similarly we define $\xi_i(T_1)$ $(i = 3,\ldots,n+1)$ in turn. Consequently, we have

(15) $\lim_{T_1 \to 0} \xi_{i+1}(T_1) = 0 \quad (i = 1,\ldots,n)$.

To determine a set $(T_1,\ldots,T_{n+1})$ satisfying eq.(10) and eq.(11) is equivalent to determine T_1 maximizing the following equation $p_A(n,T)$.

(16) $p_A(n,T) = A(n,T)/B(n,T)$

where

$$A(n,T) = \int_0^{\sum_{i=0}^{n}\xi_{i+1}(T)} \bar{F}(x)dx,$$

$$B(n,T) = r-(r-r_s)\bar{F}(\sum_{i=0}^{n}\xi_{i+1}(T))+\sum_{i=0}^{n}(\mu+\xi_{i+1}(T))\bar{F}(\sum_{j=0}^{i-1}\xi_{j+1}(T)),$$

and

$$\sum_{j=0}^{i-1} \xi_{j+1}(T) = 0 \quad \text{if } i = 0.$$

Then we have using eq.(10) and eq.(14),

$$dA(n,T)/dT = (\sum_{i=0}^{n} \xi'_{i+1}(T))\bar{F}(\sum_{i=0}^{n} \xi_{i+1}(T)) > 0 \quad \text{for all } T > 0,$$

$$dB(n,T)/dT = (\sum_{i=0}^{n} \xi'_{i+1}(T))[\bar{F}(\sum_{i=0}^{n-1} \xi_{i+1}(T))+(r-r_s)f(\sum_{i=0}^{n} \xi_{i+1}(T))] > 0 \quad \text{for all } T > 0,$$

and

(17) $dp_A(n,T)/dT = B'(n,T)[K(n,T)-p_A(n,T)]/B(n,T),$

where

(18) $K(n,T) = [(r-r_s)\lambda(\sum_{i=0}^{n} \xi_{i+1}(T))+\bar{F}(\sum_{i=0}^{n-1} \xi_{i+1}(T))/\bar{F}(\sum_{i=0}^{n} \xi_{i+1}(T))]^{-1}.$

It is easily verified that the second term of the right-hand side of eq.(18) is non-decreasing in T. Hereafter, we assume that $r > r_s$ and $\lambda(x)$ is strictly increasing. So, $K(n,T)$ is strictly decreasing. When $T \to 0$, $p_A(n,T) \to 0$ and $K(n,T) \to [1+(r-r_s)\lambda(0)]^{-1}$ by eq.(15). Consequently, $p_A(n,T)$ increases in a neighborhood of the origin. Since there exists at least one set $(T_1,\ldots,T_{n+1})$ maximizing eq.(4), there exists at least one T^* such that $dp_A(n,T)/dT = 0$. Hence, $p_A(n,T)$ will increase in the interval $(0,T^*)$ where T^* is the first root of the equation

(19) $K(n,T) - p_A(n,T) = 0.$

Further, at T^*, since $K(n,T)$ is strictly decreasing, $K(n,T^*+\varepsilon) < p_A(n,T^*+\varepsilon)$ for a sufficiently small $\varepsilon > 0$. This means that $p_A(n,T)$ is a strictly decreasing function at $T^*+\varepsilon$ and that T^* is a maximal point. Even if T^{**} is another root of eq.(19), $p_A(n,T)$ will decrease at $T^{**}+\varepsilon$. Thus T^{**} can not be a maximal point. This fact implies the uniqueness of T^*. But since T^* is a function of n, we need to write such as $T^*(n)$.

Thus a set $(T_1^*(n),\ldots,T_{n+1}^*(n))$ is optimal for any fixed n and the resulting maximum value of $p_A(n,T)$ is given by $[(r-r_s)\lambda(S_{n+1}^*(n))+\bar{F}(S_n^*(n))/\bar{F}(S_{n+1}^*(n))]^{-1}$, where $S_i^*(m) = T_0^*(m)+\ldots+T_i^*(m)$ and $T_0^*(m) = 0$ $(i=0,1,\ldots,m+1)$ for fixed $m \geq 0$.

Theorem 1. Assume that (i) $r > r_s$ (ii) a failure density $f(x)$ is PF_2 and differentiable and $f(x) > 0$ for all $x > 0$ (iii) the failure rate $\lambda(x)$ is strictly increasing. Then the optimum inspection policy is given by $(\hat{T}_1,\ldots,\hat{T}_{\hat{n}+1})$, where

$$\hat{n} = \{n;\ \max[(r-r_s)\lambda(S_{n+1}^*(n))+\bar{F}(S_n^*(n))/\bar{F}(S_{n+1}^*(n))]^{-1}\},$$

$$\hat{T}_1 = \hat{T}_1^*(\hat{n}),\ \ldots,\ \hat{T}_{\hat{n}+1} = \hat{T}^*_{\hat{n}+1}(\hat{n})$$

and

$T_1^*(n),\dots,T_{n+1}^*(n)$ is the unique root of the eq.(10) and eq.(11).

Then $p_A(n,T) = [(r-r_s)\lambda(\hat{S}_{\hat{n}+1}(\hat{n}))+\bar{F}(\hat{S}_{\hat{n}}(\hat{n}))/\bar{F}(\hat{S}_{\hat{n}+1}(\hat{n}))]^{-1}$, where $\hat{S}_i(m) = \hat{T}_0(m) + \hat{T}_1(m) + \dots + \hat{T}_i(m)$ and $\hat{T}_0(m) = 0$ $(i=0,1,\dots,m+1)$ for fixed $m \geqq 0$.

Example 1. Suppose that a failure distribution is a gamma distribution with parameter 2 and mean λ, i.e.,

$$\bar{F}(x) = (1+2x/\lambda)\exp[-2x/\lambda] \ , \ \lambda(x) = 4x/\lambda^2(1+2x/\lambda).$$

Suppose furthermore that $r/\lambda = 0.1$, $r_s/\lambda = 0.01$ and $\mu/\lambda = 0.001$. Then we have Table 1. Then the optimal policy is $T_1/\lambda = 0.1288$, $T_2/\lambda = 0.0693$ and $T_3/\lambda = 0.0600$ and the stationary availability is 0.9199. Similarly we obtain for $\mu/\lambda = 10^{-4}$, $T_1/\lambda = 0.1147$, $T_2/\lambda = 0.0619$, $T_3/\lambda = 0.0540$ and $p_A(\infty) = 0.9287$ and for $\mu/\lambda = 10^{-5}$, $T_1/\lambda = 0.1132$, $T_2/\lambda = 0.0611$, $T_3/\lambda = 0.0533$ and $p_A(\infty) = 0.9296$. When the system is always observed and the failure is detected as soon as it occurs, the optimum preventive maintenance time is 0.32 and the stationary availability is 0.9321 by Barlow and Proschan [2]. It seems that $p_A(\infty)$ approaches 0.9321 when μ decreases to zero.

Table 1 For each fixed n, the stationary availability and inspection periods when $r/\lambda = 0.1$, $r_s/\lambda = 0.01$ and $\mu/\lambda = 0.001$.

n	$p_A(\infty)$	$\hat{T}_1/\lambda$	$\hat{T}_2/\lambda$	$\hat{T}_3/\lambda$	$\hat{T}_4/\lambda$	$\hat{T}_5/\lambda$	$\hat{T}_6/\lambda$	$\hat{T}_7/\lambda$	...
0	0.9036	0.1794							
1	0.9084	0.1470	0.0803						
2	0.9199	0.1288	0.0693	0.0600					
3	0.9092	0.1182	0.0630	0.0542	0.0496				
4	0.9085	0.1112	0.0589	0.0505	0.0460	0.0429			
5	0.9075	0.1063	0.0561	0.0480	0.0435	0.0405	0.0382		
6	0.9056	0.0893	0.0468	0.0395	0.0354	0.0326	0.0304	0.0286	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

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Appendix

We derive the Laplace transform of the point-wise availability using the method of supplementary variables. We define the state of the system as follows:

State S_0 : system is operating after it is renewed.

State S_i : system is operating after the completion of i-th inspection. ($i = 1,\dots,n$)

State $S_{i,1}$: system is under $(i+1)$-st inspection from state S_i. ($i=0,1,\dots,n$)

State $S_{i,2}$: system is down from state S_i i.e., it is under undetected failure. ($i = 0,1,\dots,n$)

State $S_{i,3}$: system is under $(i+1)$-st inspection from state $S_{i,2}$. ($i=0,1,\dots,n$)

State $S_{i,4}$: system is under repair after the completion of $(i+1)$-st inspection. ($i = 0,1,\dots,n$)

State S_s : system is under preventive maintenance.

The state transition diagram is shown in Fig. 2. We introduce the following notations:

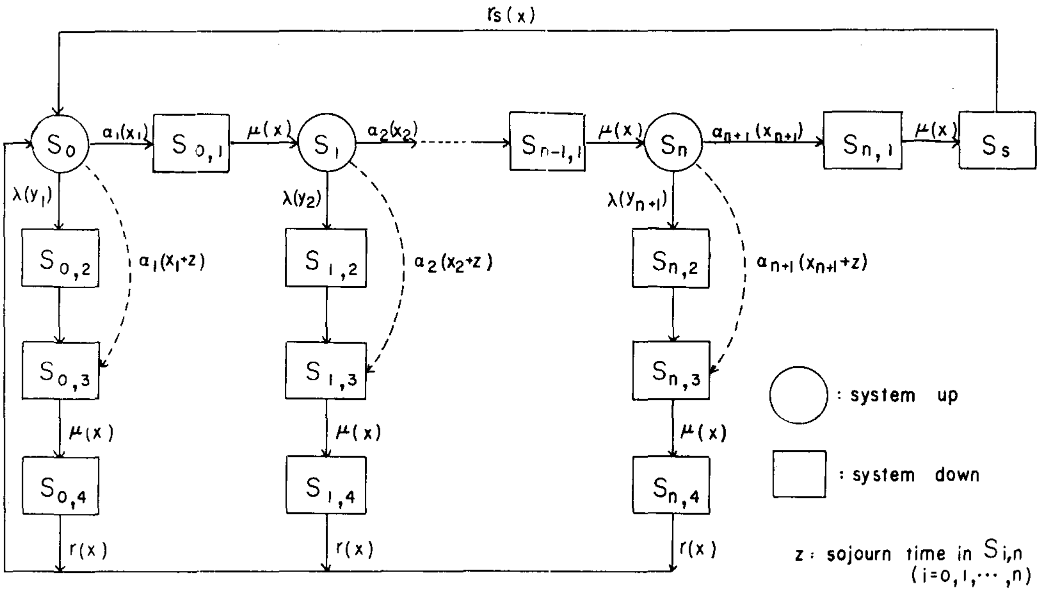

Fig. 2 The state transition diagram.

$x_i (i=1,\dots,n+1)$: system's operating time after the completion $(i-1)$-st inspection.

$y_i = y_{i-1} + x_i \ (i=1,\dots,n),\ y_0 = 0$: total operating time after the system is put in operation.

$u^*(s)$: The Laplace transform of a function $u(x)$.

$p_0(t,x)\Delta$ = Pr[at time t, system is in state S_0 and the operating time lies between x and $x + \Delta$].

$p_i(t,x_{i+1};y_i)\Delta$ = Pr[at time t, system is in state S_i and the operating time, measured from the instant at which the system has entered state S_i lies between x_{i+1} and $x_{i+1} + \Delta$ given the total operating time at which it has entered state S_i is y_i]. (i = 1,...,n)

$p_{i,1}(t,x;y_{i+1})\Delta$ = Pr[at time t, system is in state $S_{i,1}$ and the elapsed inspection time in state $S_{i,1}$ lies between x and $x+\Delta$ given the total operating time at which it has entered state $S_{i,1}$ is y_{i+1}]. (i=0,1,...,n)

$p_{i,2}(t,x;y_{i+1})\Delta$ = Pr[at time t, system is in state $S_{i,2}$ and the elapsed time in state $S_{i,2}$ lies between x and $x+\Delta$ given the total operating time at which it has entered state $S_{i,2}$ is y_{i+1}]. (i = 0,1,...,n)

$p_{i,3}(t,x)\Delta$ = Pr[at time t, system is in state $S_{i,3}$ and the elapsed inspection time lies between x and $x + \Delta$]. (i = 0,1,...,n)

$p_{i,4}(t,x)\Delta$ = Pr[at time t, system is in state $S_{i,4}$ and the elapsed repair time lies between x and $x + \Delta$]. (i = 0,1,...,n)

$p_s(t,x)\Delta$ = Pr[at time t, system is in state S_s and the elapsed preventive maintenance time lies between x and $x+\Delta$].

Keeping in view the nature of this system, the following set of partial differential equations can be set up easily:

$$[\partial/\partial t+\partial/\partial x_1+\lambda(x_1)+\alpha_1(x_1)]p_0(t,x_1) = 0,$$

$$[\partial/\partial t+\partial/\partial x_{i+1}+\lambda(x_{i+1}+y_i)+\alpha_{i+1}(x_{i+1})]p_i(t,x_{i+1};y_i) = 0 \quad (i=1,\dots,n),$$

$$[\partial/\partial t+\partial/\partial x+\mu(x)]p_{i,1}(t,x;y_{i+1}) = 0 \quad (i = 0,1,\dots,n),$$

$$\text{(A1)} \quad [\partial/\partial t+\partial/\partial x+\alpha_{i+1}(x+x_{i+1})]p_{i,2}(t,x;y_{i+1}) = 0 \quad (i = 0,1,\dots,n),$$

$$[\partial/\partial t+\partial/\partial x+\mu(x)]p_{i,3}(t,x) = 0 \quad (i = 0,1,\dots,n),$$

$$[\partial/\partial t+\partial/\partial x+r(x)]p_{i,4}(t,x) = 0 \quad (i = 0,1,\dots,n),$$

$$[\partial/\partial t+\partial/\partial x+r_s(x)]p_s(t,x) = 0,$$

where

$\alpha_i(x) = a_i(x)/\bar{A}_i(x)$, $\mu(x) = g(x)/\bar{G}(x)$, $r(x) = h(x)/\bar{H}(x)$ and

$r_s(x) = h_s(x)/\bar{H}_s(x)$.

Equations (A1) are to be solved subject to the following boundary and initial conditions:

$$p_0(t,0) = \sum_{i=0}^{n} \int_0^t r(x)p_{i,4}(t,x)dx + \int_0^t r_s(x)p_s(t,x)dx,$$

$$p_i(t,0;y_i) = \int_0^t \mu(x)p_{i-1,1}(t,x;y_i)dx \quad (i = 1,\dots,n),$$

$$p_{0,1}(t,0;y_1) = \alpha_1(x_1)p_0(t,x_1),$$
$$p_{i,1}(t,0;y_{i+1}) = \alpha_{i+1}(x_{i+1})p_i(t,x_{i+1};y_i) \quad (i = 1,\dots,n),$$
$$p_{0,2}(t,0;y_1) = \lambda(x_1)p_0(t,x_1),$$
$$p_{i,2}(t,0;y_{i+1}) = \lambda(x_{i+1}+y_i)p_i(t,x_{i+1};y_i) \quad (i = 1,\dots,n),$$
(A2)
$$p_{i,3}(t,0) = \int_0^t\cdots\int_0^{(t-y_{i+1})} \alpha_{i+1}(x+x_{i+1})p_{i,2}(t,x;y_{i+1})dxdx_{i+1}\cdots dx_1 \quad (i = 0,1,\dots,n),$$
$$p_{i,4}(t,0) = \int_0^t \mu(x)p_{i,3}(t,x)dx \quad (i = 0,1,\dots,n),$$
$$p_s(t,0) = \int_0^t\cdots\int_0^{(t-y_{n+1})} \mu(x)p_{n,1}(t,x;y_{n+1})dxdx_{n+1}\cdots dx_1,$$
and
$$p_0(0,0) = 1.$$

Taking the Laplace transforms of eqs.(A1),(A2) under the initial condition and solving, then we have after some simplification

$$p_0^*(s,x_1) = c^*(s)\exp[-sx_1]\bar{A}_1(x_1)\bar{F}(x_1),$$

$$p_i^*(s,x_{i+1};y_i) = c^*(s)g^*(s)^i\exp[-s(y_i+x_{i+1})]\bar{F}(y_i+x_{i+1})\bar{A}_{i+1}(x_{i+1}) \times \prod_{j=1}^{i} a_j(x_j) \quad (i = 1,\dots,n),$$

$$p_{i,1}^*(s,x;y_{i+1}) = c^*(s)g^*(s)^i\exp[-s(x+y_{i+1})]\bar{F}(y_{i+1})\bar{G}(x) \prod_{j=1}^{i+1} a_j(x_j) \quad (i = 0,1,\dots,n),$$

$$p_{i,2}^*(s,x;y_{i+1}) = c^*(s)g^*(s)^i\exp[-s(x+y_{i+1})]f(y_{i+1})\bar{A}_{i+1}(x+x_{i+1}) \times \prod_{j=1}^{i} a_j(x_j) \quad (i = 0,1,\dots,n),$$

$$p_{i,3}^*(s,x) = c^*(s)g^*(s)^i[a_{i+1}^*(s)\phi_i^*(s)-\phi_{i+1}^*(s)]\exp[-sx]\bar{G}(x) \quad (i=0,1,\dots,n),$$

$$p_{i,4}^*(s,x) = c^*(s)g^*(s)^{i+1}[a_{i+1}^*(s)\phi_i^*(s)-\phi_{i+1}^*(s)]\exp[-sx]\bar{H}_i(x) \quad (i = 0,1,\dots,n),$$

$$p_s^*(s,x) = c^*(s)g^*(s)^{n+1}\phi_{n+1}^*(s)\exp[-sx]\bar{H}_s(x)$$

where

$$c^*(s)=1/[1-g^*(s)^{n+1}h_s^*(s)\phi_{n+1}^*(s)-\sum_{i=0}^{n} g^*(s)^{i+1}h^*(s)\{a_{i+1}^*(s)\phi_i^*(s)-\phi_{i+1}^*(s)\}],$$

and

$$\prod_{j=1}^{i} a_j(x_j) = 1 \quad \text{if } i = 0.$$

By $p_A^*(s) = \int_0^\infty p_0^*(s,x_1)dx_1 + \sum_{i=1}^{n} \int_0^\infty\cdots\int_0^\infty p_i^*(s,x_{i+1};y_i)dx_{i+1}\cdots dx_1$, we have eq.(1).

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