Journal of the Operations Research
Society of Japan
Vol. 26, No. 1, March 1983

AN ALGORITHMIC SOLUTION TO THE M/PH/c QUEUE WITH BATCH ARRIVALS

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(Received April 28, 1982; Revised October 13, 1982)

Abstract This paper discusses algorithms for deriving the steady state features of the M/PH/c queue with batch arrivals. Many characteristic quantities such as the mean and variance of queue length in continuous time, the mean waiting time, the waiting probability, etc. are obtained in computationally tractable form. Numerical examples show that the effects of changing various parameters of queueing model may be examined at a small computational cost.

1. Introduction

Batch arrivals constitute an important class of input processes in the theory of queues. A queueing system is generally represented by $GI^X/G/c$, where GI, G, c and X denote an interarrival time distribution, a service time distribution, the number of servers and arrival group of random size, respectively. Especially, multiserver queues with batch arrivals have been investigated by a number of authors. Kabak [9,10] obtained the steady state results of $M^X/M/c$. Cromie et al. [6] extended and corrected Kabak's results. They also derived how to calculate percentiles or fractiles for the queue length distribution and the waiting time distribution. Holman et al. [8] obtained the moments of queue length of $E_k^X/M/c$ queue using the roots of the characteristic equation. Baily and Neuts [2] studied the $GI^X/M/c$ queue with bounded batch arrivals. They discussed algorithms for the computation of the steady state features and obtained queue length densities at prior to arrivals and at arbitrary times in a modified matrix-geometric form. But all of the above authors assumed that the service time distribution was exponential.

In this paper, we shall discuss algorithms for the computation of the steady state features of the $M^X/PH/c$ queue. PH denotes the phase type distribution introduced by Neuts [12]. The phase type distribution is any continuous probability distribution on $[0,\infty)$ which is obtainable as the distribution of the absorption time in a continuous time finite state Markov chain with a single absorbing state. The class of such distributions includes many well-known distribution such as generalized Erlang and hyper-exponential (i.e., a mixture of a finite number of exponential) distributions.

The random variable X represents the batch size, and obeys the probability distribution with finite support on positive integers or geometric distribution. This restriction is not essential. But, for the simplicity of algorithms, we shall only treat these cases.

Recently Lucantoni [11] derived an algorithm for computing the stationary probability vector of an infinite state Markov chain whose transition matrix has a block-partitioned structure. We shall first show that the states of $M^X/PH/c$ queue have this property by devising the representation of the states. Based on this algorithm, we shall obtain many characteristic quantities such as the mean and variance of queue length in continuous time, the mean waiting time, the waiting probability, etc. These quantities are obtained in computationally tractable forms. We shall also show various numerical examples and obtain qualitative insight into the effect of varying the parameter values. This may be done at a small fraction of the cost for the Monte Carlo simulation of the models.

2. The stationary distributions of the $M^X/PH/c$ queue

Consider the $M^X/PH/c$ queueing model, in which customers arrive in batches at a c-server unit. The c-servers have independent phase type distributed processing times with a representation $(\underline{\alpha},T)$, where T is a matrix of order M. The times between batch arrivals are independent exponentially distributed with common arrival rate λ. Consecutive batch sizes are independent and obey the probability distribution with finite support on positive integers or geometric distribution. This probability distribution is represented by $\{g_i\}_{i=1}^{\infty}$ with mean $g = \sum_{i=1}^{\infty} i g_i$.

In this section, we discuss the stationary distributions of the number of customers in the system at an arbitrary time t. In order to obtain the stationary distribution of the $M^X/PH/c$ queue, we shall consider the continuous time Markov chain which represents the transitions of the states of $M^X/PH/c$ queue at an arbitrary time. In order to define the states of this Markov chain, it is convenient to define that the index i $(i \geq 0)$ denotes the total number of customers in the system and j_k $(1 \leq k \leq M)$ the total number of customers in the k-th phase of service. The the state of the system can be represented by an ordered $(M+1)$-tuple of nonnegative integers

$$(2.1) \qquad (i; j_1, \ldots, j_M).$$

Let $\underline{i}$ be the set of all possible states such that the total number of customers in the system is equal to i, and s_i the number of states in $\underline{i}$. Since $\sum_{k=1}^{M} j_k = min(c,i)$, s_i is given by

$$(2.2) \qquad s_i = \binom{min(c,i) + M - 1}{M}.$$

We will number the states in $\underline{i}$ by a suitable rule and refer them by pairs

$$(2.3) \qquad (i;m), \quad m = 1,2,\ldots,s_i\ , \quad i = 0,1,2,\ldots\ .$$

Therefore the infinitesimal generator of this system is given by the matrix

$$(2.4) \qquad Q = \begin{array}{c|cccccccccc}
 & \underline{0} & \underline{1} & \underline{2} & \cdots & \underline{c-1} & \underline{c} & \underline{c+1} & \underline{c+2} & \cdots \\
\hline
\underline{0} & \tilde{C}_0 & \tilde{D}_{01} & \tilde{D}_{02} & \cdots & \tilde{D}_{0,c-1} & g_c\tilde{D}_0 & g_{c+1}\tilde{D}_0 & g_{c+2}\tilde{D}_0 & \cdots \\
\underline{1} & \tilde{B}_1 & \tilde{C}_1 & \tilde{D}_{12} & \cdots & \tilde{D}_{1,c-1} & g_{c-1}\tilde{D}_1 & g_c\tilde{D}_1 & g_{c+1}\tilde{D}_1 & \cdots \\
\underline{2} & 0 & \tilde{B}_2 & \tilde{C}_2 & \cdots & \tilde{D}_{2,c-1} & g_{c-2}\tilde{D}_2 & g_{c-1}\tilde{D}_2 & g_c\tilde{D}_2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
\underline{c-1} & 0 & 0 & 0 & \cdots & \tilde{C}_c & g_1\tilde{D}_{c-1} & g_2\tilde{D}_{c-1} & g_3\tilde{D}_{c-1} & \cdots \\
\underline{c} & 0 & 0 & 0 & \cdots & \tilde{B}_c & \tilde{C} & g_1\tilde{D} & g_2\tilde{D} & \cdots \\
\underline{c+1} & 0 & 0 & 0 & \cdots & 0 & \tilde{B} & \tilde{C} & g_1\tilde{D} & \cdots \\
\underline{c+2} & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{B} & \tilde{C} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots &
\end{array}\ .$$

We denote the invariant probability vector of Q by $\underline{x}$, which we may write in the partitioned form $\{\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_i, \ldots\}$, where $\underline{x}_i$ $(i \geq 0)$ are s_i-vectors.

Let τ be the real number such that

$$(2.5) \qquad \tau = max\ [\max_{1 \leq i \leq c-1} \ \max_{1 \leq j \leq s_i}\{-(C_i)_{jj}\}\ ,\ \max_{1 \leq j \leq s_c}\{-(C)_{jj}\}] > 0\ ,$$

and I the identity matrix, then the discrete time Markov chain with transition matrix

$$
(2.6)\qquad P = \begin{array}{c} \\ \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{c-1} \\ \underline{c} \\ \underline{c+1} \\ \underline{c+2} \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} \underline{0} & \underline{1} & \underline{2} & \cdots & \underline{c-1} & \underline{c} & \underline{c+1} & \underline{c+2} & \cdots \end{array} \\
\left(\begin{array}{ccccccccc}
C_0 & D_{01} & D_{02} & \cdots & D_{0,c-1} & g_cD_0 & g_{c+1}D_0 & g_{c+2}D_0 & \cdots \\
B_1 & C_1 & D_{12} & \cdots & D_{1,c-1} & g_{c-1}D_1 & g_cD_1 & g_{c+1}D_1 & \cdots \\
0 & B_2 & C_2 & \cdots & D_{2,c-1} & g_{c-2}D_1 & g_{c-1}D_2 & g_cD_2 & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & C_{c-1} & g_1C_{c-1} & g_2D_{c-1} & g_3D_{c-1} & \cdots \\
0 & 0 & 0 & \cdots & B_c & C & g_1D & g_2D & \cdots \\
0 & 0 & 0 & \cdots & 0 & B & C & g_1D & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & B & C & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots &
\end{array}\right)
\end{array},
$$

where $C_i = I + \tau^{-1}\tilde{C}_i$ $(0 \leqq i \leqq c-1)$, $C = I + \tau^{-1}\tilde{C}$, $B_i = \tau^{-1}\tilde{B}_i$ $(1 \leqq i \leqq c)$, $B = \tau^{-1}\tilde{B}$, $D_{ij} = \tau^{-1}\tilde{D}_{ij}$ $(0 \leqq i \leqq c-2,\ i+1 \leqq j \leqq c-1)$, $D_i = \tau^{-1}\tilde{D}_i$ $(0 \leqq i \leqq c-1)$ and $D = \tau^{-1}\tilde{D}$, has the same invariant probability vector as the continuous time Markov chain whose infinitesimal generator is Q. In this paper, we shall analyze about the discrete time Markov chain with transition matrix P instead of the continuous time Markov chain with infinitesimal generator Q.

Recently Lucantoni [11] derived an algorithm for computing the stationary probability vector of an infinite state Markov chain whose transition matrix has a block-partitioned structure. The transition matrix (2.6) has this property by devising the representation of the states in the following way.

Let us define dome matrices and vectors given by

$$\underline{y}_0 = (\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{c-1})$$

$$
(2.7)\qquad F = \begin{pmatrix}
C_0 & D_{01} & D_{02} & \cdots & D_{0,c-1} \\
B_1 & C_1 & D_{12} & \cdots & D_{1,c-1} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & C_{c-1}
\end{pmatrix}, \quad
G_i = \begin{pmatrix}
g_{c+i-1}D_0 \\
g_{c+i-2}D_1 \\
\vdots \\
g_iD_{c-1}
\end{pmatrix} \quad (i \geqq 1),
$$

$$E = (0, \ldots, 0, B_c) .$$

Then the stochastic matrix P is written by

$$P = \begin{pmatrix} F & G_1 & G_2 & G_3 & \cdots \\ E & C & g_1 D & g_2 D & \cdots \\ 0 & B & C & g_1 D & \cdots \\ 0 & 0 & B & C & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}. \tag{2.8}$$

Using the stochastic matrix P, we state below a practical algorithm to calculate the stationary distribution of the $M^X/PH/c$ queue by improving the algorithm discussed in Lucantoni [11].

A practical alogirithm

Step 1. Calculate the sequence of matrices $\{H(n),\ n \geq 0\}$ by

$$H(0) = B, \quad H(n+1) = B + CH(n) + \sum_{i=1}^{\infty} g_i D[H(n)]^{i+1}. \tag{2.9}$$

By above successive substitutions, the sequence $\{H(n)\}$ converges to a matrix H, which satisfies

$$H = B + CH + \sum_{i=1}^{\infty} g_i DH^{i+1}. \tag{2.10}$$

Especially the batch size distribution is the probability distribution with finite support on positive integers, we call this Case 1, where $\{g_i\}_{i=1}^{K}$ with mean $g = \sum_{i=1}^{K} ig_i$, then (2.9) reduces to

$$H(0) = B, \quad H(n+1) = B + CH(n) + \sum_{i=1}^{K} g_i D[H(n)]^{i+1}, \tag{2.11}$$

and the batch size distribution is geometric, we call this Case 2, where $g_i = (1-p)p^{i-1}$ $(i \geq 1)$ with mean $g = 1/(1-p)$, then (2.9) reduces to

$$H(0) = B, \quad H(n+1) = B + CH(n) + (1-p)D[I - pH(n)]^{-1}[H(n)]^2. \tag{2.12}$$

Step 2. Calculation of the vector $\underline{h}$ of stationary probabilities corresponding to the stochastic matrix H.

Step 3. We introduce the square matrix $\tilde{H}$ of order s_c, whose rows are all identical and equal to the vector $\underline{h}$, and the vector $\underline{\beta}$ defined by

$$\underline{\beta} = C\underline{e} + \sum_{i=2}^{\infty} ig_{i-1} D\underline{e}$$

$$(2.13)\qquad = C\underline{e} + \sum_{i=2}^{K+1} ig_{i-1}D\underline{e} \qquad \text{for Case 1}$$

$$= C\underline{e} + \frac{2-p}{1-p}D\underline{e} \qquad \text{for Case 2.}$$

Calculate the vector $\underline{\mu}$ defined by

$$(2.14)\qquad \underline{\mu} = (I - H + \tilde{H})[I - A + \tilde{H} - \Delta(\underline{\beta})\tilde{H}]^{-1}\underline{e}\ ,$$

where $A = B + C + D$ and $\Delta(\underline{\beta})$ is the diagonal matrix of order s_c, with diagonal entries $\beta_1,\beta_2,\ldots,\beta_{s_c}$.

Step 4. Calculate some matrices defined by

$$N = [I - C - \sum_{i=1}^{\infty} g_i DH^i]^{-1}E$$

$$(2.15)\qquad = [I - C - \sum_{i=1}^{K} g_i DH^i]^{-1}E \qquad \text{for Case 1}$$

$$= [I - C - (1-p)D(I - pH)^{-1}H]^{-1}E \qquad \text{for Case 2,}$$

$$L = F + \sum_{i=1}^{\infty} G_i H^{i-1}N$$

$$(2.16)\qquad = F + \sum_{i=1}^{K} G_i H^{i-1}N \qquad \text{for Case 1}$$

$$= F + G_1(I - pH)^{-1}N \qquad \text{for Case 2,}$$

$$K = E(I - F)^{-1}\sum_{i=1}^{\infty} G_i H^{i-1} + C + \sum_{i=1}^{\infty} g_i DH^i$$

$$(2.17)\qquad = E(I - F)^{-1}\sum_{i=1}^{K} G_i H^{i-1} + C + \sum_{i=1}^{K} g_i DH^i \qquad \text{for Case 1}$$

$$= E(I - F)^{-1}G_1(I - pH)^{-1} + C + (1-p)D(I - pH)^{-1}H \qquad \text{for Case 2.}$$

Step 5. The matrices N, L and K are stochastic. (see Lucantoni [11]) Calculate the invariant probability vectors $\underline{d}$ and $\underline{\kappa}$ of the matrices L and K.

Step 6. Calculate some vectors defined by

$$\underline{r}^* = [I - C - \sum_{i=1}^{\infty} g_i DH^i]^{-1}[\underline{e} + \{A - B - C - \sum_{i=1}^{\infty} g_i DH^i + \sum_{i=1}^{\infty} ig_i D\tilde{H}\}$$

$$(I - H - \tilde{H})^{-1}\underline{\mu}]$$

$$(2.18) \qquad = [I - C - \sum_{i=1}^{K} g_i DH^i]^{-1}[\underline{e} + \{A - B - C - \sum_{i=1}^{K} g_i DH^i + \sum_{i=1}^{K} ig_i D\tilde{H}\}(I - H + \tilde{H})^{-1}\underline{\mu}] \quad \text{for Case 1}$$

$$= [I - C - (1-p)D(I - pH)^{-1}H]^{-1}[\underline{e} + \{A - B - C - (1-p)D(I - pH)^{-1}H\} + (1-p)^{-1}D\tilde{H}\}(I - H + \tilde{H})^{-1}\underline{\mu}] \quad \text{for Case 2,}$$

$$\underline{d}^* = \underline{e} + \sum_{i=1}^{\infty} G_i H^{i-1}\underline{r}^* + [\sum_{i=1}^{\infty} G_i - \sum_{i=1}^{\infty} G_i H^{i-1} + \sum_{i=2}^{\infty} (i-1)G_i \tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu}$$

$$(2.19) \qquad = \underline{e} + \sum_{i=1}^{K} G_i H^{i-1}\underline{r}^* + [\sum_{i=1}^{K} G_i - \sum_{i=1}^{K} G_i H^{i-1} + \sum_{i=2}^{K} (i-1)G_i \tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu} \quad \text{for Case 1}$$

$$= \underline{e} + G_1(I - pH)^{-1}\underline{r}^* + [(1-p)^{-1}G_1 - G_1(I - pH)^{-1} + (1-p)^{-2}pG_1\tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu} \quad \text{for Case 2}$$

and

$$\underline{\kappa}^* = \underline{e} + E(I - F)^{-1}\underline{e} + [E(I - F)^{-1}\{\sum_{i=1}^{\infty} G_i - \sum_{i=1}^{\infty} G_i H^{i-1} + \sum_{i=2}^{\infty} (i-1)G_i H\} + A - B - C - \sum_{i=1}^{\infty} g_i DH^i + \sum_{i=1}^{\infty} ig_i D\tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu}$$

$$(2.20) \qquad = \underline{e} + E(I - F)^{-1}\underline{e} + [E(I - F)^{-1}\{\sum_{i=1}^{K} G_i - \sum_{i=1}^{K} G_i H^{i-1} + \sum_{i=2}^{K} (i-1)G_i H\} + A - B - C - \sum_{i=1}^{K} g_i DH^i + \sum_{i=1}^{K} ig_i D\tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu} \quad \text{for Case 1}$$

$$= \underline{e} + E(I - F)^{-1} + [E(I - F)^{-1}\{(1-p)^{-1}G_1 - G_1(I - pH)^{-1} + (1-p)^{-2}pG_1\tilde{H}\} + A - B - C - (1-p)D(I - pH)^{-1}H + (1-p)^{-1}D\tilde{H}](I - H + \tilde{H})^{-1}\underline{\mu} \quad \text{for Case 2.}$$

Step 7. The stationary probability vectors $\underline{y}_0$ and $\underline{x}_c$ are given by

$$(2.21) \qquad \underline{y}_0 = \frac{\underline{d}}{\underline{d}\underline{d}^*}, \qquad \underline{x}_c = \frac{\underline{\kappa}}{\underline{\kappa}\underline{\kappa}^*}.$$

Using this algorithm, we can obtain the stationary probability vectors $\underline{y}_0$ and $\underline{x}_c$.

3. Moments of queue Length

In this section, we obtained the stationary probability vectors $\underline{y}_0$ and $\underline{x}_c$. Using these vectors, we shall obtain the first two moments of the queue length.

Let us recall the stochastic matrix (2.8):

$$P = \begin{pmatrix} F & G_1 & G_2 & G_3 & \cdots \\ E & C & g_1D & g_2D & \cdots \\ 0 & B & C & g_1D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The steady state equations with respect to P are given by

$$\text{(3.1)} \quad \begin{aligned} \underline{y}_0 &= \underline{y}_0 F + \underline{x}_c E \\ \underline{x}_c &= \underline{y}_0 G_1 + \underline{x}_c C + \underline{x}_{c+1} B \\ \underline{x}_{c+i} &= \underline{y}_0 G_{i+1} + \sum_{j=1}^{i} \underline{x}_{c+j-1} g_{i-j+1} D + \underline{x}_{c+i} C + \underline{x}_{c+i+1} B \quad (i \geqq 1). \end{aligned}$$

We introduce the vector generating function $\underline{X}(z)$ for $0 \leqq z \leqq 1$ in the following way.

$$\text{(3.2)} \quad \underline{X}(z) = \sum_{i=c}^{\infty} \underline{x}_i z^{i-c}.$$

Multiplying both hands in (3.1) by z^i and summing over $i \geqq 1$, we can easily obtain that $\underline{X}(z)$ satisfies the equation

$$\text{(3.3)} \quad \begin{aligned} \underline{X}(z) &= \underline{y}_0 \sum_{i=1}^{\infty} G_i z^{i-1} + \sum_{i=1}^{\infty} \sum_{j=1}^{i} \underline{x}_{c+j-1} g_{i-j+1} D z^i + \sum_{i=0}^{\infty} \underline{x}_{c+i} C z^i \\ &\quad + \sum_{i=1}^{\infty} \underline{x}_{c+i} B z^{i-1} \\ &= \underline{y}_0 \sum_{i=1}^{\infty} G_i z^{i-1} + \sum_{j=1}^{\infty} \underline{x}_{c+j-1} z^{j-1} \sum_{i=j}^{\infty} g_{i-j+1} D z^{i-j+1} + \sum_{i=0}^{\infty} \underline{x}_{c+i} z^i C \\ &\quad + \sum_{i=0}^{\infty} \underline{x}_{c+i} B z^{i-1} - z^{-1} \underline{x}_c B \\ &= \underline{y}_0 \sum_{i=1}^{\infty} G_i z^{i-1} + z^{-1} \underline{X}(z) A(z) - z^{-1} \underline{x}_c B , \end{aligned}$$

where $A(z) = B + Cz + \sum_{i=1}^{\infty} g_i D z^{i+1}$.

In deriving the moments of queue length, some explicit expressions of the derivatives of the Perron-Frobenius eigenvalues and the associated eigenvectors of the $A(z)$ may be necessary.

For $0 \leqq z \leqq 1$, the matrix $A(z)$ has the Perron-Frobenius eigenvalue $\delta(z)$ uniquely. Let $\underline{u}(z)$ and $\underline{v}(z)$ be the corresponding right and left eigenvectors such as

$$[A(z) - \delta(z)I]\underline{u}(z) = \underline{v}(z)[A(z) - \delta(z)I] = 0. \tag{3.4}$$

The following relations are assumed to be hold.

$$\underline{v}(z)\underline{u}(z) = \underline{v}(z)\underline{e} = 1 \ , \quad \underline{v}(1) = \underline{\pi} \ , \quad \text{and} \quad \underline{u}(1) = \underline{e} \ , \tag{3.5}$$

where $\underline{\pi}$ is the steady state vector of the stochastic matrix $A(1)$, and $\underline{e}$ is the column vector with all entries equal to 1. We denote by $A^{(j)}(z)$ the matrix obtained by differentiating j times each entry of $A(z)$.

We have to prepare the following.

Proposition 3.1. [11] The derivatives $\delta^{(j)}(1)$, $\underline{u}^{(j)}(1)$, $\underline{v}^{(j)}(1)$, $j \geqq 0$, may be computed recursively for each j. These recursion formulae are follows.

$$\begin{aligned} &\delta^{(0)}(1) = 1 \ , \quad \underline{u}^{(0)}(1) = \underline{e} \ , \quad \underline{v}^{(0)}(1) = \underline{\pi} \\ &\delta^{(1)}(1) = \underline{\pi}A^{(1)}(1)\underline{e} \\ &\underline{u}^{(1)}(1) = [I - A(1) + \Pi]^{-1}[A^{(1)}(1) - \delta^{(1)}(1)I]\underline{e} \\ &\underline{v}^{(1)}(1) = \underline{\pi}[A^{(1)}(1) - \delta^{(1)}(1)I][I - A(1) + \Pi]^{-1} \end{aligned} \tag{3.6}$$

and for $j \geqq 2$

$$\begin{aligned} \delta^{(j)}(1) &= \sum_{i=1}^{j}\binom{j}{i}\underline{\pi}A^{(i)}(1)\underline{u}^{(j-i)}(1) - \sum_{i=1}^{j-1}\delta^{(i)}(1)\underline{\pi}\underline{u}^{(j-i)}(1) \\ \underline{u}^{(j)}(1) &= [I - A(1) + \Pi]^{-1}\sum_{i=1}^{j}\binom{j}{i}[A^{(i)}(1) - \delta^{(i)}(1)I]\underline{u}^{(j-i)}(1) \\ &\quad - [\sum_{i=1}^{j-1}\binom{j}{i}\underline{v}^{(i)}(1)\underline{A}^{(j-i)}(1)]\underline{e} \\ \underline{v}^{(j)}(1) &= \sum_{i=0}^{j-1}\binom{j}{i}\underline{v}^{(i)}(1)[A^{(j-i)}(1) - \delta^{(j-i)}(1)I][I - A(1) + \Pi]^{-1}, \end{aligned} \tag{3.7}$$

where Π is the square matrix of order s_c, and each row of it is $\underline{\pi}$. □

We want to derive the first two moments of the queue size, i.e., we shall deduce the formulae for computing $\underline{X}^{(1)}(1)\underline{e}$ and $\underline{X}^{(2)}(1)\underline{e}$. Let us recall the equation (3.3):

$$\underline{X}(z) = \underline{y}_0 \sum_{i=1}^{\infty} G_i z^{i-1} + z^{-1}\underline{X}(z)A(z) - z^{-1}\underline{x}_c B .$$

Rearranging this equation, we have

$$(3.8) \qquad \underline{X}(z)[zI - A(z)] = \underline{y}_0 \sum_{i=1}^{\infty} G_i z^i - \underline{x}_c B .$$

If we let z tend to 1, we get

$$(3.9) \qquad \underline{X}(1)[I - A(1)] = \underline{y}_0 \sum_{i=1}^{\infty} G_i - \underline{x}_c B .$$

Adding $\underline{X}(1)\Pi = (\underline{X}(1)\underline{e})\underline{\pi} = (1 - \underline{y}_0\underline{e})\underline{\pi}$ to both sides of equation (3.9) and recognizing that $\underline{\pi}[I - A(1) + \Pi]^{-1} = \underline{\pi}$, we have

$$(3.10) \qquad \underline{X}(1) = [\underline{y}_0 \sum_{i=1}^{\infty} G_i - \underline{x}_c B][I - A(1) + \Pi]^{-1} + (1 - \underline{y}_0\underline{e})\underline{\pi} .$$

Differentiating (3.8) once with respect to z yields

$$(3.11) \qquad \underline{X}^{(1)}(z)[zI - A(z)] + \underline{X}(z)[I - A^{(1)}(z)] = \underline{y}_0 \sum_{i=1}^{\infty} iG_i z^{i-1}.$$

Hence as $z \to 1$, we get

$$(3.12) \qquad \underline{X}^{(1)}(1)[I - A(1)] + \underline{X}(1)[I - A^{(1)}(1)] = \underline{y}_0 \sum_{i=1}^{\infty} iG_i .$$

We note that $I - A(1)$ is singular but $I - A(1) + \Pi$ is nonsingular and $\underline{X}^{(1)}(1)\Pi = (\underline{X}^{(1)}(1)\underline{e})\underline{\pi}$. Therefore (3.12) becomes

$$(3.13) \qquad \underline{X}^{(1)}(1) = [-\underline{X}(1)\{I - C - \sum_{i=1}^{\infty} (i+1)g_i D\} + \underline{y}_0 \sum_{i=1}^{\infty} iG_i][I - A(1) + \Pi]^{-1}$$
$$+ (\underline{X}^{(1)}(1)\underline{e})\underline{\pi} .$$

Using (3.10) and (3.13), we have the next theorem.

Theorem 3.2. The first and second factorial moments of the queue size are given by

$$(3.14) \qquad \underline{X}^{(1)}(1)\underline{e} = -\underline{X}(1)\underline{u}^{(1)}(1) + \frac{1}{2[1-\delta^{(1)}(1)]} [\delta^{(2)}(1)\underline{X}(1)\underline{e}$$
$$+ \underline{y}_0 \sum_{i=1}^{\infty} i(i-1)G_i\underline{e} + 2\underline{y}_0 \sum_{i=1}^{\infty} iG_i\underline{u}^{(1)}(1) + \underline{y}_0 \sum_{i=1}^{\infty} G_i\underline{u}^{(2)}(1)$$
$$- \underline{x}_c B\underline{u}^{(2)}(1)],$$

$$(3.15) \qquad \underline{X}^{(2)}(1)\underline{e} = -\underline{X}(1)\underline{u}^{(2)}(1) - 2\underline{X}^{(1)}(1)\underline{u}^{(1)}(1)$$

$$+ \frac{1}{3[1-\delta^{(1)}(1)]}[3\delta^{(2)}(1)\underline{x}(1)\underline{e} + 3\delta^{(2)}(1)\underline{x}(1)\underline{u}^{(1)}(1)$$

$$+ \delta^{(3)}(1)\underline{x}(1)\underline{e} + \underline{y}_0 \sum_{i=1}^{\infty} i(i-1)(i-2)G_i\underline{e} +$$

$$+ 3\underline{y}_0 \sum_{i=1}^{\infty} i(i-1)G_i\underline{u}^{(1)}(1) + 3\underline{y}_0 \sum_{i=1}^{\infty} iG_i\underline{u}^{(2)}(1)$$

$$+ \underline{y}_0 \sum_{i=1}^{\infty} G_i\underline{u}^{(3)}(1) - \underline{x}_c B\underline{u}^{(3)}(1)].$$

Proof: Multiplying (3.8) on the right by $\underline{u}(z)$, we get

$$(3.16) \qquad [z - \delta(z)]\underline{x}(z)\underline{u}(z) = \underline{y}_0 \sum_{i=1}^{\infty} G_i z^i \underline{u}(z) - \underline{x}_c B\underline{u}(z).$$

Differentiating this with respect to z and rearranging the results,

$$(3.17) \qquad \underline{x}^{(1)}(z)\underline{u}(z) = -\underline{x}(z)\underline{u}^{(1)}(z) + \frac{1}{z-\delta(z)}[-\{1 - \delta^{(1)}(z)\}\underline{x}(z)\underline{u}(z)$$

$$+ \underline{y}_0 \sum_{i=1}^{\infty} iG_i z^{i-1}\underline{u}(z) + \underline{y}_0 \sum_{i=1}^{\infty} G_i z^i \underline{u}^{(1)}(z) - \underline{x}_c B\underline{u}^{(1)}(z)].$$

Letting $z \to 1$ and using L'Hospital's rule, we have (3.14) after some tedious computations.

Differentiating (3.16) twice with respect to z and rearranging the terms, we get

$$(3.18) \qquad \underline{x}^{(2)}(z)\underline{u}(z) = -\underline{x}(z)\underline{u}^{(2)}(z) - 2\underline{x}^{(1)}(z)\underline{u}^{(1)}(z)$$

$$+ \frac{1}{z-\delta(z)}[\delta^{(2)}(z)\underline{x}(z)\underline{u}(z) - 2\{1 - \delta^{(1)}(z)\}\underline{x}^{(1)}(z)\underline{u}(z)$$

$$- 2\{1 - \delta^{(1)}(z)\}\underline{x}(z)\underline{u}^{(1)}(z) + \underline{y}_0 \sum_{i=1}^{\infty} i(i-1)G_i z^{i-2}\underline{u}(z)$$

$$+ 2\underline{y}_0 \sum_{i=1}^{\infty} iG_i z^{i-1}\underline{u}^{(1)}(z) + \underline{y}_0 \sum_{i=1}^{\infty} G_i z^i \underline{u}^{(2)}(z) - \underline{x}_c B\underline{u}^{(2)}(z)].$$

Letting $z \to 1$ and using L'Hospital's rule, we have (3.15) after tedious computations. □

Especially, we have the next corollary for Case 1 and Case 2 defined in Section 2.

Corollary 3.3. For Case 1,

$$(3.19) \qquad \underline{x}^{(1)}(1)\underline{e} = -\underline{x}(1)\underline{u}^{(1)}(1) + \frac{1}{2[1-\delta^{(1)}(1)]}[\delta^{(2)}(1)\underline{x}(1)\underline{e}$$

$$+ \underline{y}_0 \sum_{i=1}^{K} i(i-1)G_i\underline{e} + 2\underline{y}_0 \sum_{i=1}^{K} iG_i\underline{u}^{(1)}(1) + \underline{y}_0 \sum_{i=1}^{K} G_i\underline{u}^{(2)}(1)$$

$$- \underline{x}_c B\underline{u}^{(2)}(1)],$$

(3.20) $$\underline{x}^{(2)}(1)\underline{e} = -\underline{x}(1)\underline{u}^{(2)}(1) - 2\underline{x}^{(1)}(1)\underline{u}^{(1)}(1) + \frac{1}{3[1-\delta^{(1)}(1)]}[3\delta^{(2)}(1)\underline{x}^{(1)}(1)\underline{e} + 3\delta^{(2)}(1)\underline{x}(1)\underline{u}^{(1)}(1) + \delta^{(3)}(1)\underline{x}(1)\underline{e} + \underline{y}_0 \sum_{i=1}^{K} i(i-1)(i-2)G_i\underline{e} + 3\underline{y}_0 \sum_{i=1}^{K} i(i-1)G_i\underline{u}^{(1)}(1) + 3\underline{y}_0 \sum_{i=1}^{K} iG_i\underline{u}^{(2)}(1) + \underline{y}_0 \sum_{i=1}^{K} G_i\underline{u}^{(3)}(1) - \underline{x}_c B\underline{u}^{(3)}(1)].$$

For Case 2,

(3.21) $$\underline{x}^{(1)}(1)\underline{e} = -\underline{x}(1)\underline{u}^{(1)}(1) + \frac{1}{2[1-\delta^{(1)}(1)]}[\delta^{(2)}(1)\underline{x}(1)\underline{e} + 2p(1-p)^{-3}\underline{y}_0G_1\underline{e} + 2(1-p)^{-2}\underline{y}_0G_1\underline{u}^{(1)}(1) + (1-p)^{-1}\underline{y}_0G_1\underline{u}^{(2)}(1) - \underline{x}_c B\underline{u}^{(2)}(1)],$$

(3.22) $$\underline{x}^{(2)}(1)\underline{e} = -\underline{x}(1)\underline{u}^{(2)}(1) - 2\underline{x}^{(1)}(1)\underline{u}^{(1)}(1) + \frac{1}{3[1-\delta^{(1)}(1)]}[3\delta^{(2)}(1)\underline{x}^{(1)}(1)\underline{e} + 3\delta^{(2)}\underline{x}(1)\underline{u}^{(1)}(1) + \delta^{(3)}(1)\underline{x}(1)\underline{e} + 6p^2(1-p)^{-4}\underline{y}_0G_1\underline{e} + 6p(1-p)^{-3}\underline{y}_0G_1\underline{u}^{(1)}(1) + 3(1-p)^{-2}\underline{y}_0G_1\underline{u}^{(2)}(1) + (1-p)^{-1}\underline{y}_0G_1\underline{u}^{(3)}(1) - \underline{x}_c B\underline{u}^{(3)}(1)].$$

□

The higher factorial moments of the queue size can be computed in the same manner but the formulae become uninspiringly complicated. Thus they will not be shown here.

Burke [5] shows, using a result from renewal theory, that the probability of an arbitrary customer being in the n-th position is given by

(3.23) $$r_n = \frac{1}{g} \sum_{i=n}^{\infty} g_i = \frac{1}{g}(1 - \sum_{i=1}^{n-1} g_i) .$$

The probability that a randomly chosen customer is in the k-th position in line (including the ones being served) immediately following an arrival is equal to

$$(3.24)\qquad \hat{p}_k = P(k\text{-th in line}) = \sum_{i=0}^{k-1} \underline{x}_i\underline{e}r_{k-i}\ .$$

Let W_1 snd W be the time until the first customer of the batch and an arbitrary customer of the batch enter the service, respectively. Using (3.23) and (3.24), we have

$$(3.25)\qquad P(W_1 > 0) = \sum_{i=c}^{\infty} \underline{x}_i\underline{e} = 1 - \sum_{i=0}^{c-1} \underline{x}_i\underline{e}\ ,$$

$$(3.26)\qquad P(W > 0) = \sum_{i=c+1}^{\infty} \hat{p}_i = 1 - \sum_{i=0}^{c} \hat{p}_i\ .$$

4. Numerical results

Using the results obtained in section 2 and 3, we can get many characteristic quantities such as mean queue length and mean system length in continuous time, variance of queue length and variance of system length in continuous time, mean waiting time, mean system time, and the waiting probability of the first customer of the batch and an arbitrary customer of the batch. For example, mean waiting time is obtained by using (3.8) and Little's formula. We calculated mean waiting times for various service time distribution, the number of servers, mean batch size, and arrival batch type (constant or geometric).

Figure 1 gives the mean waiting times for $g = 4$ and constant and geometric batch sizes. Note that the geometric distribution leads to longer mean waiting time than the constant batch sizes. Since geometric distribution has higher variance than constant, this means that the higher the variance of the batch size, the longer the mean waiting time, which is an intuitively appealing result. When c is larger than 1, it was also found that, in the heavy traffic case, the larger the coefficient of variation of the service time distribution, the longer the mean waiting time. But, in the light traffic case, the smaller the coefficient of variation of the service time distribution, the longer the mean waiting time. This result does not hold in the single server case ($c = 1$).

Table 1 gives some other measures for the constant and geometric case if $g = 8$ and $c = 4$.

More numerical results are shown in the extended version of this paper [2].

Fig. 1. Sample charts for measures of efficiency $M^X/PH/c$ for $g = 4$ and mean waiting time. (H_2 denotes a hyperexponential distribution with two phases and its coefficient of variation is $\sqrt{2}$. ρ denotes the traffic intensity and is given by $\rho = \frac{\lambda g}{\mu c}$, where $\frac{1}{\mu}$ is mean service time.)

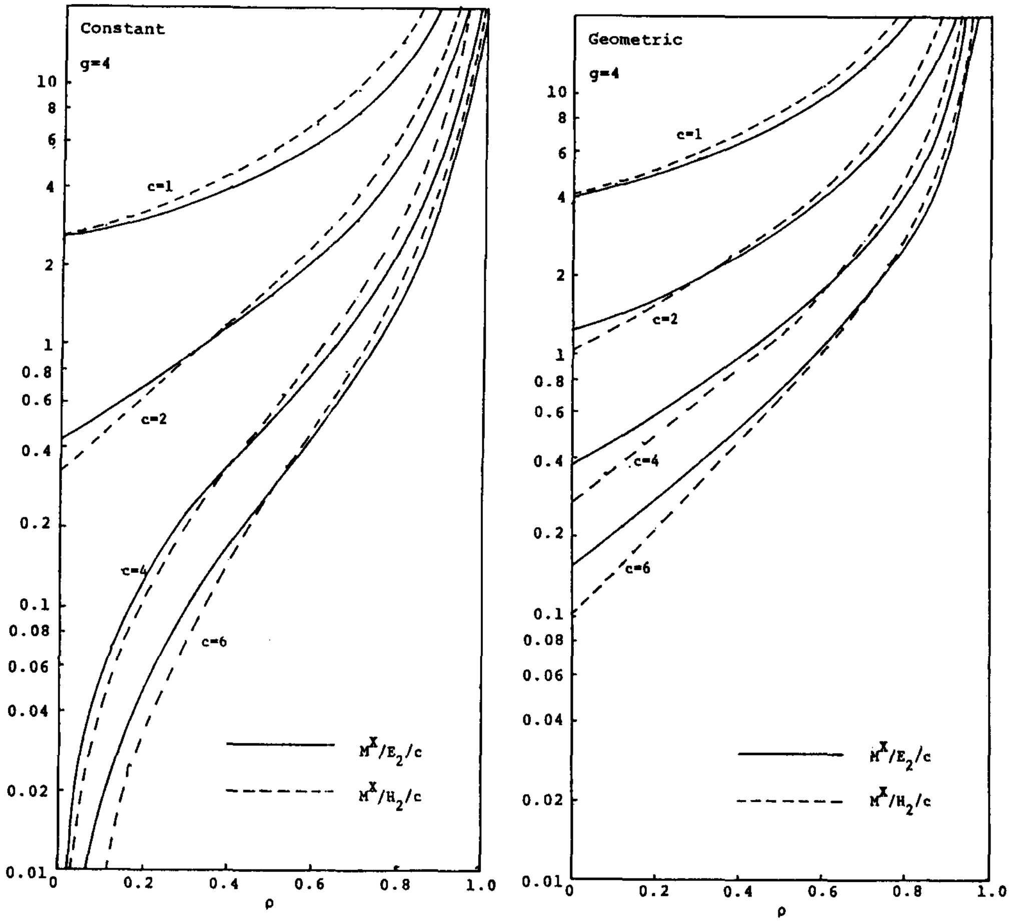

Table 1. Various measures of the $M^X/PH/c$ queue

Constant batch size $M^X/E_2/4$ Batch size 8				
ρ	$P(W > 0)$	EW	EL	VL
0.1	0.55000	0.51754	0.60702	3.66512
0.2	0.60000	0.66419	1.33135	8.67589
0.3	0.65000	0.85302	2.22362	15.91077
0.4	0.70000	1.10510	3.36816	27.07257
0.5	0.75000	1.45838	4.91677	45.79947
0.6	0.80000	1.98876	7.17303	80.98025
0.7	0.85000	2.87332	10.84530	159.08914
0.8	0.90000	4.64331	18.05859	389.65244
0.9	0.95000	9.95498	39.43792	1681.55060

Constant batch size $M^X/H_2/4$ Batch size 8				
ρ	$P(W > 0)$	EW	EL	VL
0.1	0.55000	0.36881	0.54672	2.80784
0.2	0.60000	0.51805	1.21444	7.12798
0.3	0.65000	0.72076	2.06491	14.12545
0.4	0.70000	0.99954	3.19927	26.09998
0.5	0.75000	1.39886	4.79773	48.01257
0.6	0.80000	2.00786	7.21887	92.04675
0.7	0.85000	3.03473	11.29725	194.59997
0.8	0.90000	5.10435	19.53393	506.57233
0.9	0.95000	11.34641	44.42981	2282.53285

Geometric batch size $M^X/E_2/4$ Mean batch size 8				
ρ	$P(W > 0)$	EW	EL	VL
0.1	0.62627	1.49308	0.99723	15.18963
0.2	0.66667	1.76103	2.20883	35.88925
0.3	0.70738	2.10590	3.72708	65.47637
0.4	0.74839	2.56613	5.70581	110.37438
0.5	0.78968	3.21091	8.42183	184.05185
0.6	0.83124	4.17864	12.42873	318.80830
0.7	0.87306	5.79222	19.01822	609.07628
0.8	0.91513	9.02041	32.06532	1438.25712
0.9	0.95745	18.70696	70.94507	5924.60627

Geometric batch size $M^X/H_2/4$ Mean batch size 8				
ρ	$P(W > 0)$	EW	EL	VL
0.1	0.62545	1.26382	0.90553	12.99231
0.2	0.66531	1.54429	2.03543	31.94990
0.3	0.70570	1.90971	3.49165	60.73721
0.4	0.74658	2.40224	5.44358	106.99879
0.5	0.78790	3.09776	8.19551	186.38537
0.6	0.82694	4.14805	12.35331	337.01922
0.7	0.87174	5.90725	19.34030	670.69670
0.8	0.91419	9.43785	33.40112	1643.25714
0.9	0.95695	20.05233	75.78837	6978.95779

EW : Mean waiting time, EL : Mean system length

VL : Variance of system length

Acknowledgements

I wish to express to my deep appreciation to Professor Hidenori Morimura for his helpful suggestions and criticisms. I also wish to thank Mr. Kazuhiko Isoda for his programming help in Section 4. Finally, I am also grateful to the referees for their valuable comments on this paper.

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