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EXPECTED VALUE OF SAMPLE INFORMATION IN FACILITY LOCATION

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Abstract In this paper we discuss the stochastic facility location model in which the distances between the facility and demand points are rectangular and each weight is normally distributed independent random variable with unknown mean and known variance. We are interested in finding the value of information. In general, perfect information is not attainable, but sample information may be obtained. Thus in this paper we find the expected value of sample information. In addition, the expected net gain of sampling and the optimal sample size are found.

1. Introduction

The facility location models in which the weights are random variables recently appear in the literature [1], [2], [6], [8] and [9], but in those papers the parameter values of random variables are assumed to be known.

In this paper we assume each weight is normally distributed independent random variable with unknown mean and known variance, and distances between the facility and the demand points are rectangular.

We are not interested in finding the location that minimizes expected costs. Our main interest is finding the value of information. The expected value of perfect information (EVPI) in the facility location problem is investigated in [6] and [9]. They deal with the case where the weights have a multivariate normal distribution with known means and a known covariance matrix. The EVPI is the upper bound on what one would be willing to pay for perfect information about the weights. In general, perfect information is not attainable, but sample information about the weights may be obtained to gain information about the true values of the means. Therefore we are concerned with finding the expected value of sample information (EVSI). Since sampling involves some cost, the EVSI would be helpful if we decide whether or not take an additional sample. In this paper we determine the EVSI from some samples about the weights. The EVSI in linear programming problems is studied in [3] and [4].

In Section 2 we describe the assumption and formulation of the model. In Section 3 we evaluate the EVSI by utilizing the computational method which appeared in [9]. In Section 4 we investigate the behavior of the EVSI as the sample size changes. Section 5 provides the optimal additional sample size and gives an example.

2. The Model

Let (a_i,b_i), $i=1,\ldots,n$, denote the locations of n demand points on a plane, and W_i denote the weight (cost) of distance between the i-th demand point (a_i,b_i) and the facility. We assume the distances are rectangular and W_i $(i=1,\ldots,n)$ have independent normal distributions with unknown means M_i and known variances $1/r_i$. The parameter r_i is called the precision of W_i. And we assume that the prior distribution of M_i is a normal distribution with a positive mean μ_i and a positive variance $1/\tau_i$. The parameter τ_i is the precision of M_i.

If the minimum expected cost is used as a criterion of optimality, the problem is to

$$\text{(2.1)} \qquad \underset{x,y}{\text{minimize}}\ E[\sum_{i=1}^{n} W_i\{|x-a_i|+|y-b_i|\}],$$

where (x,y) is the location of the facility.
We define $(\hat{x},\hat{y})$ to be a solution which is optimal under the prior distribution, satisfying

$$\text{(2.2)} \qquad \sum_{i=1}^{n} \mu_i\{|\hat{x}-a_i|+|\hat{y}-b_i|\}= \min_{x,y} \sum_{i=1}^{n} \mu_i\{|x-a_i|+|y-b_i|\}.$$

Now suppose that $W_i^{(1)},\ldots,W_i^{(k_i)}$ are random samples of W_i. Then the posterior distribution of M_i when $W_i^{(j)}=w_{ij}$ $(j=1,\ldots,k_i)$ is a normal distribution with mean μ_i' and precision $\tau_i+k_ir_i$ (see [5] p.167 Theorem 1), where

$$\text{(2.3)} \qquad \mu_i'=\frac{\tau_i\mu_i+k_ir_i\bar{w}_i}{\tau_i+k_ir_i}\ , \ (\bar{w}_i:\text{ sample mean}).$$

Under the posterior distribution determined by the sample mean $\bar{w}_i$, (2.1) reduces to

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$$\text{(2.4)} \qquad \underset{x,y}{\text{minimize}} \sum_{i=1}^{n} \mu_i' \{|x-a_i|+|y-b_i|\}.$$

Then the conditional value of the sample information $\bar{w}_i$ (CVSI) is as follows (see [7] pp.81-91):

$$\text{(2.5)} \qquad \text{CVSI}(\bar{w}_i) \stackrel{\Delta}{=} \sum_{i=1}^{n} \mu_i' \{|\hat{x}-a_i|+|\hat{y}-b_i|\} - \min_{x,y} \sum_{i=1}^{n} \mu_i' \{|x-a_i|+|y-b_i|\}.$$

The CVSI can be evaluated after $\bar{w}_i$ $(i=1,\dots,n)$ is known, but before $\bar{w}_i$ is known we can compute the expected value of sample information (EVSI):

$$\text{(2.6)} \qquad \text{EVSI} = E\left[\sum_{i=1}^{n} Z_i \{|\hat{x}-a_i|+|\hat{y}-b_i|\} - \min_{x,y} \sum_{i=1}^{n} Z_i \{|x-a_i|+|y-b_i|\}\right],$$

where

$$\text{(2.7)} \qquad Z_i \stackrel{\Delta}{=} \frac{\tau_i \mu_i + k_i r_i \bar{W}_i}{\tau_i + k_i r_i}.$$

Here each Z_i has an independent normal distribution with mean μ_i and variance $1/\tau_i - 1/(\tau_i + k_i r_i)$.

It will be useful to separate the EVSI as follows:

$$\text{(2.8)} \qquad \text{EVSI} = \text{EVSI}_x + \text{EVSI}_y,$$

where

$$\text{(2.9)} \qquad \text{EVSI}_x \stackrel{\Delta}{=} E\left[\sum_{i=1}^{n} Z_i |\hat{x}-a_i| - \min_{x} \sum_{i=1}^{n} Z_i |x-a_i|\right],$$

and

$$\text{(2.10)} \qquad \text{EVSI}_y \stackrel{\Delta}{=} E\left[\sum_{i=1}^{n} Z_i |\hat{y}-b_i| - \min_{y} \sum_{i=1}^{n} Z_i |y-b_i|\right].$$

Because it is easy to consider one dimension at a time, we shall deal only with finding EVSI_x hereafter. EVSI_y can be found similarly.

3. Evaluation of EVSI

According to the separability of EVSI, we consider only EVSI_x in this section.

The equation (2.9) can be reduced to

$$\text{(3.1)} \qquad \text{EVSI}_x = E\left[\sum_{i=1}^{n} Z_i |\hat{x}-a_i| - \min_{x} \sum_{i=1}^{n} Z_i |x-a_i|\right] = \sum_{i=1}^{n} \mu_i |\hat{x}-a_i| - E\left[\min_{x} \sum_{i=1}^{n} Z_i |x-a_i|\right].$$

To evaluate the second term in the right hand side of (3.1) we define $x^*(\mathbf{Z})$,

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(where $\mathbf{Z}=(Z_1,Z_2,\ldots,Z_n)$), as the optimal solution of the following

(3.2) $$\min_x \sum_{i=1}^{n} Z_i|x-a_i|.$$

Then we obtain

(3.3) $$E[\min_x \sum_{i=1}^{n} Z_i|x-a_i|]$$
$$=\sum_{j=1}^{n}[\sum_{i=1}^{n}\int_{-\infty}^{\infty}z_i|a_j-a_i|\Pr(Z_i=z_i|x^*(\mathbf{Z})=a_j)dz_i]\Pr(x^*(\mathbf{Z})=a_j)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}z_i|a_j-a_i|\Pr(x^*(\mathbf{Z})=a_j|Z_i=z_i)g_i(z_i)dz_i,$$

where $g_i(\cdot)$ is the p.d.f. of Z_i.
The second equality is derived from Bayes' theorem ([5] p.28).

Now we renumber the location of demand points according to the nonincreasing order of a_i such as $a_1\leq a_2\leq\ldots\leq a_n$. For practical purposes, we will place restrictions on μ_i and τ_i to neglect the probability of getting a negative value of each Z_i. Then the probability that $x^*(\mathbf{Z})=a_j$ becomes as follows [9]:

(3.4) $$\Pr(x^*(\mathbf{Z})=a_j)=\Pr(\{\sum_{i=1}^{j-1}Z_i\leq\sum_{i=j}^{n}Z_i\}\cap\{\sum_{i=1}^{j}Z_i>\sum_{i=j+1}^{n}Z_i\})$$
$$=\Phi(\frac{-u_j+\mu_j}{\sqrt{v}})-\Phi(\frac{-u_j-\mu_j}{\sqrt{v}}),$$

where $\Phi(\cdot)$ is the normal distribution function with mean 0 and variance 1,

(3.5) $$u_j=\sum_{i=1}^{j-1}\mu_i-\sum_{i=j+1}^{n}\mu_i$$

and

(3.6) $$v=\sum_{i=1}^{n}(\frac{1}{\tau_i}-\frac{1}{\tau_i+k_ir_i}).$$

Similarly, if i<j

(3.7) $$\Pr(x^*(\mathbf{Z})=a_j|Z_i=z_i)=\Phi(\frac{-u_j+\mu_j-z_i+\mu_i}{\sqrt{v_i}})-\Phi(\frac{-u_j-\mu_j-z_i+\mu_i}{\sqrt{v_i}}),$$

and if i>j

(3.8) $$\Pr(x^*(\mathbf{Z})=a_j|Z_i=z_i)=\Phi(\frac{-u_j+\mu_j+z_i-\mu_i}{\sqrt{v_i}})-\Phi(\frac{-u_j-\mu_j+z_i-\mu_i}{\sqrt{v_i}}),$$

where

(3.9) $$v_i=\sum_{\substack{j=1\\ \neq i}}^{n}(\frac{1}{\tau_i}-\frac{1}{\tau_i+k_ir_i}).$$

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Therefore substituting (3.7) and (3.8) into (3.3), we obtain

$$(3.10)\quad E[\min_{x} \sum_{i=1}^{n} Z_i |x-a_i|]$$
$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} Z_i [\sum_{j=1}^{i-1} (a_i-a_j)\{\Phi(\frac{-u_j+\mu_j+z_i-\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j+z_i-\mu_i}{\sqrt{v_i}})\}$$
$$+ \sum_{j=i+1}^{n} (a_j-a_i)\{\Phi(\frac{-u_j+\mu_j-z_i+\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j+z_i-\mu_i}{\sqrt{v_i}})\}] g_i(z_i)dz_i .$$

The integrals in the right-hand side of (3.10) can be calculated by numerical integration such as Simpson's rule. Since we can evaluate $EVSI_y$ similarly, we can find the EVSI by (2.8).

4. Behavior of EVSI as a Function of Sample Size

Considering the function $EVSI(\mathbf{k})$ of sample size $\mathbf{k}=(k_1,k_2,\ldots,k_n)$, we shall show some properties, treating each k_i as if it were a continuous variable.

$$(4.1)\quad EVSI_x(\mathbf{k}) = \sum_{i=1}^{n} \mu_i |\hat{x}-a_i| - \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} [\min_{x} \sum_{i=1}^{n} z_i |x-a_i|] \prod_{i=1}^{n} (g_i(z_i)dz_i).$$

Now we have the following theorem.

Theorem. The function $EVSI(\mathbf{k})$ is nondecreasing in each k_i.

To prove this theorem we give the following.

Lemma. Suppose that $\psi(x)$ is concave function in x and the random variable X has a normal distribution with mean μ and variance σ^2. Then the expected value of $\psi(X)$, $E[\psi(X)]$, is nonincreasing in σ.

Proof: Considering $E[\psi(X)]$ as a function of σ,

$$L(\sigma) \triangleq E[\psi(X)] = \int_{-\infty}^{\infty} \psi(x) \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \{\psi(\sigma x+\mu)+\psi(-\sigma x+\mu)\} \exp(-\frac{x^2}{2})dx.$$

For $0<\sigma_1<\sigma_2$ and $x>0$, we have

$$L(\sigma_1)-L(\sigma_2)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} [\{\psi(\sigma_1 x+\mu)+\psi(-\sigma_1 x+\mu)\} - \{\psi(\sigma_2 x+\mu)+\psi(-\sigma_2 x+\mu)\}] \exp(-\frac{x^2}{2})dx$$
$$= \frac{(\sigma_2-\sigma_1)}{\sqrt{2\pi}} \int_0^{\infty} x[\frac{\psi(-\sigma_1 x+\mu)-\psi(-\sigma_2 x+\mu)}{(-\sigma_1 x+\mu)-(-\sigma_2 x+\mu)} - \frac{\psi(\sigma_2 x+\mu)-\psi(\sigma_1 x+\mu)}{(\sigma_2 x+\mu)-(\sigma_1 x+\mu)}] \exp(-\frac{x^2}{2})dx \geqq 0$$

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(by the concavity of $\psi(x)$). □

Now we prove Theorem.

Proof of Theorem. Let $\boldsymbol{k}^{i}=(k_1,k_2,\ldots,k_i,\ldots,k_n)$ and $\tilde{\boldsymbol{k}}^{i}=(k_1,k_2,\ldots,\tilde{k}_i,\ldots,k_n)$. Moreover we assume $\tilde{k}_i>k_i$. Them by (4.1) we have

$$EVSI_x(\tilde{\boldsymbol{k}}^{i})-EVSI_x(\boldsymbol{k}^{i})$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}H_i(\boldsymbol{z}^{(i)})\prod_{\substack{j=1\\ \neq i}}^{n}(g_j(z_j)dz_j),$$

where

$$H_i(\boldsymbol{z}^{(i)})=\int_{-\infty}^{\infty}\min_x\sum_{j=1}^{n}z_j|x-a_j|g_i(z_i)dz_i-\int_{-\infty}^{\infty}\min\sum_{j=1}^{n}z_j|x-a_j|\tilde{g}_i(z_i)dz_i,$$

$$\boldsymbol{z}^{(i)}\triangleq(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n),$$

$g_i(\cdot)$ and $\tilde{g}_i(\cdot)$ are the density function of normal distribution with the same mean μ_i but different variances $1/\tau_i-1/(\tau_i+k_ir_i)$ and$1/\tau_i-1/(\tau_i+\tilde{k}_ir_i)$ respectively. Let

$$f(q)\triangleq\min_x\{\sum_{\substack{j=1\\ \neq i}}^{n}[z_j|x-a_j|+q|x-a_i|]\}.$$

Then it is easily shown $f(\cdot)$ is the concave function. By the lemma, $\int_{-\infty}^{\infty}f(z_i)g_i(z_i)dz_i$ is nondecreasing in the standard deviation (or variance) of Z_i. Since

$$\left(\frac{1}{\tau_i}-\frac{1}{\tau_i+\tilde{k}_ir_i}\right)-\left(\frac{1}{\tau_i}-\frac{1}{\tau_i+k_ir_i}\right)>0,$$

$H_i(\boldsymbol{z}^{(i)})\geqq 0$ holds. Therefore

$$EVSI_x(\tilde{\boldsymbol{k}}_i)-EVSI_x(\boldsymbol{k}_i)\geqq 0.$$

As to $EVSI_y$, we can show similar results. □

We define the expected value of perfect information (EVPI):

(4.2) $$\mathrm{EVPI}=\mathrm{EVPI}_x+\mathrm{EVPI}_y.$$

where

(4.3) $$\mathrm{EVPI}_x=\sum_{i=1}^{n}\mu_i|\hat{x}-a_i|-E[\min_x\sum_{i=1}^{n}\overline{Z}_i|x-a_i|],$$

(EVPI_y can be defined similarly to EVPI_x),
and assume each $\overline{Z}_i$ has independent normal distribution with mean μ_i and

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variance $1/\tau_i$. Then the following corollary holds.

Corollary. $0 \leqq EVSI \leqq EVPI$.

Proof: By (2.6) and the definition of $(\hat{x},\hat{y})$, the first inequality holds. Furthermore, as the sample size k_i increases, the variance of random variable Z_i approaches to the variance of $\bar{Z}_i$, i.e., $1/\tau_i$. Thus the thoerem implies the second inequality. □

This corollary shows that the EVPI becomes an upper bound of the EVSI.

5. Optimal Sampling

The EVSI is the value of sample information without considering sampling cost. On the other hand, if the sample information involves some costs, this sampling cost, $CS(\mathbf{k})=\sum_{j=1}^{n} c_j k_i + b$, should be substracted from the EVSI, where c_j (j=1,...,n) is a unit cost taking one sample about the i-th location and b is a fixed charge taking the sample. Then the net result called the expected net gain of sampling (ENGS) becomes as follows:

(5.1) $$ENGS(\mathbf{k})=EVSI(\mathbf{k})+CS(\mathbf{k}).$$

The optimal vector sample size is defined as that vector size $\mathbf{k}$ which maximizes $ENGS(\mathbf{k})$.

In this section, we consider the optimal sampling in which all sample sizes, $k_1,k_2,\ldots,k_n$, have same value k. If we define $c=\sum_{j=1}^{n} c_j$, then the sampling cost becomes as follows:

(5.2) $$CS(k)=ck+b.$$

In this case the ENGS is determined by the value of k and is considered as the function of k. Now the optimal sample size k^* is defined as that size k which maximizes $ENGS(k)$. In the following we give an example and find the optimal sample size.

Example

We want to locate a wholesale store in the town where there are 5 retail stores. Let (a_i,b_i) and W_i denote the location of the i-th retail store and the amount (tons) sold in a week there respectively (Table 1). The distance between the wholesale store and the i-th retail store is $|x-a_i|+|y-b_i|$ (kilometers), where the location of the wholesale store is (x,y). We assume the transportation cost per kilometer and ton is 1000 yens. Then the optimal location under the prior distribution becomes (8,6). If we assume that all sample sizes $k_1,k_2,\ldots,k_n$ have the same value k, the EVSI and the ENGS as a

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function of sample size k are shown in Fig. 1.

Table 1. Data for example.

i	1	2	3	4	5
(a_i, b_i)	(3,2)	(4,9)	(8,12)	(12,1)	(14,6)
r_i	0.01	0.01	0.01	0.01	0.01
μ_i	50	38	30	35	25
τ_i	0.1	0.1	0.1	0.1	0.1

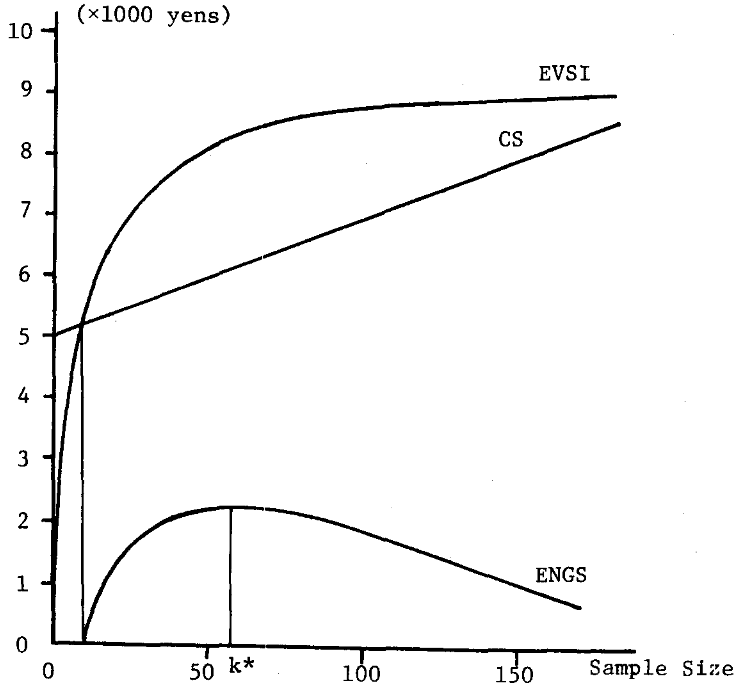

Fig. 1. EVSI and ENGS of example.

In Fig. 1 we assume the sampling cost (× 1000 yens) is as follows:

$$CS(k) = 0.02k + 5.$$

Then the optimal sample size is about 60 and ENGS(60) is 2.24 (× 1000 yens).

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