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GROUP REPLACEMENT POLICY FOR A MAINTAINED COHERENT SYSTEM

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Abstract In this paper we consider a coherent system consisting of n components. Each component is repaired upon failure. This maintained coherent system is monitored continuously, and based upon the results of monitoring a decision must be made as to whether or not to replace the system. We show that the optimal group replacement policy has a monotone property without regard to the values of failure and repair rates. Further we discuss sufficient conditions for the replacement of the maintained coherent system.

1. Introduction

This paper deals with a coherent system consisting of n components. Each component is subject to random failure. Upon failure the component is repaired and recovers its functioning perfectly. The maintained coherent system has been investigated by Barlow and Proschan [1], Chiang and Niu [2], and Ross [5]. In particular, Ross proved that the distribution of the time to first system failure has NBU (i.e., new better than used) property when all components are initially up at time zero, and have exponential uptime distributions with parameter λ_i for $i=1,2,\ldots,n$ and downtime distributions with parameter μ_i for $i=1,2,\ldots,n$. Thus efforts to replace the maintained coherent system before system failure may be advantageous. On the other hand, when compared with individual repair upon failed components, the group replacement may cause to throw away some components which are in good operating condition. However, we can expect to obtain a large discount on the purchase price and other economic attendant to large-scale undertakings.

In this paper we consider a replacement problem for a coherent system with repairable components. The above system is modeled by a continuous time Markov decision process. The system is monitored continuously, and based upon the results of monitoring a decision must be made as to whether or not to replace the maintained coherent system. The objective of this paper is to study the structure of the optimal group replacement policy minimizing the expected total discounted cost for the maintained coherent system. We show that the optimal group replacement policy has a monotone property without regard to the values of failure rate λ_i and repair rate μ_i. Further we discuss sufficient conditions for the replacement of the maintained coherent system. Finally to illustrate the optimal group replacement policy, a numerical example is presented.

2. Problem Formulation

Consider a maintained coherent system. The system consists of n components and n repair facilities. Each of its components is either up or down, indicating whether it is functioning or not, and acts independent of the behaviour of other components. When the ith component goes up [down], it remains up [down] for exponentially distributed time with parameter λ_i [μ_i] and then goes down [up]. Let uptimes and downtimes be independent. We suppose that the system state at any time depends on component states through a coherent structure function ϕ (see Barlow and Proschan [1]).

Let

$$X_i(t)=\begin{cases}1 & \text{if the } i\text{th component is up at time } t,\\ 0 & \text{otherwise,}\end{cases}$$

for any $i\in \mathrm{N}=\{1,2,\ldots,n\}$. Then the evolution of the state of the ith component is described by the stochastic process $\{X_i(t),\ t\geqq 0\}$. Let

$$X(t)=(X_1(t),X_2(t),\ldots,X_n(t))$$

and let

$$\phi(X(t))=\begin{cases}1 & \text{if the system is up at time } t,\\ 0 & \text{otherwise,}\end{cases}$$

then the state of the system is summarized by binary n-vector $X(t)$ with a state space S and the actual state of the maintained coherent system is described by the stochastic process $\{\phi(X(t)),\ t\geqq 0\}$.

In the present paper we are interested in a group replacement problem for the above system. At each time epoch $t\in T=[0,\infty)$, observing the state $X(t)$, a decision is made to replace the maintained coherent system, or to keep it. We assume that the time needed to replace the system is exponential with parameter μ_0. Let $\pi(X)\in D=\{0,1\}$ represent the decision made for the system at any time t, where $\pi(X)=0$ means to replace the system and $\pi(X)=1$ means to keep it. As the cost rate $r(X,\pi(X))$ associated with the maintained coherent system, we consider the following cost rate. At time t when the state is X and decision $\pi(X)=1$ is made on the maintained coherent system, then the cost is incurred at the rate $r(X,1)=\mathrm{P}(1-\phi(X))+\sum_{i\in \mathrm{C}_0(X)} r_i$, where P is the system down cost rate, r_i $(i\in \mathrm{N})$ is the repair cost rate of the ith component and $\mathrm{C}_0(X)$ denotes the set of currently failed components. On the other hand, when decision $\pi(X)=0$ is made, then the cost is incurred at the rate $r(X,0)=\mathrm{R}$, where R is the replacement cost rate (i.e., R/μ_0 is the expected replacement cost).

The objective is to investigate the structure of the optimal group replacement policy minimizing the expected total discounted cost with discount factor $\alpha>0$. Now let $V_\alpha(x)$ be the minimum expected total discounted cost when the state of the system is $X(0)=x=(x_1,x_2,\ldots,x_n)$ at the beginning. We begin by deriving the weak infinitesimal operator A_π of the process $\{X^\pi(t);\ t\geq 0\}$ for $\pi\in D^T$. For a function f in the domain of A_π we have

$$\mathrm{A}_\pi f(x)=\lim_{t\downarrow 0} t^{-1}[\mathrm{E}_x[f(X(t))]-f(x)]$$
$$=\begin{cases}\sum_{i\in \mathrm{C}_0(x)}\mu_i(f(1_i,x)-f(x))+\sum_{i\in \mathrm{C}_1(x)}\lambda_i(f(0_i,x)-f(x)) & \pi(x)=1,\\ \mu_0(f(\mathbb{1})-f(x)) & \pi(x)=0,\end{cases}$$

where $\mathrm{C}_j(x)=\{i\,|\,x_i=j,\ i\in\mathrm{N}\}$, $(j=0,1)$, $(\cdot_i,x)=(x_1,\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n)$, and $\mathbb{1}=(1,\ldots,1)$. Of great importance to us is Doshi's formula (see Doshi [3])

$$\alpha V_\alpha(x)=\inf_{\pi\in D}\{r(x,\pi(x))+\mathrm{A}_\pi V_\alpha(x)\}. \tag{2.1}$$

In the following section the structural properties of the optimal group replacement policy minimizing the expected total discounted cost are characterized. It is shown that a monotonic policy is optimal and sufficient conditions for replacing the whole system are presented. Here we notice that the existence of a stationary policy minimizing the expected total discounted cost is guaranteed, since all costs are bounded and the action space is finite.

3. Structure of Optimal Group Replacement Policy

In this section we discuss an optimal group replacement policy for a maintained coherent system. We can find the optimal group replacement policy by solving the functional equation (2.1). We cannot obtain, however, a solution as a function of the parameters in the model. So some properties on the optimal policy and the corresponding optimal cost of the functional equation (2.1) are discussed.

The following lemmas show the structure of optimal expected total discounted cost function, and they are used in the proof of theorems which present the structural properties of the optimal group replacement policy.

Lemma 3.1. The minimum expected total discounted cost $V_\alpha(x)$ is a non-increasing function of its component variables x_i, $i\in N$.

Proof: The functional equation (2.1) can be written as

$$V_\alpha(x)=\min\begin{cases}\{P(1-\phi(x))+\sum_{i\in C_0(x)}(r_i+\mu_i V_\alpha(1_i,x))+\sum_{i\in C_1(x)}\lambda_i V_\alpha(0_i,x)\\ \qquad +(\Lambda-\sum_{i\in C_0(x)}\mu_i-\sum_{i\in C_1(x)}\lambda_i)V_\alpha(x)\}/(\Lambda+\alpha),\\ \{R+\mu_0 V_\alpha(\mathbb{1})+(\Lambda-\mu_0)V_\alpha(x)\}/(\Lambda+\alpha),\end{cases}\tag{3.1}$$

where Λ is any value larger than $\max\{\mu_0,\max_{x\in S}\{\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i+\max_{i\in C_0(x)}\lambda_i\}\}$.
We can calculate by using the successive approximation technique:

$$V_\alpha(x;n+1)=\min\begin{cases}\{P(1-\phi(x))+\sum_{i\in C_0(x)}(r_i+\mu_i V_\alpha(1_i,x;n))+\sum_{i\in C_1(x)}\lambda_i V_\alpha(0_i,x;n)\\ \qquad +(\Lambda-\sum_{i\in C_0(x)}\mu_i-\sum_{i\in C_1(x)}\lambda_i)V_\alpha(x;n)\}/(\Lambda+\alpha),\\ \{R+\mu_0 V_\alpha(\mathbb{1};n)+(\Lambda-\mu_0)V_\alpha(x;n)\}/(\Lambda+\alpha),\end{cases}\tag{3.2}$$

where $V_\alpha(x;0)=0$ for all $x\in S$. Then proving the monotonicity of $V_\alpha(x)$ is carried out by using the mathematical induction. For $n=1$ the result follows easily from the properties of the structure function ϕ and the definitions of $C_0(x)$ and $C_1(x)$. Suppose the result is true for some n. At the $n+1$-th stage, if the optimal decision is to keep the maintained coherent system for $(0_i,x)\in S$, $i\in N$, then

$$\begin{aligned}V_\alpha(0_i,x;n+1)-V_\alpha(1_i,x;n+1)\geqq{}&[P(1-\phi(0_i,x))+\sum_{j\in C_0(0_i,x)}(r_j+\mu_j V_\alpha(0_i,1_j,x;n))\\ &+\sum_{j\in C_1(0_i,x)}\lambda_j V_\alpha(0_i,0_j,x;n)+(\Lambda-\sum_{j\in C_0(0_i,x)}\mu_j-\sum_{j\in C_1(0_i,x)}\lambda_j)V_\alpha(0_i,x;n)]\\ &/[\Lambda+\alpha]-[P(1-\phi(1_i,x))+\sum_{j\in C_0(1_i,x)}(r_j+\mu_j V_\alpha(1_i,1_j,x;n))+\sum_{j\in C_1(1_i,x)}\lambda_j\\ &\cdot V_\alpha(1_i,0_j,x;n)+(\Lambda-\sum_{j\in C_0(1_i,x)}\mu_j-\sum_{j\in C_1(1_i,x)}\lambda_j)V_\alpha(1_i,x;n)]/[\Lambda+\alpha]\end{aligned}$$

$$\geqq [P(\phi(1_i,x)-\phi(0_i,x))+r_i+\sum_{j\in C_0(1_i,x)}\mu_j(V_\alpha(0_i,1_j,x;n)-V_\alpha(1_i,1_j,x;n))$$

$$+\sum_{j\in C_1(1_i,x)}\lambda_j(V_\alpha(0_i,0_j,x;n)-V_\alpha(1_i,0_j,x;n))$$

$$+(\Lambda-\lambda_i-\sum_{j\in C_0(0_i,x)}\mu_j-\sum_{j\in C_1(1_i,x)}\lambda_j)(V_\alpha(0_i,x;n)-V_\alpha(1_i,x;n))]/[\Lambda+\alpha]$$

$$\geqq 0.$$

On the other hand, even if the optimal decision is to replace the maintained coherent system, then $V_\alpha(0_i,x;n+1)-V_\alpha(1_i,x;n+1)\geqq 0$ is proved similarly to the above. Thus for each n, $V_\alpha(x;n)$ is a nonincreasing function of its component variables x_i, $i\in N$. Then from the successive approximation technique, as

$$\lim_{n\to\infty} V_\alpha(x;n)=V_\alpha(x),$$

$V_\alpha(x)$ is a nonincreasing function of each of its component variables x_i, $i\in N$. ||

Lemma 3.2. The minimum expected total discounted cost $V_\alpha(x)$ is not larger than R/α.

The above lemma is easily proved by the functional equation (3.1). Next the structure properties of the optimal group replacement policy for a maintained coherent system are characterized.

Theorem 3.1. If all components are operating, then the optimal decision is to keep the maintained coherent system.

Proof: Follows directly from the functional equation (3.1) and Lemma 3.2. ||

Theorem 3.2. The optimal group replacement policy $\pi^*(x)$ is a nondecreasing function of each of its component variables x_i, $i\in N$.

Proof: The functional equation (2.1) can be written as

$$V_\alpha(x)=\min\begin{cases}\{P(1-\phi(x))+\sum_{i\in C_0(x)}(r_i+\mu_i V_\alpha(1_i,x))+\sum_{i\in C_1(x)}\lambda_i V_\alpha(0_i,x)\\ \quad +(\Lambda-\sum_{i\in C_0(x)}\mu_i-\sum_{i\in C_1(x)}\lambda_i)V_\alpha(x)\}/(\Lambda+\alpha),\\ \{R+\mu_0 V_\alpha(\mathbf{1})\}/(\alpha+\mu_0).\end{cases} \tag{3.3}$$

Notice that the latter quantity does not contain variable x. From Lemma 3.1 and the above fact, the result is easily obtained. ||

The following Theorems 3.3 and 3.4 are concerned with sufficient conditions for the replacement of the maintained coherent system.

Theorem 3.3. If $\mu_0 \geqq \sum_{i\in C_0(x)} \mu_i + \sum_{i\in C_1(x)} \lambda_i$ and $R \leqq P(1-\phi(x)) + \sum_{i\in C_0(x)} r_i$ for $x\in S$, then the optimal decision is to replace the maintained coherent system.

Proof: The functional equation (2.1) can be written as

$$V_\alpha(x)=\min\begin{cases}\{P(1-\phi(x))+\sum_{i\in C_0(x)}(r_i+\mu_i V_\alpha(1_i,x))+\sum_{i\in C_1(x)}\lambda_i V_\alpha(0_i,x)\} \\ \quad /\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}, \\ \{R+\mu_0 V_\alpha(11)\}/\{\alpha+\mu_0\}.\end{cases} \tag{3.4}$$

The result is shown by comparing the terms in functional equation (3.4). Thus

$$\begin{aligned}[V_\alpha(x)]_1-[V_\alpha(x)]_0 &= \{P(1-\phi(x))+\sum_{i\in C_0(x)}(r_i+\mu_i V_\alpha(1_i,x))+\sum_{i\in C_1(x)}\lambda_i V_\alpha(0_i,x)\} \\ &\quad /\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}-\{R+\mu_0 V_\alpha(11)\}/\{\alpha+\mu_0\} \\ &\geqq [1/\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}-1/\{\alpha+\mu_0\}]R \\ &\quad +[\{\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}/\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\} \\ &\quad -\mu_0/\{\alpha+\mu_0\}]V_\alpha(11) \\ &\geqq [1/\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}-1/\{\alpha+\mu_0\}]\alpha V_\alpha(11) \\ &\quad +[\{\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\}/\{\alpha+\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i\} \\ &\quad -\mu_0/\{\alpha+\mu_0\}]V_\alpha(11) \\ &=0.\end{aligned}$$

The first inequality follows from the assumptions and Lemma 3.1. The last inequality is true from Lemma 3.2. ||

Theorem 3.4. If $\mu_0 < \sum_{i\in C_0(x)} \mu_i + \sum_{i\in C_1(x)} \lambda_i$ and $R/\mu_0 \leqq [P(1-\phi(x)) + \sum_{i\in C_0(x)} r_i]/[\sum_{i\in C_0(x)} \mu_i + \sum_{i\in C_1(x)} \lambda_i]$ for $x\in S$, then the optimal decision is to replace the maintained coherent system.

Proof: The result is proved similarly to Theorem 3.3. ||

Remark 1. The results of this section remain valid even when we extend the cost rate $P(1-\phi(x))+\sum_{i\in C_0(x)} r_i$ to the general cost rate $r(x,1)$, where $r(x,1)$

is a nonincreasing function of each of its component variables x_i, $i \in N$ with $r(11,1)=0$.

Remark 2. We notice that the monotone property of the optimal policy keeps true irrespective of failure and repair rates, but it is of course that the actual policy $\pi(x)$ depends on the values of failure and repair rates.

4. Numerical Example

In this section we consider the so-called bridge structure system shown in Figure 1. To illustrate the optimal group replacement policy of the preceding section, we give a numerical example. The failure and repair rates of components are given in Table 1. The repair cost rates of components are also given in Table 1. The system down cost rate, replacement cost rate, and replacement rate are P=5.0, R=10.0, and μ_0=2.0, respectively. Then we obtain

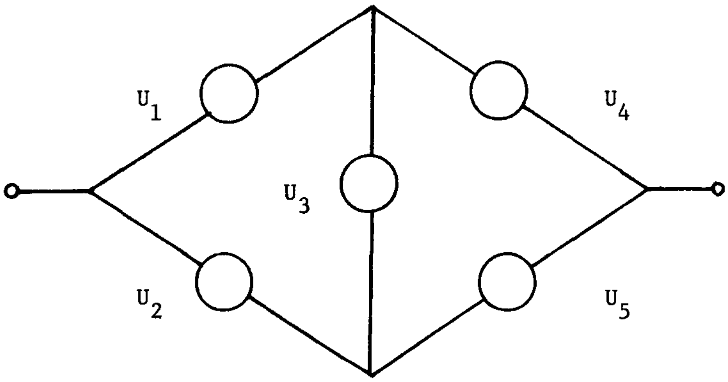

Figure 1. The bridge structure system.

Table 1. Failure rates, repair rates, and cost rates.

	λ	μ	r
U_1	0.1	1.0	2.0
U_2	0.1	2.0	2.0
U_3	0.2	1.0	2.0
U_4	0.1	1.0	2.0
U_5	0.1	2.0	2.0

Table 2. The optimal replacement policy with P=5.0, R=10.0, μ_0=2.0, and α=0.05.

No.	x_1	x_2	x_3	x_4	x_5	$\phi(x)$	$\pi(x)$	$M_1(x)$	$M_2(x)$	$M_2(x)/M_1(x)$	$V(x)$
1	0	0	0	0	0	0	0	2.80	15.0	5.35	18.60
2	0	0	0	0	1	0	0	2.04	13.0	6.37	18.60
3	0	0	0	1	0	0	0	2.44	13.0	5.33	18.60
4	0	0	0	1	1	0	0	1.68	11.0	6.55	18.60
5	0	0	1	0	0	0	0	2.48	13.0	5.24	18.60
6	0	0	1	0	1	0	0	1.72	11.0	6.40	18.60
7	0	0	1	1	0	0	0	2.12	11.0	5.19	18.60
8	0	0	1	1	1	0	0	1.36	9.0	6.62	18.60
9	0	1	0	0	0	0	0	2.04	13.0	6.37	18.60
10	0	1	0	0	1	1	0	1.28	6.0	4.69	18.60
11	0	1	0	1	0	0	0	1.68	11.0	6.55	18.60
12	0	1	0	1	1	1	0	0.92	4.0	4.35	18.60
13	0	1	1	0	0	0	0	1.72	11.0	6.40	18.60
14	0	1	1	0	1	1	0	0.96	4.0	4.17	18.60
15	0	1	1	1	0	1	0	1.36	4.0	2.94	18.60
16	0	1	1	1	1	1	1	0.60	2.0	3.33	17.64
17	1	0	0	0	0	0	0	2.44	13.0	5.33	18.60
18	1	0	0	0	1	0	0	1.68	11.0	6.55	18.60
19	1	0	0	1	0	1	0	2.08	6.0	2.88	18.60
20	1	0	0	1	1	1	0	1.32	4.0	3.03	18.60
21	1	0	1	0	0	0	0	2.12	11.0	5.19	18.60
22	1	0	1	0	1	1	0	1.36	4.0	2.94	18.60
23	1	0	1	1	0	1	1	1.76	4.0	2.27	18.13
24	1	0	1	1	1	1	1	1.00	2.0	2.00	16.15
25	1	1	0	0	0	0	0	1.68	11.0	6.55	18.60
26	1	1	0	0	1	1	0	0.96	4.0	4.17	18.60
27	1	1	0	1	0	1	0	1.32	4.0	3.03	18.60
28	1	1	0	1	1	1	1	0.56	2.0	3.57	17.38
29	1	1	1	0	0	0	0	1.36	9.0	6.62	18.60
30	1	1	1	0	1	1	1	0.60	2.0	3.33	17.46
31	1	1	1	1	0	1	1	1.00	2.0	2.00	16.15
32	1	1	1	1	1	1	1	0.24	0.0	0.00	14.07

the optimal group replacement policy for the maintained coherent system by the value iteration method. Also to illustrate the results of Theorems 3.3 and 3.4, the values $M_1(x)=\sum_{i\in C_0(x)}\mu_i+\sum_{i\in C_1(x)}\lambda_i$, $M_2(x)=P(1-\phi(x))+\sum_{i\in C_0(x)}r_i$, and $M_2(x)/M_1(x)$ are computed. The results of these computations are given in Table 2 in the case of discount factor α=0.05. No. 4, 6, 11, 13, 18, and 25 satisfy the condition of Theorem 3.3, and No. 1, 2, 3, 5, 7, 9, 17, and 21 satisfy the condition of Theorem 3.4. But No. 8, 10, 12, 14, 15, 19, 20, 22, 26, 27, and 29 don't satisfy these conditions, while the optimal decision is to replace the maintained coherent system. This shows that the conditions of Theorems 3.3 and 3.4 are not necessary for replacing the system. No. 10, 12, 14, 15, 19, 20, 22, 26, and 27 show that a preventive replacement is optimal.

5. Conclusion

In this paper we have been examined the structure of the optimal group replacement policy for the maintained coherent system. We showed that the optimal group replacement policy minimizing the expected total discounted cost is a monotone policy without regard to the failure and repair rates. Further we discussed the sufficient conditions for the group replacement of the maintained coherent system. It is a future problem to find the structure of the optimal group replacement policy in the case where a group replacement policy in the presence of fixed costs for turning on or turning off repair facilities.

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