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ON N-JOB, 2-MACHINE FLOW-SHOP SCHEDULING PROBLEM WITH ARBITRARY TIME LAGS AND TRANSPORTATION TIMES OF JOBS

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Abstract This short paper gives solution algorithm of obtaining an optimal sequence to give minimum total elapsed time in a 'n-job, 2-machine' flow-shop scheduling problem in which jobs involve Arbitrary Time Lags (i.e. Start Lags and Stop Lags) and Transportation Times. All the times (Processing Times, Arbitrary Lags, and Transportation times) are given prior and are of deterministic nature.

1. Introduction

Mitten [3,4] and Johnson [2] separately gave solution algorithm of obtaining an optimal sequence for a 'n-job, 2-machine' flow-shop scheduling problem in which each job involves arbitrary time lags (Start-lag, Stop-lag). Further, very recently Maggu and Das [5] give solution algorithm of obtaining optimal sequence for a 'n-job, 2-machine' flow-shop scheduling problem wherein each job involves transportation-time. This paper is designed to study a 'n-job, 2-machine' flow-shop scheduling problem in which there are considered,

(i) Start-lag and Stop-lag times of jobs

(ii) Transportation times of jobs.

Now we give Key-definitions:

Consider Johnson's [1] 'two-machine (say A,B), n-job' flow-shop scheduling Problem. Let each job i has a "Start-lag" D_i (≥ 0), a "Stop-lag" E_i (≥ 0) and transportation time t_i (≥ 0). Thus, as defined by Mitten in [4] (see, page 131), Start-lag (D_i) is the minimum time which must elapse between starting job i on first machine A and Starting it on second machine B, while stop-lag

(E_i) is the minimum time which must elapse between completing job i on machine A and completing it on second machine B. The lag times may be smaller than the respective processing times.

Transportation time (t_i) as defined by Maggu and Das [5] is the minimum time which must elapse after completion of job i on the first machine A and then starting it on the second machine B. The physical situation corresponding to the problem mathematically can be modelled as follows.:

(i) The manufacturing system i.e. flow-shop consists of two different machines A and B installed at distant places which are ordered as AB according to the order of production stage. Let every machine remain continuously available with the subject that it can process only one job at a time.

(ii) Every job is completed through the same production stage that A $\rightarrow$ B.

(iii) Let A_i and B_i denote processing times of jobs i on machines A and B, respectively.

(iv) The same job-sequence occurs on each machine, in other words no passing is allowed in the flow-shop.

(v) Let $D_i \geq 0$ be the start-lag for job i.

(vi) Let $E_i \geq 0$ be the stop-lag for job i.

(vii) Let $t_i \geq 0$ be the transportation time for job i.

The question before us is to find an optimal rule to minimize the performance measure as total elapsed-time for the above stated 'n-job, 2-machine' flow-shop scheduling problem.

2. The Optimal Algorithm

The optimal algorithm is decomposed into following steps.

Step 1: Let t_i' denote the effective transportation time, defined by

$$t_i' = \max\ (D_i - A_i,\ E_i - B_i,\ t_i)$$

Step 2: Let G and H be two fictitious machines having respective processing times for job i as G_i and H_i, where G_i and H_i are defined by

$$G_i = A_i + t_i'$$

$$H_i = B_i + t_i'$$

Step 3: Find optimal sequence by Johnson's [1] procedure for '2-machine, n-job' problem on the reduced problem in step 2.

Step 4: The optimal sequence obtained in Step 3 gives the optimal sequence for the original problem.

3. Particular Studies

For every job i,

(1) If $D_i = A_i$ and $E_i = B_i$
then the algorithm reduces to Maggu and Das [5] problem algorithm.

(2) If either $t_i = 0$, or $D_i \geqq A_i + t_i$, $E_i \geqq t_i + B_i$, then the algorithm reduces to the Mitten-Johnson's [2] problem.

(3) If $t_i = 0$, $D_i = A_i$, $E_i = B_i$, then the algorithm reduces to Bellman's [6] and Johnson's [1] problem.

4. Proof of the Optimal Algorithm

Let U_{ix} and T_{ix} denote Starting and Completion times of any job i on machine X (X=A,B, i=1,2,3,...,n) respectively in a sequence S. From definition of Statt-lag D_i, we have

$$U_{iB} - U_{iA} \geqq D_i$$

Now

$$T_{iA} = U_{iA} + A_i$$

i.e.,

$$U_{iA} = T_{iA} - A_i$$

Hence, we have

$$U_{iB} - (T_{iA} - A_i) \geqq D_i$$

i.e.,

$$U_{iB} - T_{iA} \geqq D_i - A_i \tag{1}$$

From definition of Stop-lag E_i, we have

$$T_{iB} - T_{iA} \geqq E_i,$$

Where,

$$T_{iB} = U_{iB} + B_i$$

Hence, we have,

$$U_{iB} + B_i - T_{iA} \geqq E_i$$

i.e.,

$$U_{iB} - T_{iA} \geqq E_i - B_i \tag{2}$$

Also, from the definition of transportation time t_i, we have

(3) $U_{iB} - T_{iA} \geqq t_i$

Let

(4) $t'_i = \max \{D_i - A_i, E_i - B_i, t_i\}$

From (1), (2) & (3), it is obvious that

(5) $U_{iB} - T_{iA} \geqq t'_i$

Let us form a reduced problem 'two-machine, n-job problem' with transportation times from our given original problem replacing three times (Start-lag, Stop-lag, transportation time) by single time t'_i (which has been referred to here as effective time and is as defined in (4)).

Now in the original problem an optimal ordering of jobs to minimize total elapsed time is given by the following rule. Job i precedes job $i + 1$ if

$$\min (t_{iA} + t'_i, t_{i+1B} + t'_{i+1})$$

(6) $$\leqq \min (t_{i+1A} + t'_{i+1}, t_{iB} + t'_i),$$

where t_{ix} denotes the processing time of i-th job on machine X.

For this, let S and S' be the sequences of jobs given by

$$S = (J_1, J_2,\dots,J_{i-1}, J_i, J_{i+1}, J_{i+2},\dots,J_n)$$

$$S' = (J_1, J_2,\dots,J_{i-1}, J_{i+1}, J_i, J_{i+2},\dots,J_n)$$

Let $(U_{p\,X}, U'_{p\,X})$ and $(Y_{p\,X}, y'_{p\,X})$ denote the processing times and completion times of any p-th job on machine X in the process of sequences (S, S') respectively.

Let (U_p, U'_p) denote the transportation times of job p from machine A to the machine B in the process of sequences (S, S') respectively. That it is obvious that

(7) $Y_{p\,B} = \max (Y_{p\,A} + U_p, Y_{p-1,B}) + U_{p\,B}$

Now sequence S is preferable to S' if

(8) $Y_{n\,B} \leqq Y'_{n\,B}$

that is,

$$\max \quad (y_{n\,A} + U_n, Y_{n-1\,B}) + U_{n\,B}$$

$$\leqq \max \ (y'_{n\,A} + U'_n, Y'_{n-1\,B}) + U'_{n\,B}$$

Now $Y_{n\,A} + U_n = \sum_{i-1}^{n} t_{i\,A} + t'_n = Y'_{n\,A} + U'_n$, and

$$U_{n\ B} = U'_{n\ B} = t_{n\ B},$$

Inequality (8) is true, if

$$Y_{n-1\ B} \leqq Y'_{n-1\ B}$$

Continuing in this way, one can easily get:

$$Y_{p\ B} \leqq Y'_{p\ B}, \quad (p = i+2,\ i+3, \ldots, n)$$

and

$$\text{(9)} \qquad Y_{i+1\ B} \leqq Y'_{i+1\ B}$$

from (6) as shown below.

We proceed to calculate values of $Y_{i+1\ B}$ and $Y'_{i+1\ B}$

Now

$$\begin{aligned}
Y_{i+1\ B} &= \max\ (\, Y_{i+1\ A} + U_{i+1},\ Y_{i\ B}) + U_{i+1\ B} \\
&= \max \quad Y_{i+1\ A} + U_{i+1}, \\
&\quad\ \max\ (\, Y_{i\ A} + U_i,\ Y_{i-1\ B}\,) + U_{i\ B} \quad + U_{i+1\ B} \\
&= \max\ (\, Y_{i+1\ A} + U_{i+1} + U_{i+1\ B}, \\
&\qquad\quad Y_{i\ A} + U_i + U_{i\ B} + U_{i+1\ B}, \\
&\qquad\quad Y_{i-1\ B} + U_{i\ B} + U_{i+1\ B}\,) \\
&= \max\ (\, Y_{i-1\ A} + U_{i\ A} + U_{i+1\ A} + U_{i+1} + U_{i+1\ B}, \\
&\qquad\quad Y_{i-1\ A} + U_{i\ A} + U_i + U_{i\ B} + U_{i+1\ B}, \\
&\qquad\quad Y_{i-1\ B} + U_{i\ B} + U_{i+1\ B}\,)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Y'_{i+1\ B} &= \max\ (\, Y'_{i-1\ A} + U'_{i\ A} + U'_{i+1\ A} + U'_{i+1} + U'_{i+1\ B}, \\
&\qquad\quad Y'_{i-1\ A} + U'_{i\ A} + U'_i + U'_{i\ B} + U'_{i+1\ B}, \\
&\qquad\quad Y'_{i-1\ B} + U'_{i\ B} + U'_{i+1\ B}\,)
\end{aligned}$$

Now by comparing S and S', we can easily have:

$$Y_{i-1\ A} = Y'_{i-1\ A}$$

$$Y_{i-1\ B} = Y'_{i-1\ B}$$

$$U_{i\ X} = U'_{i+1\ X} = t_{i\ X},\ U_i = U'_{i+1} = t'_i$$

$$U_{i+1\ X} = U'_{i\ X} = t_{i+1\ X},\ U_{i+1} = U'_i = t'_{i+1}$$

Now using the corresponding values in (9), we have

$$\begin{aligned}&\max\ (\ Y_{i-1\ A} + t_{i\ A} + t_{i+1\ A} + t'_{i+1} + t_{i+1\ B},\\ &\qquad Y_{i-1\ A} + t_{i\ A} + t'_{i} + t_{i\ B} + t_{i+1\ B},\\ &\qquad Y_{i-1\ B} + t_{i\ B} + t_{i+1\ B}\ )\\ \leqq\ &\max\ (\ Y_{i-1\ A} + t_{i+1\ A} + t_{i\ A} + t'_{i} + t_{i\ B},\\ &\qquad Y_{i-1\ A} + t_{i+1\ A} + t'_{i+1} + t_{i+1\ B} + t_{i\ B},\\ &\qquad Y_{i-1\ B} + t_{i+1\ B} + t_{i\ B}\ )\end{aligned}$$

or, we have in order to yield (9).

$$\begin{aligned}&\max\ (\ Y_{i-1\ A} + t_{i\ A} + t_{i+1\ A} + t'_{i+1} + t_{i+1\ B},\\ &\qquad Y_{i-1\ A} + t_{i\ A} + t'_{i} + t_{i\ B} + t_{i+1\ B}\ )\\ \leqq\ &\max\ (\ Y_{i-1\ A} + t_{i+1\ A} + t_{i\ A} + t'_{i} + t_{i\ B},\\ &\qquad Y_{i-1\ A} + t_{i+1\ A} + t'_{i+1} + t_{i+1\ B} + t_{i\ B}\ )\end{aligned}$$

Now deduct $Y_{i-1\ A} + t_{i\ A} + t_{i+1\ A} + t'_{i} + t'_{i+1} + t_{i\ B} + t_{i+1\ B}$ from each term, we have

$$\begin{aligned}&\max\ (\ -\ t'_{i} - t_{i\ B},\ -\ t'_{i+1} - t_{i+1\ A}\ )\\ \leqq\ &\max\ (\ -\ t'_{i+1} - t_{i+1\ B},\ -t'_{i} - t_{i\ A}\ )\end{aligned}$$

Or,

$$\begin{aligned}&\min\ (\ t'_{i} + t_{i\ B},\ t'_{i+1} + t_{i+1\ A}\ )\\ \geqq\ &\min\ (\ t'_{i+1} + t_{i+1\ B},\ t'_{i} + t_{i\ A}\ )\end{aligned}$$

Or,

$$\begin{aligned}&\min\ (\ t'_{i\ A} + t'_{i},\ t'_{i+1} + t_{i+1\ B}\ )\\ \leqq\ &\min\ (\ t_{i+1\ A} + t'_{i+1},\ t'_{i} + t_{i\ B}\ )\end{aligned}$$

5. Numerical Example

Obtain optimal sequence for '5-job, 2-machine' problem given by the following tableau:

Job	Machine A	Machine B	Transportation time	Start-lag	Stop-lag
(i)	(A_i)	(B_i)	(t_i)	(D_i)	(E_i)
1	5	6	1	7	9
2	1	5	6	3	2
3	4	2	5	8	7
4	6	3	2	1	4
5	5	8	9	7	6

Define effective transportation-time t_i' as per step 1 of the algorithm as:

$$t_i' = \max(D_i - A_i,\ E_i - B_i,\ t_i)$$

$$\begin{aligned} t_1' &= \max(7-5,\ 9-6,\ 1) \\ &= \max(2,\ 3,\ 1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} t_2' &= \max(3-1,\ 2-5,\ 6) \\ &= \max(2,\ -3,\ 6) \\ &= 6 \end{aligned}$$

$$\begin{aligned} t_3' &= \max(8-4,\ 7-2,\ 5) \\ &= \max(4,\ 5,\ 5) \\ &= 5 \end{aligned}$$

$$\begin{aligned} t_4' &= \max(1-6,\ 4-3,\ 2) \\ &= \max(-5,\ 1,\ 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} t_5' &= \max(7-5,\ 6-8,\ 9) \\ &= \max(2,\ -2,\ 9) \\ &= 9 \end{aligned}$$

Let G and H be two fictitious machines introduced as per Step 2, with G_i and H_i given in the following table for job i.

Job	Machine G	Machine H
(i)	(G_i)	(H_i)
1	5+3=8	6+3=9
2	1+6=7	5+6=11
3	4+5=9	2+5=7
4	6+2=8	3+2=5
5	5+9=14	8+9=17

Now as per Step 3, applying Johnson's procedure for the above reduced problem, we have

21534 as the optimal schedule/sequence.

The total elapsed time for the schedule 21534 is given as below:

Job (i)	Machine A in - out	t'_i	Machine B in - out
2	0 - 1	6	7 - 12
1	1 - 6	3	12 - 18
5	6 - 11	9	20 - 28
3	11 - 15	5	28 - 30
4	15 - 21	2	30 - 32

Here T = total elapsed time = 32 for this optimal sequence 21534.
Here it may be observed as follows:

$D_1 = 7 \leq 12-1 = 11$
$D_2 = 3 \leq 7-0 = 7$
$D_3 = 8 \leq 28-11 = 17$
$D_4 = 1 \leq 30-15 = 15$
$D_5 = 7 \leq 20-6 = 14$

$E_1 = 9 \leq 18-6 = 12$
$E_2 = 2 \leq 12-1 = 11$
$E_3 = 7 \leq 30-15 = 15$
$E_4 = 4 \leq 32-21 = 11$
$E_5 = 6 \leq 28-11 = 17$

$t_1 = 1 \leq 12-6 = 6$
$t_2 = 6 \leq 7-1 = 6$
$t_3 = 5 \leq 28-15 = 13$
$t_4 = 2 \leq 30-21 = 9$
$t_5 = 9 \leq 20-11 = 9$

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