

OPTIMAL SEARCH FOR AN OBJECT WITH A RANDOM LIFETIME

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Abstract One stationary object is in one of n boxes with the distribution $\langle p_1, \dots, p_n \rangle$. Let $F_i(t)$ be the distribution of the lifetime of the object in box i and suppose that $F_i(t)$ is composed of two probability masses α_i at $t = 0$, β_i at $t = \infty$ and a probability density function $f_i(t)$ on the interval $(0, \infty)$ which is differentiable in t almost everywhere. Let c be the search cost per unit time. If the object is in box i and box i is searched for t hours, the object is detected with probability $1 - \exp(-\lambda_i t)$ whether it is alive or not. We suppose that the search is continued until the object is detected whether it is alive or not. If the searcher detects the living object in box i , he obtains a reward $r_i (> 0)$. If the searcher detects the died object, no reward is obtained. The criterion is to maximize the expected return (reward minus cost) until the object is detected. We obtain necessary and sufficient conditions for a policy to be optimal. Furthermore we obtain the optimal search rate in the implicit form. Specially we obtain the optimal search policy explicitly in the case that $f_i(t)$ ($i = 1, \dots, n$) are differentiable in t . We consider two numerical examples and give the explicit solutions. One of them is the case of the exponential lifetime distribution and another is the case of the uniform lifetime distribution. Finally we deal with a stopping problem in which the searcher is permitted to stop the search at any time. Some results about the optimal search policy and the optimal stopping time are obtained.

1. Introduction

In winter many alpinists are lost in a snowy mountain and the search for them is carried out. In such a search it is important to detect the alpinist while he lives. Since the capability of his survival depends exceedingly on the geographical feature and the weather in the accident place, his lifetime seems to be a random variable depended on the place. Therefore we must search first in the place where he cannot live long even if the efficiency of search is bad there. In this point our search model differs from the common one. If the alpinist dies, his family wants to detect his dead body. Therefore it is

supposed that the search is continued until the alpinist is detected whether he lives or not.

The search problem in such a situation is modelled as follows: One object is in one of n boxes and does not move among these boxes. Let p_i be the prior probability that the object is in box i ($\sum_{i=1}^n p_i = 1$). The lifetime T_i of the object in box i is a random variable according to the probability distribution $F_i(t)$ which is composed of two probability masses α_i at $t = 0, \beta_i$ at $t = \infty$ and a probability density function $f_i(t)$ on the interval $(0, \infty)$. The mass α_i is a probability that the object in box i dies before the search is begun. The mass β_i is a probability that the object in box i is alive eternally. Note that $\alpha_i + \int_0^{\infty} f_i(t) dt + \beta_i = 1$ for $i = 1, \dots, n$. We suppose that the density function $f_i(t)$ is differentiable in t almost everywhere. Let $c(>0)$ be the search cost per unit time which is supposed to be independent of the box. If the object is in box i and the box i is searched for t hours, the object is detected with probability $1 - \exp(-\lambda_i t)$ where $\lambda_i(>0)$ is given constant. Suppose that if the object is died, its dead body is detected with the same probability. If the live object is detected in box i , the searcher obtains a reward $r_i(>0)$. If the dead body of the object is detected, no reward is obtained. The problem is to find the optimal search policy, that is, the allocation of search time maximizing the expected return (i.e. reward minus cost) until the object is detected whether it is alive or not.

The next search problem can be modelled in the same mathematical form. The object does not die eternally. But the reward is diminished with time, that is, the reward of detecting the object in box i at time t is $r_i \beta_i(t)$ where $\beta_i(t)$ is a discounted rate in box i at time t and is supposed to be non-increasing in t . For example, if we put that $\beta_i(t) = 1 - F_i(t)$, we obtain the above-mentioned model.

A search policy may be denoted by the function $\phi(t) \equiv \{\phi_1(t), \dots, \phi_n(t)\}$ where $\phi_i(t)$ is search time in box i until time t and satisfies that $\sum_{i=1}^n \phi_i(t) = t$ and $\phi_i(t) \geq 0$ ($i = 1, \dots, n$) for any $t (>0)$. The meaning of a policy ϕ is that if we search in box i for $\phi_i(t)$ hours ($i = 1, \dots, n$) until time t and cannot detect the object, in the next time interval $[t, t+\Delta t]$ we search in box i for $\phi_i^1(t)\Delta t$ hours ($i = 1, \dots, n$) knowing the failure of search until now. First we define some quantities.

$$g_i(t) \equiv \text{the expected return of detecting the object in box } i \text{ at time } t \\ = r_i \left\{ \int_t^{\infty} f_i(s) ds + \beta_i \right\} - ct$$

$P_i(t|\phi) \equiv$ the probability of detecting the object by time t given that it is in box i and that a polity ϕ is used

$$= 1 - \exp\{-\lambda_i \phi_i(t)\}.$$

$R[\phi] \equiv$ the expected return by using a policy ϕ

$$= \sum_{i=1}^n p_i \int_0^{\infty} g_i(t) P_i'(t|\phi) dt$$

where $P_i'(t|\phi)$ is the derivative by t . Integrating by parts,

$$(1.1) \quad R[\phi] = \sum_{i=1}^n p_i \alpha_i (1 - \alpha_i) + \sum_{i=1}^n p_i \int_0^{\infty} g_i'(t) \exp\{-\lambda_i \phi_i(t)\} dt.$$

Since the first term of the equation (1.1) is independent of the policy ϕ , the problem is given as follows:

$$(1.2) \quad R[\phi] \equiv \sum_{i=1}^n p_i \int_0^{\infty} g_i'(t) \exp\{-\lambda_i \phi_i(t)\} dt \rightarrow \max_{\phi}$$

subject to

$$(1.3) \quad \sum_{i=1}^n \phi_i(t) = t \text{ for any } t(\geq 0)$$

$$(1.4) \quad \phi_i(t) \text{ is nonnegative, continuous and nondecreasing in } t(\geq 0) \\ (i = 1, \dots, n).$$

Note that the integration of the equation (1.2) exists since the density $f_i^*(t)$ is continuous almost everywhere. The assumption that the search cost c is independent of the box is a very strong restriction, but it is necessary for formulating the problem in the above form. If the search cost depends on the box, it seems to be more difficult to analyze the problem. On the other hand, it seems that we can replace the exponential detection function with the more general form.

The search problem for a stationary object is investigated in many literatures (for example [1], [2], ..., [12]). A model which has been analyzed by de Guenin [5] is given as follows: One stationary object exists in a continuous search space X with the prior distribution $g(x)$. This object does not die eternally. Let $P[\phi(x)]$ be the conditional probability of detecting the object with an effort $\phi(x)$ when the object is indeed at x . The objective is to obtain an allocation of search effort maximizing the probability of detecting the object by using the given total effort Φ . This problem is formulated as follows:

$$\left\{ \begin{array}{l} \int_X g(x) P[\phi(x)] dx \quad \max \\ \text{subject to} \quad \phi \\ \int_X \phi(x) dx = \Phi, \quad \phi(x) \geq 0. \end{array} \right.$$

Under the suitable regular conditions, de Guenin [5] gives a necessary condition for a policy being optimal which is denoted by the form of the Neyman-Pearson lemma, that is, if a policy $\phi^*(t)$ is optimal, there is a positive constant λ such that

$$g(x)P'[\phi^*(x)] \begin{cases} \geq \\ \leq \end{cases} \lambda \quad \text{if } \phi^*(t) \begin{cases} > \\ \leq \end{cases} 0$$

where $P'[\phi^*(x)]$ is the derivative of $P[\phi]$ at $\phi = \phi^*(x)$.

In Section 2, using the similar method to de Guenin [5], we find necessary and sufficient conditions for a policy to be optimal in our model (Theorem 1). Furthermore we obtain the optimal search rates at any time (Theorem 2). Theorem 2 contains a quantity which cannot be determined explicitly and therefore the optimal search rates cannot be obtained really. In Section 3, we obtain explicitly the optimal search policy in the case of the differentiable lifetime density function, that is, $f_i(t)$ ($i = 1, \dots, n$) are differentiable in $t \in (0, \infty)$ (Theorem 3). We give a numerical example with the exponential lifetime distribution in three-box case. In Section 4, we treat the case of the uniform lifetime distribution and give a numerical example in two-box case.

In the above model for the purpose of modelling the alpinist problem, we suppose that the search is continued until the object is detected whether it is alive or not. But usually if the search cost c is positive, it is useless to continue the search for the object when it is perhaps died. Therefore we must consider a stopping problem with permitting to stop the search at any time. In Section 5, we deal such a stopping problem and obtain some properties of the optimal search policy (Theorem 4). Specially in the case of the differentiable lifetime density function, a necessary condition for the optimal stopping time is obtained (Theorem 5).

2. The Properties of the Optimal Search Policy

In the next theorem we give necessary and sufficient conditions for a policy to be optimal in the Neyman-Pearson type. Necessary condition is proved by using the de Guenin's method with respect to time instead of space. Sufficient condition is proved by considering the Gâteaux differential.

Theorem 1. Necessary and sufficient conditions for a policy $\phi^*(t) \equiv \{ \phi_1^*(t), \dots, \phi_n^*(t) \}$ to be optimal are given as follows: There is a non-negative function $\mu(t)$ for any $t(\geq 0)$ such that

$$(2.1) \quad p_i \lambda_i \int_t^\infty [-g'_i(s)] \exp\{-\lambda_i \phi_i^*(s)\} ds \left\{ \begin{array}{l} = \\ \leq \end{array} \right\} u(t) \quad \text{if } \phi_i^*(t) \left\{ \begin{array}{l} > \\ = \end{array} \right\} 0.$$

Proof: The proof of the necessity. Suppose that the policy ϕ^* is optimal. We consider any time $t_1 (\geq 0)$ such that $\phi_i^*(t_1) > 0$ and define a policy ϕ for any box $j (\neq i)$ and any positive constants $\epsilon, \Delta t$ as follows:

$$\phi_i(t) \equiv \begin{cases} \phi_i^*(t) & \text{if } 0 < t < t_1 \\ \phi_i^*(t) - \frac{\epsilon}{\Delta t} (t - t_1) & \text{if } t_1 \leq t \leq t_1 + \Delta t \\ \phi_i^*(t) - \epsilon & \text{if } t_1 + \Delta t \leq t < \infty, \end{cases}$$

$$\phi_j(t) \equiv \begin{cases} \phi_j^*(t) & \text{if } 0 \leq t \leq t_1 \\ \phi_j^*(t) + \frac{\epsilon}{\Delta t} (t - t_1) & \text{if } t_1 \leq t \leq t_1 + \Delta t \\ \phi_j^*(t) + \epsilon & \text{if } t_1 + \Delta t \leq t < \infty \end{cases}$$

and $\phi_k(t) \equiv \phi_k^*(t)$ if any $k (\neq i, j)$. It can be proved that if $\epsilon = o(\Delta t)$, the policy ϕ satisfies restrictions (1.3), (1.4) for a sufficiently small Δt .

For ϕ^* and ϕ , we obtain

$$(2.2) \quad R[\phi^*] - R[\phi] = p_i \int_{t_1}^\infty g'_i(t) [\exp\{-\lambda_i \phi_i^*(t)\} - \exp\{-\lambda_i \phi_i(t)\}] dt \\ + p_j \int_{t_1}^\infty g'_j(t) [\exp\{-\lambda_j \phi_j^*(t)\} - \exp\{-\lambda_j \phi_j(t)\}] dt.$$

By the mean value theorem,

$$(2.3) \quad \exp\{-\lambda_i \phi_i^*(t)\} - \exp\{-\lambda_i \phi_i(t)\} \\ = \begin{cases} -\frac{\lambda_i \epsilon}{\Delta t} (t - t_1) \exp[-\lambda_i \{\phi_i^*(t) - \frac{\theta_1 \epsilon}{\Delta t} (t - t_1)\}] & \text{if } t_1 \leq t \leq t_1 + \Delta t \\ -\lambda_i \epsilon \exp[-\lambda_i \{\phi_i^*(t) - \theta_2 \epsilon\}] & \text{if } t_1 + \Delta t \leq t < \infty \end{cases}$$

and

$$(2.4) \quad \exp\{-\lambda_j \phi_j^*(t)\} - \exp\{-\lambda_j \phi_j(t)\} \\ = \begin{cases} \frac{\lambda_j \epsilon}{\Delta t} (t - t_1) \exp[-\lambda_j \{\phi_j^*(t) + \frac{\theta_3 \epsilon}{\Delta t} (t - t_1)\}] & \text{if } t_1 \leq t \leq t_1 + \Delta t \\ \lambda_j \epsilon \exp[-\lambda_j \{\phi_j^*(t) + \theta_4 \epsilon\}] & \text{if } t_1 + \Delta t \leq t < \infty \end{cases}$$

where $0 \leq \theta_i \leq 1$ for $i = 1, 2, 3, 4$. Substituting the relations (2.3), (2.4) into the equation (2.2),

$$(2.5) \quad R[\phi^*] - R[\phi] =$$

$$- p_i \frac{\lambda_i \varepsilon}{\Delta t} \int_{t_1}^{t_1 + \Delta t} g_i(t) (t - t_1) \exp[-\lambda_i \{\phi_i^*(t) - \frac{\theta_1 \varepsilon}{\Delta t} (t - t_1)\}] dt$$

$$- p_i \lambda_i \varepsilon \int_{t_1 + \Delta t}^{\infty} g_i'(t) \exp[-\lambda_i \{\phi_i^*(t) - \theta_2 \varepsilon\}] dt$$

$$+ p_j \frac{\lambda_j \varepsilon}{\Delta t} \int_{t_1}^{t_1 + \Delta t} g_j(t) (t - t_1) \exp[-\lambda_j \{\phi_j^*(t) + \frac{\theta_3 \varepsilon}{\Delta t} (t - t_1)\}] dt$$

$$+ p_j \lambda_j \varepsilon \int_{t_1 + \Delta t}^{\infty} g_j'(t) \exp[-\lambda_j \{\phi_j^*(t) + \theta_4 \varepsilon\}] dt \geq 0.$$

The last inequality of (2.5) follows from the optimality of the policy ϕ^* . Since $\varepsilon = o(\Delta t)$, the first and third terms of (2.5) converge to zero as Δt approaches to zero. Dividing the second and fourth terms of (2.5) by $\varepsilon (> 0)$ and letting Δt approach to zero,

$$(2.6) \quad p_i \lambda_i \int_{t_1}^{\infty} [-g_i'(t)] \exp\{-\lambda_i \phi_i^*(t)\} dt$$

$$\geq p_j \lambda_j \int_{t_1}^{\infty} [-g_j'(t)] \exp\{-\lambda_j \phi_j^*(t)\} dt.$$

If we select the box j ($\neq i$) satisfying $\phi_j^*(t_1) > 0$, the discussion obtained by exchanging i for j in the above discussion can be developed and therefore we obtain the opposite inequality to (2.6). Then if $\phi_i^*(t_1)$ and $\phi_j^*(t_1)$ are positive, the equality is satisfied in (2.6). In other words, if $\phi_i^*(t)$ is positive, the left hand side of (2.6) becomes independent of i . Then a function $\mu(t)$ exists such that

$$(2.7) \quad p_i \lambda_i \int_t^{\infty} [-g_i'(s)] \exp\{-\lambda_i \phi_i^*(s)\} ds = \mu(t) \quad \text{if } \phi_i^*(t) > 0.$$

On the other hand if $\phi_j^*(t_1) = 0$, the opposite inequality of (2.6) is not satisfied. Then

$$(2.8) \quad p_j \lambda_j \int_t^{\infty} [-g_j'(s)] \exp\{-\lambda_j \phi_j^*(s)\} ds \leq \mu(t) \quad \text{if } \phi_j^*(t) = 0.$$

The relation (2.1) is derived from (2.7), (2.8).

The proof of the sufficiency. Suppose that the policy ϕ^* satisfies the relation (2.1) but is not optimal. Therefore there is a policy ϕ such that

$R[\phi] > R[\phi^*]$. We put

$$\xi(t) \equiv \phi(t) - \phi^*(t) = \{\phi_1(t) - \phi_1^*(t), \dots, \phi_n(t) - \phi_n^*(t)\}$$

and consider the Gâteaux differential $R'[\phi : \xi]$ of the functional $R[\phi]$ at the point $\phi = \phi^*$ in the direction of ξ which is defined by

$$R'[\phi^* : \xi] \equiv \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \{R[\phi^* + \varepsilon\xi] - R[\phi^*]\}.$$

By the equation (1.2) and the mean value theorem, we obtain that for $0 \leq \theta \leq 1$,

$$\begin{aligned} & \varepsilon^{-1} \{R[\phi^* + \varepsilon\xi] - R[\phi^*]\} \\ &= \sum_{i=1}^n p_i \int_0^\infty g_i'(t) \varepsilon^{-1} [\exp\{-\lambda_i[\phi_i^*(t) + \varepsilon\xi_i(t)]\} - \exp\{-\lambda_i\phi_i^*(t)\}] dt \\ &= \sum_{i=1}^n p_i \int_0^\infty g_i'(t) \xi_i(t) (-\lambda_i) \exp\{-\lambda_i[\phi_i^*(t) + \theta\varepsilon\xi_i(t)]\} dt. \end{aligned}$$

Letting ε approach to zero, we obtain the Gâteaux differential.

$$\begin{aligned} (2.9) \quad R'[\phi^* : \xi] &= \sum_{i=1}^n p_i \lambda_i \int_0^\infty [-g_i'(t)] \{\phi_i(t) - \phi_i^*(t)\} \exp\{-\lambda_i\phi_i^*(t)\} dt \\ &= \sum_{i=1}^n \int_0^\infty \int_0^t p_i \lambda_i [-g_i'(t)] \{\phi_i'(s) - \phi_i^{*\prime}(s)\} \exp\{-\lambda_i\phi_i^*(t)\} ds dt \\ &= \sum_{i=1}^n \int_0^\infty \int_s^\infty p_i \lambda_i [-g_i'(t)] \{\phi_i'(s) - \phi_i^{*\prime}(s)\} \exp\{-\lambda_i\phi_i^*(t)\} dt ds \\ &= \sum_{i=1}^n \int_0^\infty [p_i \lambda_i \int_s^\infty [-g_i'(t)] \exp\{-\lambda_i\phi_i^*(t)\} dt] \{\phi_i'(s) - \phi_i^{*\prime}(s)\} ds \\ &\leq \sum_{i=1}^n \int_0^\infty \mu(s) \{\phi_i'(s) - \phi_i^{*\prime}(s)\} ds \quad (\text{by the relation (2.1)}) \\ &= \int_0^\infty \mu(s) \left\{ \sum_{i=1}^n \phi_i'(s) - \sum_{i=1}^n \phi_i^{*\prime}(s) \right\} ds \\ &= 0. \quad (\text{by the condition (1.3)}) \end{aligned}$$

On the other hand since $g_i'(t) \leq 0$ ($i = 1, \dots, n$), the functional $R[\phi]$ is concave in ϕ . Therefore

$$\begin{aligned} R[\phi^* + \varepsilon\xi] - R[\phi^*] &= R[(1 - \varepsilon)\phi^* + \varepsilon\phi] - R[\phi^*] \\ &\geq (1 - \varepsilon)R[\phi^*] + \varepsilon R[\phi] - R[\phi^*] = \varepsilon \{R[\phi] - R[\phi^*]\}, \end{aligned}$$

that is,

$$(2.10) \quad \epsilon^{-1}\{R[\phi^* + \epsilon\xi] - R[\phi^*]\} \geq R[\phi] - R[\phi^*] > 0.$$

The last inequality follows from the assumption of the contradiction method. Letting ϵ approach to zero in (2.10), we obtain that $R'[\phi^*; \xi] > 0$ which contradicts to (2.9). Then the policy ϕ^* is optimal. (q.e.d.)

In the next theorem, we obtain the optimal search rate at any time.

Theorem 2. If the derivatives $f_i'(t)$ ($i = 1, \dots, n$) exist almost everywhere, then the optimal policy $\phi^*(t) = \{\phi_1^*(t), \dots, \phi_n^*(t)\}$ is given almost everywhere as follows. For the function $\mu(t)$ in Theorem 1, we define

$$(2.11) \quad I(t) \equiv \{i | p_i \lambda_i [-g_i'(t)] \exp[-\lambda_i \phi_i^*(t)] = -\mu'(t) \text{ and } \frac{g_i''(t)}{g_i'(t)} > \frac{\mu''(t)}{\mu'(t)}\}.$$

Then $I(t) = \{i | \phi_i^*(t) > 0\}$ and the optimal search rate is given as follows:

$$(2.12) \quad \phi_i^*(t) = \begin{cases} \lambda_i^{-1} \left[\frac{g_i''(t)}{g_i'(t)} + \left(1 - \sum_{j \in I(t)} \lambda_j^{-1} \frac{g_j''(t)}{g_j'(t)} \right) \left(\sum_{j \in I(t)} \lambda_j^{-1} \right)^{-1} \right] & \text{if } i \in I(t) \\ 0 & \text{if } i \notin I(t). \end{cases}$$

Proof: Put $T_i \equiv \{t | \phi_i^*(t) > 0\}$ ($i = 1, \dots, n$). By Theorem 1,

$$(2.13) \quad p_i \lambda_i \int_t^\infty [-g_i'(s)] \exp[-\lambda_i \phi_i^*(s)] ds = \mu(t) \quad \text{if } t \in T_i.$$

Differentiating both sides of (2.13) with respect to t ,

$$(2.14) \quad p_i \lambda_i [-g_i'(t)] \exp[-\lambda_i \phi_i^*(t)] = -\mu'(t) \quad \text{if } t \in T_i$$

or

$$\phi_i^*(t) = \lambda_i^{-1} \{\log(p_i \lambda_i [-g_i'(t)]) - \log[-\mu'(t)]\} \quad \text{if } t \in T_i.$$

Differentiating once more,

$$(2.15) \quad \dot{\phi}_i^*(t) = \lambda_i^{-1} \left[\frac{g_i''(t)}{g_i'(t)} - \frac{\mu''(t)}{\mu'(t)} \right] \quad \text{if } t \in T_i.$$

Therefore if $t \in T_i$,

$$\frac{g_i''(t)}{g_i'(t)} > \frac{\mu''(t)}{\mu'(t)}.$$

On the other hand, if the equation (2.14) holds in the complement of the closure of the set T_i , we obtain by the same method that $g_i''(t)/g_i'(t) =$

$\mu''(t)/\mu'(t)$. Therefore $I(t)$ defined by (2.11) coincides with the set $\{i | \phi_i^*(t) > 0\}$ almost everywhere. Next substituting (2.15) into $\sum_{j=1}^n \phi_j^*(t) = 1$,

$$1 = \sum_{j \in I(t)} \phi_j^*(t) = \sum_{j \in I(t)} \lambda_j^{-1} \left[\frac{g_j''(t)}{g_j'(t)} - \frac{\mu''(t)}{\mu'(t)} \right]$$

or

$$(2.16) \quad \frac{\mu''(t)}{\mu'(t)} = - \left(1 - \sum_{j \in I(t)} \lambda_j^{-1} \frac{g_j''(t)}{g_j'(t)} \right) \left(\sum_{j \in I(t)} \lambda_j^{-1} \right)^{-1}.$$

Substituting (2.16) into (2.15), we obtain (2.12). (q.e.d.)

Remark 1. Since the function $\mu(t)$ is not known, the optimal search rate $\phi_i^*(t)$ is not explicit.

Remark 2. Let $p_i(t|\phi)$ be the posterior probability that the object is in box i given that it is not detected until time t by using a policy ϕ . By the Baye's rule,

$$(2.17) \quad p_i(t|\phi^*) = \frac{p_i \exp\{-\lambda_i \phi_i^*(t)\}}{\sum_{j=1}^n p_j \exp\{-\lambda_j \phi_j^*(t)\}}.$$

If $i \in I(t)$, by (2.14) and (2.17), we know that $p_i(t|\phi^*) \lambda_i [-g_i'(t)]$ is independent of i . In other words, the optimal search policy is to allocate the search time to equalize the value of $p_i(t|\phi^*) \lambda_i [-g_i'(t)]$ for all boxes which are searched at time t . But we must note that the optimal policy is not necessarily to search in all boxes which maximize the value of $p_i(t|\phi^*) \lambda_i [-g_i'(t)]$. We shall give an example indicating this notice in Section 4. On the other hand, in the case of the object being immortal, $[-g_i'(t)]$ is independent of i and it is well-known that the optimal policy is to search in all boxes maximizing the value of $p_i(t|\phi^*) \lambda_i$.

3. The case of the Differentiable Lifetime Density Function

In this section we suppose that the lifetime density functions $f_i(t)$ ($i = 1, \dots, n$) are differentiable in $t \in (0, \infty)$. We consider a following policy ϕ : We put

$$h_i(t|\phi) \equiv p_i \lambda_i [-g_i'(t)] \exp\{-\lambda_i \phi_i(t)\} \quad (i = 1, \dots, n)$$

and

$$J(t) \equiv \{i | h_i(t|\phi) = \max_{1 \leq j \leq n} h_j(t|\phi)\} \quad \text{for } t \in [0, \infty).$$

The policy ϕ searches at time t in all boxes maximizing $h_i(t|\phi)$ with rates given by

$$(3.1) \quad \phi_i'(t) = \begin{cases} \lambda_i^{-1} \left[\frac{g_i'(t)}{g_i(t)} + \left(1 - \sum_{j \in J(t)} \lambda_j^{-1} \frac{g_j'(t)}{g_j(t)} \right) \left(\sum_{j \in J(t)} \lambda_j^{-1} \right)^{-1} \right] & \text{if } i \in J(t) \\ 0 & \text{if } i \notin J(t). \end{cases}$$

Lemma 1. Suppose that $f_i(t)$ ($i=1, \dots, n$) are differentiable in t . If the above policy i searches in box t at time t , it searches in box i at any time in the interval $[t, \infty)$.

Proof: Suppose that though the above policy ϕ searches in box i at time t , it does not necessarily search in box i at all time t in $[t, \infty)$. Then there is a time s ($s \geq t$) such that $i \in J(s)$ and $i \notin J(s+\Delta t)$ for a sufficiently small Δt (> 0). We consider box j such that $j \in J(s+\Delta t)$. Therefore

$$(3.2) \quad h_i(s+\Delta t|\phi) < h_j(s+\Delta t|\phi).$$

If $j \notin J(s)$, $h_i(s|\phi) > h_j(s|\phi)$ which is contradictory to the continuity of the function $h_i(t|\phi)$ in t . Then $j \in J(s)$. Therefore

$$(3.3) \quad h_i(s|\phi) = h_j(s|\phi).$$

On the other hand, we obtain

$$(3.4) \quad h_i'(s|\phi) = p_i \lambda_i \exp\{-\lambda_i \phi_i(s)\} [-g_i''(s) + g_i'(s) \lambda_i \phi_i'(s)]$$

Substituting (3.1) into (3.4) [note that $i \in J(s)$], we obtain

$$(3.5) \quad \begin{aligned} h_i'(s|\phi) &= h_i(s|\phi) \times [\text{constant in } i] \\ &= h_j(s|\phi) \times [\text{constant in } i] \quad (\text{by the relation (3.3)}) \\ &= h_j'(s|\phi) \end{aligned}$$

Two equations (3.3) and (3.5) lead us to the fact that $h_i(s+\Delta t|\phi) = h_j(s+\Delta t|\phi)$ which contradicts to (3.2). Then by the contradiction method, the proof is completed. (q.e.d.)

Theorem 3. If $f_i(t)$ ($i=1, \dots, n$) are differentiable in $t \in (0, \infty)$, a policy which searches at time t in all boxes maximizing $h_i(t|\phi)$ with rates given by (3.1) is optimal.

Proof: We prove that ϕ satisfies the sufficient condition in Theorem 1, that is,

$$(3.6) \quad \text{if } i \in J(t), \int_t^\infty h_i(s|\phi) ds \geq \int_t^\infty h_j(s|\phi) ds \quad \text{for any } t, j.$$

Noting that the policy ϕ searches all boxes in the long run, let t_1 be the first time such that $\phi_j(t_1) > 0$ and $t_1 > t$.

By Lemma 1, we obtain

$$h_i(s|\phi) \begin{cases} > \\ = \end{cases} h_j(s|\phi) \quad \text{if } s \begin{cases} < \\ \geq \end{cases} t_1$$

which leads us to (3.6).

(q.e.d)

Corollary 1. If $f_i(t) = 0$ ($0 < t < \infty$) or $r_i = 0$ for $i=1, \dots, n$, then a policy ϕ which searches at time t in all boxes maximizing $p_i \lambda_i \exp\{-\lambda_i \phi_i(t)\}$ in proportion to λ_i^{-1} is optimal.

Proof: Noting that $[-g_i(t)] = c$ for any i, t in this case, we obtain the conclusion by Theorem 3.

(q.e.d)

Remark 3. It is interesting that the optimal search policy is independent of r_i ($i=1, \dots, n$) in Corollary 1. Specially the result of the case that $\beta_i = 1$ for $i=1, \dots, n$ is well-known.

Numerical example 1. We consider the 3-box case with the exponential lifetime distribution, that is $F_i(t) = 1 - \exp(-\theta_i t)$ ($\theta_i > 0$) for $i=1, 2, 3$. We give the values of parameters as follows: $\langle p_1, p_2, p_3 \rangle = \langle 0.5, 0.3, 0.2 \rangle$, $(\lambda_1, \lambda_2, \lambda_3) = (0.6, 0.9, 1.2)$, $(\theta_1, \theta_2, \theta_3) = (0.1, 0.3, 0.5)$, $c = 1$ and $r_i = 2$ ($i=1, 2, 3$). We can obtain the optimal search policy ϕ^* by Theorem 3 as follows:

$$h_i(t|\phi^*) = p_i \lambda_i \{r_i \theta_i \exp(-\theta_i t) + c\} \exp\{-\lambda_i \phi_i^*(t)\}.$$

$$h_i(0|\phi^*) = p_i \lambda_i (r_i \theta_i + c) = \begin{cases} 0.36 & (i = 1) \\ 0.432 & (i = 2) \\ 0.48 & (i = 3). \end{cases}$$

Since $h_3(0|\phi^*) > h_2(0|\phi^*) > h_1(0|\phi^*)$, ϕ^* must search in box 3 only until time t_1 such that $h_3(t_1|\phi^*) = h_2(t_1|\phi^*)$, that is, t_1 is the unique positive root of the equation

$$0.24[\exp(-0.5t) + 1] \exp(-1.2t) = 0.27[0.6\exp(-0.3t) + 1].$$

If $t_1 < t < t_2$, ϕ^* must search in box 2, 3 only with rates given by (3.1), that is,

$$\phi_i^*(t) = \begin{cases} \frac{4}{7} - \frac{3}{7} [3 + 5\exp(0.3t)]^{-1} + \frac{5}{21} [1 + \exp(0.5t)]^{-1} & (i = 2) \\ \frac{3}{7} + \frac{3}{7} [3 + 5\exp(0.3t)]^{-1} - \frac{5}{21} [1 + \exp(0.5t)]^{-1} & (i = 3), \end{cases}$$

The time t_2 is given by $h_2(t_2|\phi^*) = h_1(t_2|\phi^*)$, that is, t_2 is the unique positive root of the equation

$0.27[0.6\exp(-0.3t) + 1] \exp\{-0.9\phi_2^*(t)\} = 0.3[0.2\exp(-0.1t) + 1]$
 where $\phi_2^*(t)$ is given by

$$\phi_2^*(t) = \int_{t_1}^t \left\{ \frac{4}{7} - \frac{3}{7} [3 + 5\exp(0.3s)]^{-1} + \frac{5}{21} [1 + \exp(0.5s)]^{-1} \right\} ds.$$

If $t_2 < t < \infty$, ϕ^* must search in all boxes with rates given by

$$\phi_i^*(t) = \begin{cases} \frac{6}{13} - \frac{7}{78} [1+5\exp(0.1t)]^{-1} + \frac{6}{13} [3+5\exp(0.3t)]^{-1} + \frac{15}{78} [1+\exp(0.5t)]^{-1} \\ \frac{4}{13} + \frac{4}{78} [1+5\exp(0.1t)]^{-1} - \frac{9}{13} [3+5\exp(0.3t)]^{-1} + \frac{10}{78} [1+\exp(0.5t)]^{-1} \\ \frac{3}{13} + \frac{3}{78} [1+5\exp(0.1t)]^{-1} + \frac{3}{13} [3+5\exp(0.3t)]^{-1} - \frac{25}{78} [1+\exp(0.5t)]^{-1} \end{cases}$$

($i = 3$).

Remark 4. If the object can live for ever, the order of the start of the optimal search is box 1, 2, 3 since $p_1\lambda_1 > p_2\lambda_2 > p_3\lambda_3$ which is a well-known result. But in this model the order must be box 3, 2, 1 since the expected lifetime in each box is $(\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1}) = (10, 10/3, 2)$. Namely, though the efficiency of search is worst in box 3, the optimal search policy must search in box 3 first since the expected lifetime is very small there.

4. The Case of the Uniform Lifetime Distribution

In this section, we suppose that the lifetime distribution of the object is uniform, that is,

$$F_i(t) = \begin{cases} t/a_i & \text{for } 0 \leq t \leq a_i \\ 1 & \text{for } a_i < t \leq \infty \end{cases} \quad (i = 1, \dots, n).$$

Theorem 2 does not give the optimal search policy explicitly. Theorem 3 cannot be held in this case since $f_i^*(t)$ is not continuous at $t = a_i$. The author made efforts to find an algorithm to obtain the optimal policy explicitly, but could not find the available algorithm. It seems that we follow Theorem 3 in the general situation but must make a suitable modification in the neighborhood of the discontinuous point of $f_i^*(t)$ ($i=1, \dots, n$). To see this point, we give a numerical example.

Numerical example 2. We consider the 2-box problem with following values of parameters: $\langle p_1, p_2 \rangle = \langle 0.6, 0.4 \rangle$, $(\lambda_1, \lambda_2) = (0.7, 0.9)$, $(a_1, a_2) = (6, 3)$, $(r_1, r_2) = (3, 4)$ and $c = 1$. We put $h_i(t|\phi^*)$

$= p_i \lambda_i [-g_i'(t)] \exp[-\lambda_i \phi_i^*(t)]$. Theorem 1 states that the optimal policy searches at time t in boxes which attain the maximum of the area of the region under the function $h_i(s|\phi^*)$ on the interval $[t, \infty)$. Then in the general situation, ϕ^* searches in boxes maximizing $h_i(t|\phi^*)$, but in the neighborhood of $t = a_i$ ($i=1, 2$), ϕ^* must be chosen such that the above property is satisfied. After trial and error, we offer the following policy ϕ^* as the optimal policy (which will be proved to be optimal later). The policy ϕ^* searches in boxes in accordance with Table 1 in which $t_2 < 3 < t_3$ and $t_4 < 6 < t_5$. See Fig. 1. When ϕ^* searches both boxes, the proportion of the search in box 1 is $\lambda_1^{-1}/(\lambda_1^{-1} + \lambda_2^{-1}) = 9/16$ by Theorem 2.

Table 1.

time interval	$(0, t_1)$	(t_1, t_2)	(t_2, t_3)	(t_3, t_4)	(t_4, t_5)	(t_5, ∞)
boxes to be searched	2	1,2	1	1,2	2	1,2

For this policy ϕ^* , $h_i(t|\phi^*)$ ($i=1, 2$) are given as follows:

$$h_1(t|\phi^*) = \begin{cases} 0.63 & \text{if } 0 \leq t \leq t_1 \\ 0.63 \exp[-0.7 \times \frac{9}{16}(t-t_1)] & \text{if } t_1 \leq t \leq t_2 \\ 0.63 \exp[-0.7 \{ \frac{9}{16}(t_2-t_1) + (t-t_2) \}] & \text{if } t_2 \leq t \leq t_3 \\ 0.63 \exp[-0.7 \{ \frac{9}{16}(t_2-t_1) + (t_3-t_2) + \frac{9}{16}(t-t_3) \}] & \text{if } t_3 \leq t \leq t_4 \\ 0.63 \exp[-0.7 \{ \frac{9}{16}(t_2-t_1) + (t_3-t_2) + \frac{9}{16}(t_4-t_3) \}] & \text{if } t_4 \leq t \leq 6 \\ 0.42 \exp[-0.7 \{ \frac{9}{16}(t_2-t_1) + (t_3-t_2) + \frac{9}{16}(t_4-t_3) \}] & \text{if } 6 < t < t_5 \\ 0.42 \exp[-0.7 \{ \frac{9}{16}(t_2-t_1) + (t_3-t_2) + \frac{9}{16}(t_4-t_3) + \frac{9}{16}(t-t_5) \}] & \text{if } t_5 < t < \infty \end{cases}$$

$$h_2(t|\phi^*) = \begin{cases} 0.94 \exp[-0.9t] & \text{if } 0 \leq t \leq t_1 \\ 0.94 \exp[-0.9 \{ t_1 + \frac{7}{16}(t_2-t_1) \}] & \text{if } t_2 \leq t \leq 3 \\ 0.36 \exp[-0.9 \{ t_1 + \frac{7}{16}(t_2-t_1) \}] & \text{if } 3 \leq t \leq t_3 \\ 0.36 \exp[-0.9 \{ t_1 + \frac{7}{16}(t_2-t_1) + \frac{7}{16}(t_4-t_3) + (t-t_4) \}] & \text{if } t_4 \leq t \leq t_5 \\ h_1(t|\phi^*) & \text{otherwise.} \end{cases}$$

which are described in Fig. 1. Here five threshold time t_i ($i=1, 2, \dots, 5$) are determined as follows: If box 2 only is searched, the function $h_2(t|\phi^*)$ decreases in t and intersects the function $h_1(t|\phi^*)$ ($= h_1(0|\phi^*) = 0.63$) at

last. Then let t_1 be the first time at which $h_2(t|\phi^*) = h_1(t|\phi^*)$, that is, t_1 is a unique root of the equation

$$0.94\exp[-0.9t] = 0.63.$$

If we fix the time t_2 , then t_3 can be determined as a function of t_2 by the equation

$$0.63\exp[-0.7\{\frac{9}{16}(t_2-t_1) + (t_3-t_2)\}] = 0.36\exp[-0.9\{t_1 + \frac{7}{16}(t_2-t_1)\}]$$

which states that two functions $h_1(t|\phi^*)$, $h_2(t|\phi^*)$ separating from each other at time t_2 become equivalent to each other again at time t_3 . We determine t_2 (therefore t_3 also) such that two region A_1 , A_2 in Fig. 1 have the same areas, that is,

$$\int_{t_2}^{t_3} [h_1(t|\phi^*) - h_2(t|\phi^*)] dt = 0.$$

Furthermore we determine t_4 , t_5 such that two regions B_1 , B_2 in Fig. 1 have the same areas. Thus we obtain following values:

$$\begin{aligned} t_1 &= \frac{10}{9} \log \frac{4}{3} (\cong 0.32) & t_2 &= 3 - \frac{5}{7} \log \frac{7}{3} (\cong 2.395) \\ t_3 &= 3 + \frac{5}{7} \log \frac{7}{3} (\cong 3.605) & t_4 &= 6 - \frac{10}{9} \log \frac{21}{10} (\cong 5.176) \\ t_5 &= 6 + \frac{10}{9} \log \frac{21}{10} (\cong 6.824). \end{aligned}$$

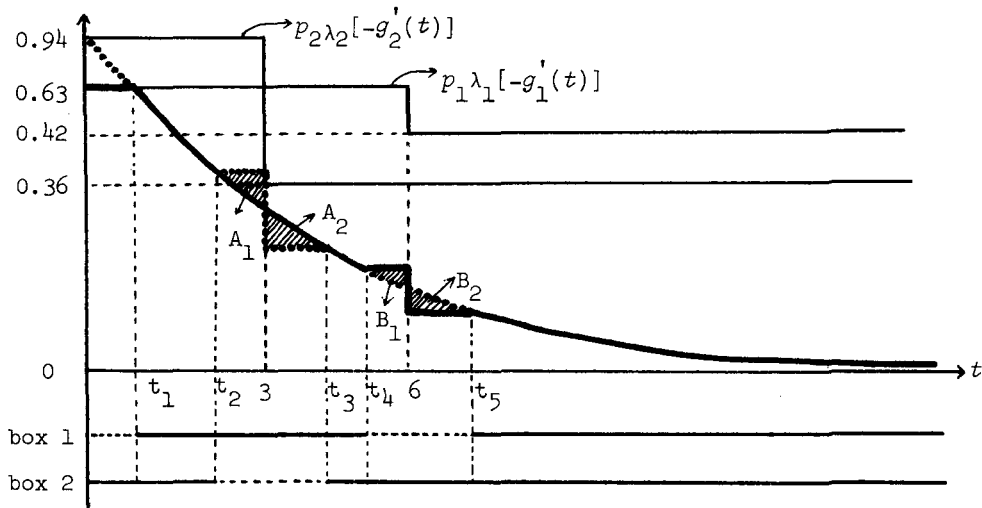


Fig. 1. — $h_1(t|\phi^*)$, $h_2(t|\phi^*)$

Finally we prove that the above policy ϕ^* is optimal. Let $\mu(t)$ be a function such that $-\mu'(t)$ is defined by

$$-\mu'(t) = \begin{cases} h_1(t|\phi^*) & \text{if } t \in (t_1, t_4), (t_5, \infty) \\ h_2(t|\phi^*) & \text{otherwise.} \end{cases}$$

Then the function $\mu(t)$ is expressed by the area of the region which is formed below the function $-\mu'(t)$. See Fig. 1. By the method of structuring of functions $h_i(t|\phi^*)$ ($i=1, 2$) and $\mu(t)$, it is evident that ϕ^* satisfies the sufficient condition of Theorem 1. Then ϕ^* is optimal.

Remark 5. Though $\mu(t) = \max_i \int_t^\infty h_i(s|\phi^*) ds$, it is not satisfied that $-\mu'(t) = \max_i h_i(t|\phi^*)$, for example, see the graph on the intervals $[t_2, 3]$, $[t_4, 6]$ in Fig. 1. Then Theorem 3 is not held in this case. See Remark 2.

5. Optimal Stopping Problem

In the model treated in the former section, we supposed that the search is continued until the object is detected because in the alpinist problem the family of the alpinist wants to find him though he is died. But in the usual problem if the search cost c is positive, it is reasonable to stop the search at a certain time. Then in this section we treat the stopping problem with permitting to stop the search at any time.

We modify the model in the former section about the following points:

1. The searcher is permitted to stop the search at any time.
2. The objective of the searcher is to maximize the expected return (reward minus cost) until the searcher detects the object or stops the search.

A policy in the modified model is expressed by (ϕ, τ) where ϕ is the search policy and τ is the stopping time. Let $V(\phi, \tau)$ be the expected return by using a policy (ϕ, τ) .

$$V(\phi, \tau) = \sum_{i=1}^n p_i \int_0^\tau g_i(t) P_i'(t|\phi) dt + \{1 - \sum_{i=1}^n p_i P_i(\tau|\phi)\} (-c\tau)$$

Integrating by parts, we obtain

$$(5.1) \quad V(\phi, \tau) = \sum_{i=1}^n p_i r_i [1 - F_i(\tau)] [1 - \exp\{-\lambda_i \phi_i(\tau)\}] - c\tau + \sum_{i=1}^n p_i \int_0^\tau [-g_i'(t)] [1 - \exp\{-\lambda_i \phi_i(t)\}] dt + \max_{\tau} \max_{\phi}$$

Following the outline of the proof of the former theorems, we obtain the

following theorems.

Theorem 4. For a fixed stopping time τ , necessary and sufficient conditions for a search policy ϕ^* to be optimal are given as follows: There is a nonnegative function $\mu(t)$ for any $t (\geq 0)$ such that if $\phi_i^*(t) \{ \begin{matrix} > \\ = \end{matrix} \} 0$,

$$(5.2) \quad p_i \lambda_i \int_t^\tau [-g_i'(s)] \exp\{-\lambda_i \phi_i^*(s)\} ds + p_i \lambda_i r_i [1 - F_i(\tau)] \exp\{-\lambda_i \phi_i^*(\tau)\} \{ \begin{matrix} = \\ \leq \end{matrix} \} \mu(t).$$

Proof: The proof is similar to that of Theorem 1. (q.e.d.)

We can prove that Theorem 2 and Lemma 1 hold without modifications. But Theorem 3 cannot hold since maximizing $h_i(t|\phi)$ does not guarantee the sufficient condition in Theorem 4.

Theorem 5. Suppose that $f_i'(t)$ ($i=1, \dots, n$) are differentiable at all $t \in [0, \infty)$. (i) For a fixed stopping time τ , a policy ϕ which searches at time t in all boxes maximizing $h_i(t|\phi)$ with rates given by (3.1) is optimal if $\phi_i'(\tau) > 0$ for $i=1, \dots, n$. (ii) For the optimal policy ϕ given by (i), the optimal stopping time τ^* is a root of the equation

$$(5.3) \quad \sum_{i=1}^n p_i \{ \lambda_i r_i [1 - F_i(\tau)] \phi_i'(\tau) - c \} \exp\{-\lambda_i \phi_i(\tau)\} = 0,$$

if $\phi_i'(\tau^*) > 0$ for $i=1, \dots, n$.

Proof: (i) Since $\phi_i'(\tau) > 0$ ($i=1, \dots, n$), we obtain by Theorem 4

$$p_i \lambda_i r_i [1 - F_i(\tau)] \exp\{-\lambda_i \phi_i(\tau)\} = \mu(\tau) \quad (i=1, \dots, n).$$

Then the relation (5.2) becomes

$$p_i \lambda_i \int_t^\tau [-g_i'(s)] \exp\{-\lambda_i \phi_i(s)\} ds \{ \begin{matrix} = \\ \leq \end{matrix} \} \mu(t) - \mu(\tau) \quad \text{if } \phi_i'(t) \{ \begin{matrix} > \\ = \end{matrix} \} 0$$

which has the same form as the relation (2.1), except that the integral region \int_t^τ is replaced by \int_t^τ . Therefore Theorem 3 can hold. Then ϕ is optimal. (ii) If we note that the optimal search policy ϕ given in (i) is independent of τ , $\frac{d}{d\tau} V(\phi, \tau) = 0$ leads us to the equation (5.3) (q.e.d.)

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