Journal of the Operations Research Society of Japan
Vol. 25, No. 2, June 1982

A SIMPLE CLOSED FORM OF THE LONG-RUN DISTRIBUTION OF STATIONARY INVENTORY POSITION PROCESSES IN ⟨R, r, T⟩ INVENTORY SYSTEMS

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(Received February 25, 1981; Final January 22, 1982)

Abstract A simple-and-general closed form of the long-run distribution of stationary inventory position processes in $<R, r, T>$ inventory systems is formulated in recurrence relations, the computational procedure of which is also applicable to determine the long-run distribution of non-stationary inventory position processes in those systems as long as their limits exist. Moreover, its computation on computer is economically plausible so that a significant contribution to the analytical study on such inventory problems is greatly anticipated.

1. Introduction

As a procurement policy for periodic-review inventory problems, the $<R,r,T>$ model is often implemented as in Hadley and Whitin [1], under which a sufficient quantity is ordered to bring the inventory position (IP) (which is defined to be the amount of on hand (OH) plus on order (OO) minus backorders (BO)) up to an inventory level R. A procurement in such inventory systems is made at each incremental review time $T \equiv T_{k+1} - T_k$ (T_k denoting the k^{th} review time kT) only if the inventory position $\{IP_{T_k}\}$ at a review time $T_k (k=0,1,2,\cdots)$ is less then or equal to the reorder point r, given $T_o \equiv 0$.

In view of the above procurement mechanism and under the assumptions of discrete and independent inter-review-period demands, it is realized that the process $\{IP_{T_k}\}_{k=1}^{\infty}$ just depends on the immediately preceding state of inventory level but not on any further inventory history, and so satisfies the so-called Markov chain property. Therefore, once the associated inter-review-period demand processes, denoted by $D_{(T_k, T_{k+1}]} (k=0,1,2,\cdots)$, are identified, it is possible to describe $\{IP_{T_k}\}_{k=1}^{\infty}$ accordingly.

Under the assumption of stationary Poisson demand processes in dealing with $<R,r,T>$ systems reviewed at the fixed incremental time point of T over infinite time horizon, an inventory cost expression was derived in Hadley and Whitin [1], where the derivation was initiated based on Markov chain theory but ended up with the limiting distribution of $\{IP_{T_k}\}_{k=1}^{\infty}$ obtained as a byproduct of the computation for the average annual inventory cost. Furthermore, the limiting distribution was formulated in rather inconvenient expression for any practical applications.

By the way, such $<R,r,T>$ systems lends itself to an explicit computation of the long-run distribution of $\{IP_{kT}\}_{k=1}^{\infty}$ from the associated stationary transition matrix, say $P_T = \{p_{T,ij}\}$ defined on the state space $S = \{r+1, r+2,\cdots,R\}$, where denoting IP^+_{KT} the inventory position immediately after the k^{th} review,

$$(1) \qquad p_{T,ij} = P_r\{IP^+_{(k+1)T} = r+j \mid IP^+_{kT} = r+i\}$$
$$= P_r\{IP^+_{2T} = r+j \mid IP^+_{T} = r+i\}, \text{ for } k=(0,1,2,\cdots)$$

and all $(i,j) \in S^* = \{1,2,\cdots,R-r\}$

Therefore, through this study, a simple closed form of the longrun distribution of stationary inventory position processes $\{IP^+_{kT}\}_{k=1}^{\infty}$ in $<R,r,T>$ inventory systems shall be specified in terms of recurrence relations, whereupon the formulation of any associated inventory cost will get more practical.

As to the application of Markov chain theory, the transition matrices to be created from the aforementioned stationary process $\{IP^+_{kT}\}_{k=1}^{\infty}$ shall first be investigated.

2. Transition Matrix

The procurement mechanism involved in $<R,r,T>$ systems will direct to giving rise to the operation of inventory position processes $\{IP^+_{kT}\}_{k=1}^{\infty}$ such that the positive stationary transition probabilities $\{p_{T,ij}\}$ exist only when at least one of the following combinations is satisfied; for all $(i,j) \in S^* = \{1,2,\cdots, R-r\}$ and $\ell=(0,1,2,\cdots)$,

$$(2) \qquad \begin{array}{ll} \text{(i)} & j \le i \text{ and } j \ne R-r, \text{ and } D_{(kT,(k+1)T]} = i-j, \\ \text{(ii)} & i \ne R-r \text{ and } j = R-r, \text{ and } D_{(kT,(k+1)T]} = \ell+i, \\ \text{(iii)} & i = j = R-r, \text{ and } D_{(kT,(k+1)T]} = \{0, \text{ or } \ell + (R-r)\}. \end{array}$$

Otherwise, the transition probabilities are zeros. Thereupon, such variations of demand materializations lead to the construction of the transition matrix $P_T = \{p_{T,ij}\}$ shown in Table 1, where θ_i representing $P_r\{D_{(kT,(k+1)T]}=i\}$ with the constant review-period $T = T_{k+1}-T_k$ for $k=(0,1,2,\cdots)$ and $i=(0,1,2,\cdots)$.

Since the matrix P_T is found primitive in Table 1, the next theorem is stated without proof (see Kemeny and Snell [3], and Isaacson and Madsen [2] for its proof), which insures that the unique limiting probability vector π for a finite primitive stationary transition matrix P is determined by $\pi P = \pi$.

Theorem 1: If a finite $N \chi N$ stationary transition matrix P is primitive, then the powers p^m for $m\geq 1$ approach a constant stochastic matrix G such that each row of G is the unique probability vector $\pi = \pi_1, \pi_2, \cdots, \pi_N$ satisfying $\pi P = \pi$ and hence $PG = GP = G$.

Even if Theorem 1 guarantees that the long-run distribution $\pi = \{\pi_{r+1}, \pi_{r+2}, \cdots, \pi_R\}$ for $P_T = \{p_{T,ij}\}$ can be determined by $\pi P_T=\pi$, it was concluded in Hadley and Whitin [1] that its explicit computation is not analytically obvious. However, its closed form of solution shall be derived in terms of recurrence relations in the next section.

3. Long-Run Distribution

From the matrix in Table 1, it will be easier for us to solve the systems of $(R-r)$ equations $\pi P_T^* = \pi$ and $\sum_{i=r+1}^{R} \pi_i = 1$ for π_i's $(i=r+1, r+2,\cdots,R)$, where P_T^* is the $(R-r) \chi (R-r-1)$ matrix reduced by removing the column "R". Therefore, with this reduced system of equations, the closed form of solution voctor π shall be determined. Following is the system of equations the computation will start with;

$$\sum_{i=r+1}^{R} \pi_i = 1 \quad , \quad \text{and from } P_T^*,$$

$$(3) \qquad \begin{aligned} &\theta_1\pi_R + \theta_o\pi_{R-1} = \pi_{R-1} \\ &\theta_2\pi_R + \theta_1\pi_{R-1} + \theta_o\pi_{R-2} = \pi_{R-2} \\ &\theta_3\pi_R + \theta_2\pi_{R-1} + \theta_1\pi_{R-2} + \theta_o\pi_{R-3} = \pi_{R-3} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$\theta_{R-2-r}\pi_R + \theta_{R-3-r}\pi_{R-1} + \cdots + \theta_1\pi_{r+3} + \theta_o\pi_{r+2} = \pi_{r+2}$$

$$\theta_{R-1-r}\pi_R + \theta_{R-2-r}\pi_{R-1} + \cdots + \theta_1\pi_{r+2} + \theta_o\pi_{r+1} = \pi_{r+1}$$

Table 1. Transition matrix of $[IP^+_{T_k}]$ for <R,r,T> model

$IP^+_{T_k}$ \ $IP^+_{T_{k+1}}$		$\{r+j\}$ $(j=1,2,\cdots,Q)$					
		R	$R-1$	$R-2$	....	$r+2$	$r+1$
$\{r+i\}$	R	$P\{D_{I_k}=0\}+\sum_{\ell=0}^{\infty}P\{D_{I_k}=\ell+(R-r)\}$ $=\theta_0+\sum_{\ell=0}^{\infty}\theta_{\ell+R-r}$	$P\{D_{I_k}=1\}$ $=\theta_1$	$P\{D_{I_k}=2\}$ $=\theta_2$	....	$P\{D_{I_k}=R-2-r\}$ $=\theta_{R-2-r}$	$P\{D_{I_k}=R-1-r\}$ $=\theta_{R-1-r}$
	$R-1$	$\sum_{\ell=0}^{\infty}P\{D_{I_k}=\ell+(R-1-r)\}$ $=\sum_{\ell=0}^{\infty}\theta_{\ell+(R-1-r)}$	$P\{D_{I_k}=0\}$ $=\theta_0$	$P\{D_{I_k}=1\}$ $=\theta_1$	....	$P\{D_{I_k}=R-3-r\}$ $=\theta_{R-3-r}$	$P\{D_{I_k}=R-2-r\}$ $=\theta_{R-2-r}$
	$R-2$	$\sum_{\ell=0}^{\infty}P\{D_{I_k}=\ell+(R-2-r)\}$ $=\sum_{\ell=0}^{\infty}\theta_{\ell+(R-2-r)}$	0	$P\{D_{I_k}=0\}$ $=\theta_0$	....	$P\{D_{I_k}=R-4-r\}$ $=\theta_{R-4-r}$	$\{D_{I_k}=R-3-r\}$ $=\theta_{R-3-r}$
	...	....	....	....	....	....	....
	$r+1$	$\sum_{\ell=0}^{\infty}P\{D_{I_k}=\ell+1\}$ $=\sum_{\ell=0}^{\infty}\theta_{\ell+1}$	0	0	....	0	$P\{D_{I_k}=0\}$ $=\theta_0$

Note: $I_k=(T_k,T_{k+1}]$

Solving Eq. (3) for π_{R-i} $(i=1,2,\cdots,R-1-r)$ in terms of π_R, we can get the following set of equations for which the coefficients have a recurrence relation;

$$(4)\qquad (1-\theta_o)\pi_{R-i} = \sum_{j=1}^{i} \theta_j\, \pi_{R-i+j}, \qquad \text{for } i=1,2,\cdots,R-r-1.$$

Let K_i denote the coefficient of π_R in equation π_{R-i} $(i=1,2,\cdots,R-1-r)$. Then, Eq. (4) can be simplified as follows;

$$(5)\qquad (1-\theta_o)\,\frac{\pi_{R-i}}{\pi_R} = \sum_{j=1}^{i} \theta_j \cdot \frac{\pi_{R-i+j}}{\pi_R}, \quad \text{for } i=1,2,\cdots, R-r-1.$$

Thus, letting $\frac{\pi_{R-i}}{\pi_R} \equiv K_i$ and $K_o \equiv 1$,

$$(6)\qquad (1-\theta_o)K_i = \sum_{j=1}^{i} \theta_j \cdot K_{i-j}, \text{ for } i=(1,2,\cdots, R-1-r),$$

and so

(7) $$K_i = \frac{1}{1-\theta_o} \sum_{j=1}^{i} \theta_j K_{i-j} \text{ for } i=(1,2,\cdots,R-1-r).$$

Thence,

$$1 = \sum_{i=r+1}^{R} \pi i$$

$$= \{ \sum_{i=1}^{R-1-r} K_i + 1\} \pi_R$$

Hence,

(8)
$$\pi_R = \frac{1}{1 + \sum_{i=1}^{R-1-r} K_i} \text{ , and}$$

$$\pi_{R-i} = K_i \cdot \pi_R \text{ for } i=(1,2,\cdots,R-1-r).$$

These $\pi_{R-i}(i=0,1,2,\cdots,R-1-r)$ in Eq. (8) can be easily computed on a digital computer once the probabilities θ_i's $(i=0,1,2,\cdots,R-1-r)$ are determined. In other words, to compute $\pi_{r+j} = \lim_{k\to\infty} \mathrm{P_r}\{IP^+_{kT} = r+j\}$ we need only the finite number of probabilities $\theta_i = \mathrm{P_r}\{D_{(kT,(k+1)T]}=i\}$ for $i=0,1,2,\cdots,R-1-r$. Therefore, their computational time can be greatly reduced and also the more accurate values be determined in comparison of the result in Hadley and Whitin [1] where the infinite series of high order probability convolutions are involved.

4. Long-Run Expected Average Inventory Cost

A long-run expected average inventory cost in $\langle R,r,T\rangle$ systems with a constant procurement lead time τ shall be discussed for various demand processes under a Poisson type of demand occurrences.

By the definition of $\{IP_{kT};T > 0\}_{k=1}^{\infty}$ stated for this study, the so-called net inventory position process $\{NIS_{kT+u}\}_{k=1}^{\infty}$ for $\tau \le u < T+\tau$ is expressed in the following arithmetical relations of stochastic variates;

(9)
$$NIS_{kT+u} = IP^+_{kT} - D_{(kT,\ kT+u]}$$
$$= OH_{kT+u} - BO_{kT+u} \text{ , and hence}$$
$$NIS_{kT+u} = OH_{kT+u} \text{ , if } NIS_{kT+u} \ge 0$$
$$= BO_{kT+u} \text{ , otherwise,}$$

where the range of u between τ and $T+\tau$ is considered, since it is assumed that everything on order immediately after the review at time kT will arrive in the system by $kT + \tau$, but nothing not on order cay arrive before time $(k+1)T + \tau$.

Then, under the Poisson demand process assumption it follows that

$$P_r\{NIS_{kT+u} = r+s\} = \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j,\ D_{(kT,kT+u]} = j-s\}^+$$
$$\text{for } s=(R-r,R-r-1,\cdots,0,-1,-2,\cdots),$$

$$(10)\qquad = \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\}\ P_r\{D_{(kT,kT+u]} = j-s\}^+ \qquad \text{for } 0 \le u \le T\ ,$$

where $P_r\{D_{(kT,kT+u]} = j-s\}^+ = P_r\{D_{(kT,kT+u]} = j-s\}$, if $j \ge s$,
$= 0$, otherwise.

Thus, $P_r\{OH_{kT+u} = x\} = P_r\{NIS_{kT+u} = x\}$, for $x=(0,1,2,\cdots)$,

$$(11)\qquad = \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\}\ P_r\{D_{(kT,kT+u]} = r+j-x\}^+\ .$$

Therefore, from Eq. (11),

$$E\{OH_{kT+u}\} = \sum_{x=0}^{\infty} x \cdot P_r\{OH_{kT+u} = x\}$$

$$(12)\qquad = \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\} \sum_{x=0}^{r+j} x \cdot P_r\{D_{(kT,kT+u]} = r+j-x\}$$

$$= \sum_{j=1}^{R-r} Pr\{IP^+_{kT} = r+j\} \sum_{n=0}^{r+j} (r+j-n)\ P_r\{D_{(kT,kT+u]} = n\}$$

Similarly, since

$$P_r\{BO_{kT+u} = x\} = P_r\{NIS_{kT+u} = -x\}, \quad \text{for } x=(1,2,\cdots),$$

$$= \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\}\ P_r\{D_{(kT,kT+u]} = r+j+x\},$$

$$(13)\qquad E\{BO_{kT+u}\} = \sum_{x=1}^{\infty} x \cdot P_r\{BO_{kT+u} = x\}$$

$$= \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\} \sum_{x=1}^{\infty} x \cdot P_r\{D_{(kT,kT+u]} = r+j+x\}.$$

$$= \sum_{j=1}^{R-r} P_r\{IP^+_{kT} = r+j\} \Big[E\{D_{(kT,kT+u]}\} - (r+j) +$$

$$\sum_{n=0}^{r+j} (r+j-n)\ P_r\{D_{(kT,kT+u]} = n\} \Bigg]$$

Now, denote by ΔBO_{kT} the number of backorders incurred between $kT+\tau$ and $(k+1)T+\tau$, given the inventory position of $r+j$ immediately after the k^{th} review. Then, it results in the next expression;

$$\Delta BO_{kT} = BO_{(kT,(k+1)T+\tau]} - BO_{(kT,kT+\tau]},$$

where $BO_{(kT,(k+1)T+\tau]}$ denotes the number of backorders at time $(k+1)T+\tau$, before items ordered at $(k+1)T$ (if order is made) are delivered, while $BO_{(kT,kT+\tau]}$ denotes the number of backorders at time $kT+\tau$, after items ordered at time kT (if order is placed).

Therefore, from the result of Eq. (13),

$$\text{(14)}\quad \begin{aligned} E\{\Delta BO_{kT}\} = \sum_{j=1}^{R-r} P_r\{IP_{kT}^{+} = r+j\} \Bigg[& E\{D_{(kT,(k+1)T+\tau]} - D_{(kT,kT+\tau)}\} \\ & + \sum_{n=0}^{r+j} (r+j-n)(P_r\{D_{(kT,(k+1)T+\tau]}=n\} - P_r\{D_{(kT,kT+\tau]}=n\}) \Bigg]. \end{aligned}$$

For ordering cost consideration, let $P_{od}(k)$ be the probability that an order will be placed at a given review time $(k+1)T$ $(k=0,1,2,\cdots)$. It follows that given $IP_{kT}^{+} = r+j$ $(j=1,2,\cdots,R-r)$,

$$P_r\{IP_{(k+1)T} \le r\} = \sum_{j=1}^{R-r} P_r\{IP_{kT}^{+} = r+j\}\ P_r\{IP_{(k+1)T} \le r \mid IP_{kT}^{+}=r+j\}.$$

Under the assumption that inter-review-period demands are independent,

$$\begin{aligned} P_{od}(k) &= P_r\{IP_{(k+1)T} \le r\} \\ &= \sum_{j=1}^{R-r} P_r\{IP_{kT}^{+} = r+j\} \Big[1-P_r\{D_{(kT,(k+1)T]} \le j-1\} \Big]. \end{aligned}$$

In addition, we need define some cost parameters for the formulation of our long-run expected average annual inventory cost function.

As in the literature, let \$$A$ be a fixed ordering cost, \$$C$ the unit cost of each item independent of the quantity ordered, \$$W$ the cost of review, and I the inventory carrying charge constant to give rise to the instantaneous charge rate $IC \cdot x$ from carrying the on hand inventory level of x. Furthermore, denote by B and $\hat{B}$ the fixed cost per unit backordered and the cost per unit year of the shortage, respectively.

Thus, for the $\langle R,r,T\rangle$ type of inventory systems in which demands occurring when the system is out of stock are backordered, the general long-run expected average annual inventory cost function $H(R,r,T)$ can be derived from Eqs. (12), (13), (14) and (15) as follows;

$$(16)\quad H(R,r,T) = \frac{W}{T} + \frac{1}{T}\cdot A\,[\lim_{k\to\infty} P_{od}(k)] + IC\cdot\frac{1}{T}\int_{\tau}^{T+\tau}[\lim_{k\to\infty} \mathrm{E}\{OH_{kT+u}\}]du + B\cdot\frac{1}{T}[\lim_{k\to\infty}\mathrm{E}\{\Delta BO_{kT}\}] + \hat{B}\cdot\frac{1}{T}\int_{\tau}^{T+\tau}[\lim_{k\to\infty}\mathrm{E}\{BO_{kT+u}\}]du.$$

Note: It is known that under the above stated cost systems, the steady state $\langle R,r,T\rangle$ policy is optimal when the backorders are allowed.

For a specific example of standard Poisson demand process with λ_t, the limit terms in Eq. (16) can be determined as follows;

$$(17)\quad \theta_i = \mathrm{P_r}\{D_{(kT,(k+1)T]} = i\} = \frac{e^{-\lambda T}(\lambda T)^i}{i!} \quad (i=0,1,2,\cdots;\ k=0,1,2,\cdots).$$

$$(18)\quad \pi_{r+j} = \lim_{k\to\infty}\mathrm{P_r}\{IP^{+}_{kT} = r+j\} \qquad (j=1,2,\cdots,R-1-r)$$

$$= K_{R-r-j}\cdot\pi_R, \text{ from Eq. (8)}$$

$$= \frac{e^{-\lambda T}}{1-e^{-\lambda T}}\sum_{n=1}^{R-r-j}\{\frac{(\lambda T)^n}{n!}\}K_{R-r-j-n}\cdot\pi_R,$$

where π_R and $K_{R-r-j-n}$ can be finitely determined from both the recurrence relations of Eqs. (7) and (8).

$$(19)\quad \lim_{k\to\infty}\mathrm{E}\{OH_{kT+u}\} = \sum_{j=1}^{R-r}\pi_{r+j}\sum_{n=0}^{r+j}(r+j-n)\{\frac{e^{-\lambda u}(\lambda u)^n}{n!}\}, \text{ from Eq. (12).}$$

$$(20)\quad \lim_{k\to\infty}\mathrm{E}\{BO_{kT+u}\} = \sum_{j=1}^{R-r}\pi_{r+j}\left[\lambda u-(r+j) + \sum_{n=0}^{r+j}(r+j-n)\{\frac{e^{-\lambda u}(\lambda u)^n}{n!}\}\right], \text{ from Eq. (13).}$$

$$(21)\quad \lim_{k\to\infty}\mathrm{E}\{\Delta BO_{kT}\} = \sum_{j=1}^{R-r}\pi_{r+j}\left[\lambda T + \sum_{n=0}^{r+j}(r+j-n)\{\frac{e^{-\lambda(T+\tau)}(\lambda(T+\tau))^n}{n!} - \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}\}\right], \text{ from Eq. (14).}$$

$$(22)\quad \lim_{k\to\infty}P_{od}(k) = \sum_{j=1}^{R-r}\pi_{r+j}\left[1 - \sum_{n=0}^{j-1}\frac{e^{-\lambda T}(\lambda T)^n}{n!}\right], \text{ from Eq. (15).}$$

Similar derivations can also be easily made for compound Poisson demand processes $\{D_t\}$ associated with Poisson demand occurrences $\{N_t\}$ and mutually independent geometrically (or uniformly) distributed demand sizes $\{X_i;\ i=1,2,\cdots,N_t\}$.

It is generally suggested to use a digital computer along with an appropriate search routine to find the optimal values of R^*, r^*, and T^* which minimize such long-run expected average annual cost functions $H(R,r,T)$. As another computational procedure of solving such $<R,r,T>$ systems for R^*, r^* and T^*, Dynamic Programming approach has been dealt with in Hadley and Whitin [1] under the assumptions of the constant unit cost $\$C$ and of the convex expected cost of carrying inventory and backorders in the period from τ to $\tau+T$.

5. Conclusion

The computation for the long-run distribution π of a stationary inventory position process $\{IP^+_{kT};\ T\geq 0\}^\infty_{k=1}$ was completed recurrently. In view of the transition matrix P_T constructed in Table 1, the recurrent computation approach is realized as a general approach successfully applicable to determine the long-run distribution of inventory positions associated with any kind of demand processes in $<R,r,T>$ inventory systems.

The closed form of the long-run distribution π determined in recurrent format appears much simpler and more suitable to make practical applications than the complicated solution given in Hadley and Whitin [1], and further seems to save much time in their computation on computer, so that its significant contribution to the analytical study on such inventory problems is greatly expected. Moreover, this approach can be directly applied to the analysis of nonstationary inventory position processes in those inventory systems as long as their limit distribution exist.

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