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STABILITY OF THE CRITICAL QUEUEING SYSTEMS

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Abstract This paper considers the stability of the waiting time in the queueing system with the critical condition: $ES_i = sET_i$. It is shown that the waiting time is stable in the model with special moving average input and the unstability condition is obtained in the strictly stationary input with the finite variance.

1. Introduction

This paper deals with the s-server queuing model with FIFO discipline. We assume that the first customer arrives at time zero. Let T_n be the time interval between the arrival of the nth customer and that of the $(n + 1)$th customer and S_n be the service time of the nth customer. We assume that $(T_n, S_n)(n \geqq 1)$ is the strictly stationary process and extend it to form the strictly stationary process $(T_n, S_n)(-\infty < n < \infty)$. Let $w_n = (w_{n,1}, \cdots, w_{n,s})$ be the waiting time vector at the arrival point of the nth customer. We regard w_1 to be fixed. That is, each component of the vector w_1 is the corresponding server's work given at the starting point. We use the terminologies defined by Loynes [6], i.e. stable, substable, unstable, honest, dishonest, $U_n = S_n - sT_n$, etc.

The asymptotic behavior of the waiting time in the queueing model has been discussed by Lindley [5], Loynes [6], Miyazawa [7] and others. They proved that if $S_i - sT_i$ is the strictly stationary process such as $ES_i > sET_i$, then w_n is unstable and that if $ES_i < sET_i$, the w_n is substable. In the case of $ES_i = sET_i$, Lindley proved that the w_n is unstable in the single server queue satisfying the conditions: (*i*) the two sequences T_n and S_n are independent, (*ii*) the sequence T_n is identically and independently distributed, (*iii*) the sequence S_n is identically and independently distributed. Kiefer and Wolfowitz [3] obtained the same result in the many-server queue with (*i*), (*ii*) and (*iii*). Loynes considered the queue without these assumptions and showed that if $S_i - sT_i$ is the strictly stationary process and if the servers are initially unoccupied, the distribution of $w_n = (w_{n,1}, \cdots, w_{n,s})$ tends monotonically to some distribution, i.e., w_n is stable or unstable. He also obtained the unstability of the critical queue with both assumptions (*i*) and (*ii*) or both (*i*) and (*iii*).

Though Loynes suggested the existence of stable model with this critical condition, he gave few examples. The purpose of this paper is to give such examples and to discuss some unstable cases. So, throughout the paper, it is assumed that $EU_n = 0$. The unstability of w_n is obtained under the more relaxed condition than Loynes'.

2. Examples of the Stable Single Server Queue

In the single server queueing model with FIFO discipline and $w_1 = 0$, the waiting time of the nth customer is represented as

$$w_n = [\sup_{1 \leqq r \leqq n-1} \sum_{r}^{n-1} U_k]^{\dagger},$$

where $[x]^{\dagger} = \max(x, 0)$, so that the asymptotic behavior of w_n depends only on the process U_n. It will be useful to note that for any given strictly stationary process U_n there is a two-variate nonnegative valued stationary process (T_n, S_n) such as $U_n = S_n - T_n$, for example, $S_n = [U_n]^{\dagger}$ and $T_n = [U_n]^{\dagger} - U_n$.

Now, the w_n has the same distribution as $[\sup_{1 \leqq r \leqq n-1} \sum_{1}^{r} U_{-K}]^{\dagger}$. Hence the limiting distribution of w_n is honest if and only if the distribution of the random variable

$$M = [\sup_{r \geqq 1} \sum_{1}^{r} U_{-k}]^{\dagger}$$

is honest, and M is honest if and only if $\Pr\{\sup_{r \geqq 1} \sum_{1}^{r} U_{-k} < \infty\} = 1$.

In this section we give the following two stable examples.

Example 1. The circular stationary process such as

$$U_{mp+i} = U_i, \qquad i=1,\cdots,p, \ m=0,\pm 1,\pm 2,\cdots.$$

In this case, if $r = mp + i$,

$$\sum_{1}^{r} U_{-k} = m\sum_{1}^{p} U_{-k} + \sum_{1}^{i} U_{-k}.$$

Hence, M is honest if and only if $\Pr\{\sum_{1}^{p} U_{-k} \leq 0\} = 1$.

Example 2. The moving average input

$$U_n = \varepsilon_n - \theta_1\varepsilon_{n-1} - \cdots - \theta_p\varepsilon_{n-p}$$

with the condition $1 - \sum_{1}^{p}\theta_i = 0$, where ε_n is the strictly stationary process with mean zero. In this model the equation

$$\text{(1)} \qquad \sup_{r\geq 1} \sum_{1}^{r} U_{-k} = \varepsilon_{-1} + \sum_{i=1}^{p-1} (1 - \theta_1 - \cdots - \theta_i)\varepsilon_{-1-i} - \inf_{r} \sum_{i=1}^{p} (\theta_i + \cdots + \theta_p)\varepsilon_{-r-i}$$

holds, so that M is honest if the following (a) or (b) holds.

(a) There is a constant K such as $\Pr\{|\varepsilon_n| < K\} = 1$.

(b) There is a constant K such as $\Pr\{\varepsilon_n > -K\} = 1$, and $\theta_\eta + \cdots + \theta_p \geq 0$ for all η.

Now we assume that ε_n is identically independently distributed. If we let τ_1 and τ_2 be the numbers such as

$$\tau_1 = \inf\{\tau: \Pr\{\varepsilon_n < \tau\} > 0\}$$

and

$$\tau_2 = \sup\{\tau: \Pr\{\varepsilon_n > \tau\} > 0\},$$

then the third term of the right-hand side of (1) is

$$\sum_{i=1}^{p} [\theta_i + \cdots + \theta_p]^{+}\tau_1 - \sum_{i=1}^{p} [-(\theta_i + \cdots + \theta_p)]^{+}\tau_2.$$

Then, if M is honest, (a) or (b) holds, that is, the above condition is the necessary and sufficient condition, and moreover we can find the distribution of M from that of the first and the second terms of (1).

The condition of $1 - \sum_{i=1}^{p} \theta_i = 0$ of Example 2 is necessary, as is shown in the following theorem.

Theorem 1. In the moving average input

$$U_n = \varepsilon_n - \theta_1\varepsilon_{n-1} - \cdots - \theta_p\varepsilon_{n-p}, \quad \varepsilon_n \sim i.i.d. \quad E\varepsilon_n = 0,$$

if $1 - \sum_{i=1}^{p} \theta_i \neq 0$, if $\Pr\{\varepsilon_n = 0\} < 1$ and if $E|\varepsilon_n| < \infty$, then M is dishonest.

Proof:

$$\sum_{1}^{r} U_{-k} = x_r + y_r$$

where

$$x_r = (1 - \theta_1 - \cdots - \theta_p) \sum_{i=-r}^{-1-p} \varepsilon_i \qquad (r \geqq p+1),$$

and

$$y_r = \varepsilon_{-1} + \sum_{i=1}^{p-1} (1 - \theta_1 - \cdots - \theta_i)\varepsilon_{-1-i} - \sum_{i=1}^{p} (\theta_i + \cdots + \theta_p)\varepsilon_{-r-i}.$$

By Chung Fuchs' theorem ([3] or [9] p.196), $\sup_n x_n = \infty$ with probability one. Hence for any positive numbers m and K

$$\Pr\{ \sup_{r \geqq p+1} \sum_1^r U_{-k} \leqq K\}$$

$$\leqq \sum_{n=p+1}^{\infty} \Pr\{x_i < mK,\ (p+1 \leqq i \leqq n-1),\ x_n \geqq mK,\ y_n \leqq (1 - m)K\}$$

The random variable y_n is independent of the random vector $(x_{p+1},\cdots,x_n)$ and its distribution does not depend on n. Hence we continue with

$$= \Pr\{y_{p+1} \leqq (1 - m)K\} \xrightarrow{m \to \infty} 0.$$

Q.E.D.

3. Examples of the Stable Many-Server Queue

First we will prove the following theorem. Kiefer and Wolfowitz or Loynes proved that $w_{n,1}$ is stable if and only if w_n is stable, so that the following theorem also holds with respect to $w_{n,1}$ instead of w_n.

Theorem 2. Suppose that $s \geqq 2$ and that there are honest random variables V_1 and V_2 such that $V_1 < \sum_{n=i}^{j} U_n < V_2$ for all i and j, and suppose that $w_1 = 0$. Then w_n is stable if and only if $\sup_n S_n < \infty$ with probability one.

Proof: Since $w_1 = 0$, the distribution of w_n tends monotonically to some distribution which is honest or dishonest. First we suppose that $\sup S_n < \infty$ with probability one. For an arbitrary positive integer k there is an integer j $(\leqq k)$ such as

$$w_{i,1} \begin{cases} = 0 & : i = j \\ > 0 & : j + 1 \leqq i \leqq k. \end{cases}$$

Then

$$w_{j,r} \begin{cases} = 0 & r = 1 \\ \leqq \sup_n S_n : & r \geqq 2 \end{cases}$$

and

$$Sw_k = Sw_j + \sum_{i=j}^{k-1} U_i < (s - 1)\sup S_n + V_2$$

where $Sw_n = \sum_{i=1}^{s} w_{n,i}$. Therefore the limiting distribution of Sw_n is honest and w_n is stable.

Next suppose that $\sup_n S_n = \infty$. Then $\sup_n Sw_n = \infty$ from the inequality

$$Sw_{n+1} \geqq w_{n+1,s} \geqq S_n - T_n = \frac{s-1}{s} S_n + \frac{1}{s} U_n \geqq \frac{s-1}{s} S_n + \frac{1}{s} V_1.$$

On the other hand

$$Sw_n = \sum_{i=1}^{n-1} S_i - s \sum_{i=1}^{n-1} T_i + O_n = \sum_{i=1}^{n-1} U_i + O_n$$

where O_n is the sum of the total idle times of servers from the original starting point to the arrival point of the nth customer. Hence,

$$(2) \qquad V_1 + O_n \leqq Sw_n \leqq V_2 + O_n.$$

Since O_n is non-decreasing with respect to n and $\sup Sw_n = \infty$, $\lim_{n\to\infty} O_n = \infty$ from the right inequality of (2), so that $\lim_{n\to\infty} Sw_n = \infty$ from the left inequality of (2). The unstability of w_n follows from this.

Q.E.D.

From this theorem, the w_n in the following three cases is stable if and only if $\sup S_n < \infty$ with probability one.

Example 3. $S_n = sT_n = x_n$ where x_n is the strictly stationary process. In this example $U_n = 0$ with probability one.

Example 4. $U_n = S_n - sT_n$ is the circular stationary process such that $U_{mp+i} = U_i$ $(i=1,\cdots,p,\ m=0,\pm1,\pm2,\cdots)$ and that $\Pr\{\sum_1^p U_i = 0\} = 1$. In this example,

$$\inf_{1\leqq k\leqq p-1} \sum_1^k U_n \leqq \sum_{n=i}^{j} U_n \leqq \sup_{1\leqq k\leqq p-1} \sum_1^k U_n.$$

Example 5. $U_n = S_n - sT_n = \varepsilon_n - \theta_1\varepsilon_{n-1} - \cdots - \theta_p\varepsilon_{n-p}$, where the parameters satisfy the equation; $1 - \theta_1 - \cdots - \theta_p = 0$ and ε_n is the strictly stationary process such that $\Pr\{|\varepsilon_n| \leqq K\} = 1$ for some number K. In this example

$$\left|\sum_{n=1}^{j} U_n\right| \leqq K\{1 + \sum_{i=1}^{p-1} |1 - \theta_1 - \cdots - \theta_i| + \sum_{i=1}^{p} |\theta_i + \cdots + \theta_p|\}.$$

4. Strictly Stationary Input with the Finite Variance

In the examples of the previous sections the finiteness of Var U_n is not supposed. If Var $U_n < \infty$, i.e., if U_n is the strictly and weakly stationary process, the autocovariance of U_n is characterized by the spectral distribution function F, that is,

$$EU_nU_{n-k} = \int_{-\pi}^{\pi} e^{ik\lambda}dF(\lambda).$$

Although the stability of w_n, as is seen in Example 2, is not determined only by the spectral distribution function, the stable processes given by Example 1 ~ 5 have the special spectral distribution functions. For example, the spectral distribution of the circular stationary process has masses only on the points $\lambda_j = \frac{2j\pi}{p}$, $(j = [\frac{2-p}{2}], \cdots, -1, 0, 1, \cdots, [\frac{p}{2}])$ and the spectral density of the moving average process of Example 2, if ε_n is identically independently distributed, is

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left|1 - \sum_{j=1}^{p} \theta_j e^{-ij\lambda}\right|^2, \quad \sigma^2 = E\varepsilon_n^2.$$

Moreover, the condition $\Pr\{\sum_{1}^{p} U_i = 0\} = 1$ of Example 1 or 4 is corresponding to $F(0) - F(0-) = 0$ and the condition $1 - \sum_{i=1}^{p} \theta_i = 0$ in Example 2 or 5 means that

$$f(\lambda)\left|1 - e^{-i\lambda}\right|^{-2} = f(\lambda)(2\sin\frac{\lambda}{2})^{-2} \xrightarrow{\lambda \to 0} (\text{constant}).$$

These facts make us infer that the stable process has some special spectral distribution and that its spectral distribution has small masses in the neighborhood of $\lambda = 0$. In this section we will consider these points.

First, the following theorem shows the various possibility of existence of the stable model in the strictly stationary process with the finite variance.

Theorem 3. Let $f(\lambda)$ be a nonnegative function on $[-\pi, \pi]$ such that $f(\lambda)$ is symmetric with respect to $\lambda = 0$ and that $f(\lambda)(\sin\frac{\lambda}{2})^{-2}$ is integrable. Let the Fourier series of $\sqrt{f(\lambda)(2\sin\frac{\lambda}{2})^{-2}}$ be $\sum_{j=-\infty}^{\infty} \theta_j e^{ij\lambda}$. If $\sum_{j=-\infty}^{\infty} |\theta_j| < \infty$, then we can construct the stable model with s servers in which $f(\lambda)$ is the spectral density function of a strictly stationary process U_n.

Proof. Let ε_n be the sequence of identically independently distributed random variables with mean zero and variance one. Moreover we suppose that $\Pr\{|\varepsilon_n| < K\} = 1$ for some number K. Then the spectral density function of

$x_n = \sum_{j=-\infty}^{\infty} \theta_j \varepsilon_{n-j}$ is $f(\lambda)(2\sin\frac{\lambda}{2})^{-2}$. Let $U_n = x_n - x_{n-1}$ and let $S_n = [U_n]^{\dagger}$ and $sT_n = [U_n]^{\dagger} - U_n$. Then the spectral density function of U_n is $|1-e^{-i\lambda}|^2 f(\lambda) \times (2\sin\frac{\lambda}{2})^{-2} = f(\lambda)$ (see Nakatsuka [8], Theorem 1, 11). Since $|x_n| < K \sum_{j=-\infty}^{\infty} |\theta_j| < \infty$, this is the case of Example 2 or 5 with $p = 1$.

Q.E.D.

Next, we consider about unstability. If the single server queue with U_n is unstable, the many-server queue with the same U_n is unstable. This is easily seen by the relation $Sw_{n+1} \geqq [Sw_n + U_n]^{\dagger}$.

Theorem 4. Let $\sigma_n^2 = \mathrm{Var}(\sum_{k=1}^{n} U_{-k})$. If there is a sequence n_i such that $\lim_{i\to\infty} \sigma_{n_i}^2 = \infty$ and $(\frac{1}{\sigma_{n_i}} \sum_{k=1}^{n_i} U_{-k})^2$ is uniformly integrable, then the s-server queue with this U_n is unstable.

Proof: Without loss of generality we assume that $\lim_{n\to\infty} \sigma_n^2 = \infty$ and $(\frac{1}{\sigma_n} \sum_{k=1}^{n} U_{-k})^2$ is uniformly integrable. If we put $y_n = \sigma_n^{-1} \sum_1^n U_{-k}$, y_n distributes with mean 0 and variance 1. By the Helly's sellection lemma there is a subsequence n_j such that the distribution function of y_{n_j} converges weakly to some right continuous nondecreasing function G. G is a probability distribution function because of the Tchebychev's inequality

$$\Pr\{y_n > k\} < \frac{1}{k^2} \text{ for any } k.$$

From the uniform integrability of $y_{n_j}^2$ the distribution G has mean zero and variance 1 (see [2] p.32), which means that G has the positive probability on the open half line $(0, \infty)$. Hence there are positive numbers α, ε and $m(\alpha, \varepsilon)$ such that for any j larger than $m(\alpha, \varepsilon)$

$$\Pr\{\sum_1^{n_j} U_{-k} > \sigma_{n_j}\alpha\} = \Pr\{y_{n_j} > \alpha\} > \varepsilon > 0.$$

Therefore $\Pr\{\sup \sum_{k=1}^{n} U_{-k} = \infty\} > 0$.

Q.E.D.

Loynes proves that the G/G/1 queue with $EU_n = 0$ is unstable if one of S_n and T_n is a mutually independent sequence. If the weakly stationary process is mutually independent, its spectral density is a uniform function of $[-\pi, \pi]$. Therefore, in the strictly ϕ-mixing stationary input with the finite variance, the Loynes' result is included in the following corollary.

Corollary 5. Suppose that U_{-k} is ϕ-mixing with $\sum_n \phi_n^{\frac{1}{2}} < \infty$. Let $f(\lambda)$ be the spectral density function of the absolutely continuous part of the spectral distribution function of U_{-k}. If $f(\lambda)$ is continuous at $\lambda = 0$ and if $f(0) > 0$, then w_n is unstable.

Proof: $\liminf_n \frac{1}{n} \sigma_n^2 \geq 2\pi f(0)$ (see [8], p.274 or [1], p.459). By [2], $\frac{1}{n}\left(\sum_{k=1}^{n} U_{-k}\right)^2$ is uniformly integrable. Hence, $\frac{1}{\sigma_n^2}\left(\sum_{k=1}^{n} U_{-k}\right)^2$ is uniformly integrable. The corollary follows from Theorem 4.

Q.E.D.

If U_n is Gaussian, $\frac{1}{\sigma_n^2}\left(\sum_1^n U_{-k}\right)^2$ is uniformly integrable. In such cases the queue is unstable if $\limsup_{n\to\infty} \sigma_n^2 = \infty$. The following theorem means that $\limsup \sigma_n^2 = \infty$ if and only if $\int_{-\pi}^{\pi} (\sin \frac{\lambda}{2})^{-2} dF = \infty$.

Theorem 6. Let F be the spectral distribution function of U_{-k}. Then,

$$(3) \qquad \frac{1}{2} \int_{-\pi}^{\pi} (\sin \frac{\lambda}{2})^{-2} dF \leq \limsup_{n\to\infty} \operatorname{Var}\left(\sum_1^n U_{-k}\right) \leq \int_{-\pi}^{\pi} (\sin \frac{\lambda}{2})^{-2} dF$$

where $\int_{-\pi}^{\pi} (\sin \frac{\lambda}{2})^{-2} dF = \infty$ if $F(0) - F(0-) > 0$.

Particularly if F is the absolutely continuous function,

$$(4) \qquad \lim_{n\to\infty} \operatorname{Var}\left(\sum_1^n U_{-k}\right) = \frac{1}{2} \int_{-\pi}^{\pi} (\sin \frac{\lambda}{2})^{-2} dF.$$

Proof:

$$(5) \qquad \operatorname{Var}\left(\sum_1^n U_{-k}\right) = \int_{-\pi}^{\pi} \left| \sum_{t=1}^{n} e^{it\lambda} \right|^2 dF = \int_{-\pi}^{\pi} \left\{ \frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right\}^2 dF$$

$$= -\frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos n\lambda}{(\sin \frac{\lambda}{2})^2} dF + \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{(\sin \frac{\lambda}{2})^2} dF.$$

The right inequality of (3) is easily derived from this. Next we prove the left inequality. First suppose that $\int (\sin \frac{\lambda}{2})^{-2} dF < \infty$. Unless left inequality of (3) holds, there are an integer m and a positive number ε such that for any $n \geqq m$

(6) $$\int_{-\pi}^{\pi} \frac{\cos n\lambda}{(\sin \frac{\lambda}{2})^2} dF > \varepsilon.$$

Therefore

$$\liminf_{n\to\infty} \frac{1}{n-m-1} \sum_{j=m}^{n} \int_{-\pi}^{\pi} \frac{\cos j\lambda}{(\sin \frac{\lambda}{2})^2} dF \geqq \varepsilon.$$

This is impossible by the Lebesgue's dominated convergence theorem and by the equation

$$\lim_{n\to\infty} \frac{1}{n-m-1} \sum_{j=m}^{n} (\cos j\lambda + i\sin j\lambda) = \lim_{n\to\infty} \frac{1}{n-m-1} \sum_{j=m}^{n} e^{ij\lambda} = \begin{cases} 1 : \lambda = 0 \\ 0 : \lambda \neq 0. \end{cases}$$

Next suppose that $\int(\sin \frac{\lambda}{2})^{-2}dF = \infty$. Then $F(0) - F(0-) > 0$ or $\int_{\{\pi\geqq|\lambda|>0\}} (\sin \frac{\lambda}{2})^{-2}dF = \infty$. In the former case

(7) $$\int_{-\pi}^{\pi} \left|\sum_{t=1}^{n} e^{it\lambda}\right|^2 dF \geqq n\{F(0) - F(0-)\} \xrightarrow{n\to\infty} \infty.$$

In the latter case if $\delta > 0$,

(8) $$\limsup_{n\to\infty} \int_{-\pi}^{\pi} \left|\sum_{t=1}^{n} e^{it\lambda}\right|^2 dF \geqq \limsup_{n\to\infty} \int_{\{|\lambda|>\delta\}} \left|\sum_{t=1}^{n} e^{it\lambda}\right|^2 dF$$

By the fact proved first,

$$\geqq \frac{1}{2} \int_{\{|\lambda|>\delta\}} (\sin \frac{\lambda}{2})^{-2}dF \xrightarrow{\delta \downarrow 0} \infty.$$

Next let F be the absolutely continuous function such as $\int_{-\pi}^{\pi}(\sin \frac{\lambda}{2})^{-2}dF < \infty$. From (5) it is sufficient to prove that the equation

(9) $$\lim_{n\to\infty} \int_{-\pi}^{\pi} g(\lambda)\cos n\lambda d\lambda = 0$$

holds for any integrable function $g(\lambda)$ on $[-\pi, \pi]$.

Let B be the set of all subsets E on $[-\pi, \pi]$ such as $\lim_{n\to\infty} \int_E \cos n\lambda d\lambda = 0$. B contains the any interval and is a σ additive field, so that B contains the any Borel set on $[-\pi, \pi]$. Hence, (9) holds for any simple function and therefore for any integrable function.

When $\int_{-\pi}^{\pi}(\sin\frac{\lambda}{2})^{-2}dF = \infty$, the equality; $\lim_{n\to\infty}\int_{-\pi}^{\pi}|\sum_{t=1}^{n} e^{it\lambda}|^2 dF = \infty$ is proved by (7) and (8).

Q.E.D.

5. Initial Condition

Loynes ([6] p.506) states that the w_n with a special circular process and a positive initial value is properly substable. As other example we will consider Example 2 with the process $U_n = \varepsilon_n - \varepsilon_{n-1}$ where ε_n is identically independently distributed. Since we have

$$\begin{aligned} w_{n+1} &= \max\{w_n + U_n, 0\} \\ &= \max\{w_1 + U_1 + \cdots + U_n, U_2 + \cdots + U_n, \cdots, U_n, 0\} \\ &= \varepsilon_n + \max\{w_1 - \varepsilon_0, -\varepsilon_1, \cdots, -\varepsilon_{n-1}, -\varepsilon_n\}, \end{aligned}$$

the limiting distribution is the same as that of $\varepsilon_n + \max\{w_1 - \varepsilon_0, -\tau_1\}$. Hence, although w_n is stable for any initial condition, its limiting distribution is affected by the initial condition. In this case the variance of $\sum_{i=1}^{n} U_i$ is bounded from Theorem 6 and generally the following theorem holds.

Theorem 7. There is no queue satisfying the following three conditions.

(10) w_n is stable for any initial condition.

(11) The variance of $\sum_{i=1}^{n} U_i$ is bounded with respect to n.

(12) The limiting distribution of w_n is not affected by the initial condition.

Proof: Let's assume that a w_n satisfies these conditions and let G be the limiting distribution of Sw_n. Then for any real number K,

$$\begin{aligned} \Pr\{\sum_{i=1}^{n-1} U_i \leqq K\} &= \Pr\{Sw_n - Sw_1 - O_n \leqq K\} \\ &\geqq \Pr\{Sw_n \leqq K + Sw_1\} \end{aligned}$$

If we select w_1 such that $K + Sw_1$ is the continuity point of G, then we can continue with

$$\xrightarrow{n\to\infty} G(K + Sw_1)$$

$$\xrightarrow{Sw_1\to\infty} 1.$$

This is impossible because of Tchebychev's inequality and the condition (11).

Q.E.D.

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