

AN APPROXIMATION METHOD USING CONTINUOUS MODELS FOR QUEUEING PROBLEMS II (MULTI-SERVER FINITE QUEUE)

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Abstract An approximation method based on the diffusion theory is proposed for solving multi-server finite queueing problems having general independent inter-arrival time and general service time distributions. The discrete customer flow through the system is approximated by a continuous one, and a diffusion equation for the process of the number of customers in the system is constructed by using means and variances of the inter-arrival time and service time distributions. Two reflecting boundaries are imposed at the origin and m , the maximum number of customers being allowed in the system. Later, the boundary conditions are modified to improve the approximation. Approximate formulas for P_n , probability of finding n customers in the system, and for N , mean number of lost customers from the system per mean service time, are given for steady state. Numerical examples for mean number of customers in the system are presented for some $E_k/E_k/s(m)$ systems to show the effectiveness of the proposed method.

1. Introduction

Multi-server finite queueing problems are found in many practical situations. Exact results, however, are known only for simple models with special distributions for arrival and service processes and approximation techniques are expected to work for the problems where exact analysis is difficult. Various approximation techniques for solving queueing problems have been devised by many authors [1] and the diffusion method is one of them. Considerable research works have been presented for this approximation technique, e.g., in [4], [5], [7] ~ [11], but most of them treated queues with no limitation to the waiting room and under certain heavy traffic conditions. In [11], the authors

proposed a diffusion approximation method for multi-server queueing problems with possibly short queues. Using the same idea, we propose here a similar approximation method for solving finite queueing problems having general independent inter-arrival time and general service time distributions.

In the next section, our method is described, and in Section 3, modifications of the boundary conditions are discussed. Numerical examples are given in Section 4.

Assumptions and applicability of diffusion approximation techniques to queueing problems have been discussed in detail in [8], [10] and [11]. For the convenience of later discussions, we summarize here the diffusion approximation technique proposed in [11] for G/G/s(∞) system in short. The following notations are used in the sequel.

- $L(t)$ = the number of customers in the system at time t
- $P_n(t)$ = the probability that $L(t)$ is n
- λ = the mean arrival rate
- μ = the mean service rate of each server
- σ_a^2 = the variance of the inter-arrival time distribution
- σ_s^2 = the variance of the service time distribution
- s = the number of parallel servers
- m = the system capacity (= number of servers + maximum queue length) ≥ 2
- $l = 1/(\lambda^2 \sigma_a^2) \geq 1$
- $k = 1/(\mu^2 \sigma_s^2) \geq 1$
- $\rho = \lambda/s = \lambda/(\mu s)$

For a diffusion approximation, we should replace the discrete valued variable $L(t)$ by a continuous variable $X(t)$. We assume that $X(t)$ has a density function $f(x,t)$ defined as $f(x,t)dx = \Pr\{x \leq X(t) \leq x+dx\}$. We shall confine ourselves here to the steady state solution. Let $P_n = \lim_{t \rightarrow \infty} P_n(t)$ and $f(x) = \lim_{t \rightarrow \infty} f(x,t)$. If $X(t)$ is a diffusion process, then $f(x)$ satisfies the following diffusion equation

$$(1.1) \quad 0 = -\frac{d}{dx}\{F(x) f(x)\} + \frac{1}{2} \frac{d^2}{dx^2}\{D(x) f(x)\} \quad ,$$

where $F(x)$ and $D(x)$ are respectively the mean and the variance of instantaneous change in X , i.e.,

$$F(x) = \lim_{\Delta t \rightarrow 0} E\{(X(t+\Delta t) - X(t)) \mid X(t) = x\} / \Delta t \quad ,$$

$$D(x) = \lim_{\Delta t \rightarrow 0} \text{Var} \{ (X(t + \Delta t) - X(t)) \mid X(t) = x \} / \Delta t .$$

As was shown in [8], an integration of (1.1) with a reflecting boundary at the origin and with the condition $f(\infty) = 0$, $df(\infty)/dx = 0$ at $x = \infty$ leads to

$$(1.2) \quad 0 = - F(x) f(x) + \frac{1}{2} \frac{d}{dx} \{ D(x) f(x) \} ,$$

and hence

$$(1.3) \quad f(x) = \frac{\text{const.}}{D(x)} e^{2 \int_0^x \{ F(x)/D(x) \} dx} .$$

It will be natural to set, for $x < s$,

$$F(x) = \lambda - \mu x \quad \text{and} \quad D(x) = \lambda / \ell + \mu x / k$$

and for $x \geq s$,

$$F(x) = \lambda - \mu s \quad \text{and} \quad D(x) = \lambda / \ell + \mu s / k .$$

Then, $f(x)$ of (1.3) coincides with $f_1(x)$ for $x < s$ and with $f_2(x)$ for $x \geq s$ as given below.

$$(1.4) \quad f_1(x) = c \left(\frac{\lambda}{\ell} + \frac{\mu x}{k} \right)^{2\alpha k(k/\ell+1) - 1} e^{-2kx} \equiv c g_1(x) ,$$

$$(1.5) \quad f_2(x) = c \left\{ \frac{g_1(s)}{g_2(s)} \right\} g_2(x) ,$$

where

$$g_2(x) = e^{-\alpha x} \quad \text{and} \quad \alpha = 2(1-\rho)/(\rho/\ell+1/k) .$$

(1.5) reflects the continuity of $f(x)$ at $x = s$. The constant of integration c is determined by the normalization criterion that the integrated value of $f(x)$ over the region $(0, \infty)$ is unity.

In [8], at first, values of P_n 's are calculated by discretizing $f(x)$ and then values of other system parameters are obtained from them. In [11], an approximation formula for the mean queue length was derived directly. But both methods give almost the same results. The former is useful if one needs the value of P_n . If one needs only the value of the mean queue length, the later method is simpler.

2. Diffusion Approximation Method for Finite Queueing System

We assume that, from an infinite source, customers arrive at a multi-server service station with a finite capacity waiting room, that the total capacity of the system (i.e., the number of servers plus the capacity of the waiting room) is m , and that customers waiting for service form a single queue. The service is achieved on the first come - first served basis. If the waiting room is full, then newly arriving customers are not allowed to enter the system and are lost.

The same steady state diffusion equation (1.1) may be used for this finite queueing model with slightly different boundary conditions. In this case, it might be natural to consider that the diffusion process is restricted within two reflecting boundaries at both ends $x=0$ and $x=m$. Using either of the following reflecting boundary conditions

$$(2.1) \quad 0 = -F(x)f(x) + \frac{1}{2} \frac{d}{dx} \{D(x)f(x)\} \Big|_{\text{at } x=0 \text{ or at } x=m},$$

we can derive (1.2) from (1.1). Any solution of (1.2) satisfies both of the boundary conditions of (2.1). Two reflecting boundary conditions have earlier been used in [4] and [7] for the approximate solution of queueing problems. More detailed descriptions about the boundary conditions are available in [2], [3].

Depending on the number of customers waiting in the queue, $F(x)$ and $D(x)$ in (1.1) take different forms. Here we explain them together with the corresponding solution $f(x)$.

(A) The case $m \geq s + 1$

The mean entering rate of customers to the system remains equal to the mean arrival rate λ until n becomes equal to $m - 1$ and then it drops to zero at $n = m$. In order to represent this situation from the standpoint of continuous model, we assume that the mean in-flow rate and its variance decrease linearly to zero in the region $(m - 1, m)$ (see Fig. 1(a)). On the other hand, the mean service rate is μn for $n \leq s$, and μs for $n \geq s$. Hence, the mean out-flow rate for the continuous model may be assumed as shown in Fig. 1(b).

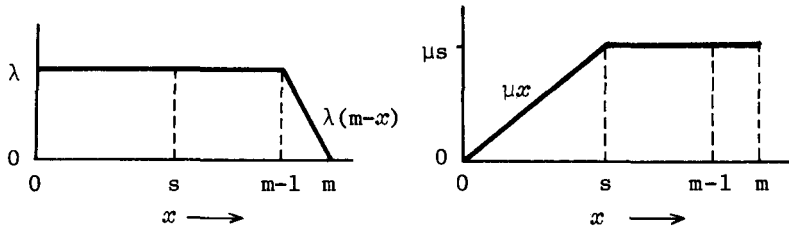


Fig. 1(a) Mean in-flow rate Fig. 1(b) Mean out-flow rate

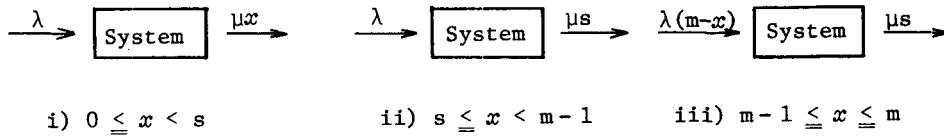


Fig. 2 The system conditions

i) For $0 \leq x < s$:

$F(x)$ and $D(x)$ remain the same as those described in the last section and the solution $f(x)$ in this interval is given by $f_1(x)$ in (1.4).

ii) For $s \leq x < m-1$:

$F(x)$ and $D(x)$ remain the same as those described in the last section for $x \geq s$, and the solution $f(x)$ in this interval is given by $f_2(x)$ in (1.5).

iii) For $m-1 \leq x \leq m$:

$$F(x) = \lambda(m-x) - \mu s \quad \text{and} \quad D(x) = \lambda(m-x)/\ell + \mu s/k ,$$

and from the continuity of the solution, $f(x)$ for $m-1 \leq x \leq m$ is given by

$$(2.2) \quad f_3(x) = c \left\{ \frac{g_1(s)}{g_2(s)} \right\} \left\{ \frac{g_2(m-1)}{g_3(m-1)} \right\} g_3(x) ,$$

where

$$(2.3) \quad g_3(x) = \left\{ \frac{\lambda(m-x)}{\ell} + \frac{\mu s}{k} \right\} e^{\frac{2\ell}{\sigma} \left(\frac{\ell}{k} + 1 \right) - 1} \cdot 2\ell x .$$

The total $f(x)$ is shown in Fig. 3.

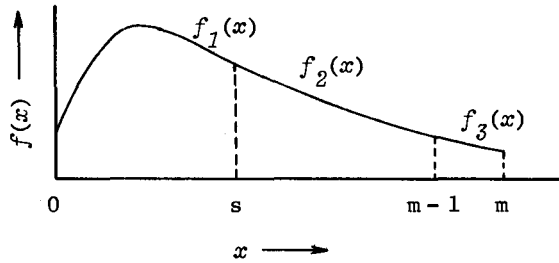


Fig. 3 The total solution

The constant c is determined so that the integrated value of $f(x)$ over the region $(0, m)$ is unity, i.e.,

$$(2.4) \quad 1 = \int_0^s f_1(x) dx + \int_s^{m-1} f_2(x) dx + \int_{m-1}^m f_3(x) dx .$$

An approximate value of the probability of finding n customers in the system in the steady state can be obtained by discretizing $f(x)$ as

$$(2.5) \quad P_n = \int_{n-.5}^{n+.5} f(x) dx \quad (1 \leq n \leq m-1) .$$

Then, an approximate formula L_1 for the mean number of customers in the system is derived from (1.4), (1.5), (2.2), (2.4) and (2.5) as

$$(2.6) \quad L_1 = \sum_{n=1}^m n P_n$$

$$= \sum_{n=1}^{s-1} n \int_{n-.5}^{n+.5} f_1(x) dx + s \left[\int_{s-.5}^s f_1(x) dx + \int_s^{s+.5} f_2(x) dx \right] + \sum_{n=s+1}^{m-2} n \int_{n-.5}^{n+.5} f_2(x) dx$$

$$+ (m-1) \left[\int_{m-1.5}^{m-1} f_2(x) dx + \int_{m-1}^{m-.5} f_3(x) dx \right] + m \int_{m-.5}^m f_3(x) dx ,$$

where the approximation

$$(2.7) \quad P_m = \int_{m-.5}^m f(x) dx$$

is used. The above formula (2.6) is valid for $m \geq s+2$. For $m = s+1$, the terms with $f_2(x)$ vanish.

An approximate formula D_1 for the mean number of departing customers per unit time after receiving service is given by

$$(2.8) \quad D_1 = \sum_{n=1}^{s-1} \mu n P_n + \sum_{n=s}^m \mu s P_n ,$$

which may be written in a form similar to (2.6). Hence, an approximate formula N_1 for the mean number of lost customers per mean service time is obtained by the relation

$$(2.9) \quad N_1 = (\lambda - D_1) / \mu = \alpha - D_1 / \mu .$$

(B) The case $s = m \geq 2$

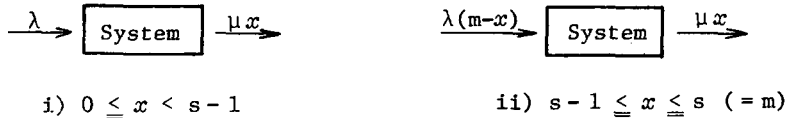


Fig. 4 The system conditions

i) For $0 \leq x < s - 1$:

$F(x)$ and $D(x)$ remain the same as those described in Section 1 and the solution $f(x)$ in this interval is given by $f_1(x)$ in (1.4).

ii) For $s - 1 \leq x \leq s (=m)$:

$$F(x) = \lambda(m-x) - \mu x \quad \text{and} \quad D(x) = \lambda(m-x) / \ell + \mu x / k .$$

From the continuity of the solution, $f(x)$ in this interval is given by

$$(2.10) \quad f_4(x) = c \left\{ \frac{g_1(s-1)}{g_4(s-1)} \right\} g_4(x) ,$$

where

$$(2.11) \quad g_4(x) = \left\{ \frac{\lambda(m-x)}{\ell} + \frac{\mu x}{k} \right\} e^{Ux} ,$$

$$U = \frac{-2(\alpha + 1)}{-\alpha/\ell + 1/k} \quad \text{and} \quad W = \frac{2m\alpha(1/\ell + 1/k)}{(-\alpha/\ell + 1/k)^2} - 1$$

L_1 , D_1 and N_1 can be obtained by the procedure shown in (A).

3. Modifications of the Boundary Conditions

The discretization (2.5) is not applicable for $n = 0$ and $n = m$ if reflecting boundaries are set at $x = 0$ and at $x = m$. In [7], different types of assumptions were made for P_0 and P_m in solving two stage cyclic queueing problem by the diffusion approximation method.

It seems, however, from our numerical experience that the shifting of reflecting boundaries, at $x = 0$ to $x = -0.5$ and at $x = m$ to $x = m + 0.5$, so that (2.5) is applicable for $n = 0$ and $n = m$, gives better approximate results.

We assume that the mean in-flow rate and the mean out-flow rate in the region $(-0.5, 0)$ are λ and 0 respectively and, in the region $(m, m + 0.5)$, 0 and μs respectively. The flow rates for the case $m \geq s + 1$ are shown in Fig. 5 .

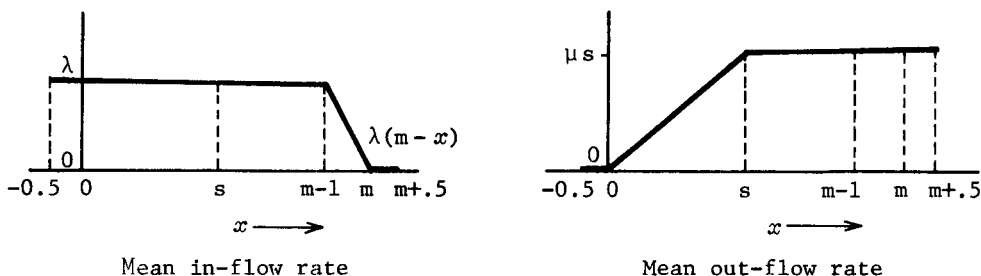


Fig. 5 Modifications of mean flow rates

Then,

i) for $-0.5 \leq x \leq 0$:

$$F(x) = \lambda - 0 \quad \text{and} \quad D(x) = \lambda/\ell + 0/k ,$$

and

ii) for $m \leq x \leq m + 0.5$:

$$F(x) = 0 - \mu s \quad \text{and} \quad D(x) = 0/\ell + \mu s/k .$$

Hence, $f(x)$ of (1.3) is given by

$$(3.1) \quad f_0(x) = c_0 e^{2\lambda x} \equiv c_0 g_0(x) \quad \text{for } -0.5 \leq x \leq 0, \text{ and}$$

$$(3.2) \quad f_e(x) = c_e e^{-2kx} \equiv c_e g_e(x) \quad \text{for } m \leq x \leq m + 0.5 ,$$

where c_0 and c_e are constants.

In the following, for the simplicity of discussion, we confine ourselves to (A), the case $m \geq s + 1$. A similar improvement procedure can be applied

to (B). Due to the change in the range of $f(x)$ as considered above, the constant c of (1.4), (1.5) and (2.2) should be changed. The continuity of $f(x)$ at $x = 0$ requires that

$$(3.3) \quad c = c_0 \left\{ \frac{g_0(0)}{g_1(0)} \right\},$$

and the continuity at $x = m$ requires that

$$(3.4) \quad c_e = c_0 \left\{ \frac{g_0(0)}{g_1(0)} \right\} \left\{ \frac{g_1(s)}{g_2(s)} \right\} \left\{ \frac{g_2(m-1)}{g_3(m-1)} \right\} \left\{ \frac{g_3(m)}{g_e(m)} \right\}.$$

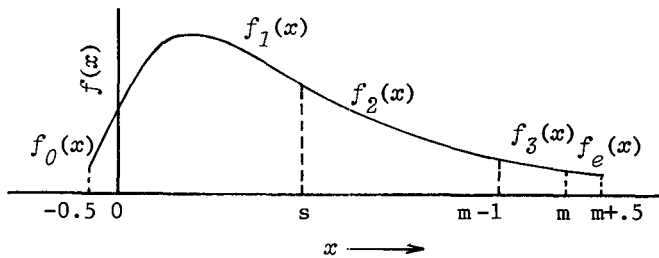


Fig. 6 The total solution

The normalization criterion that the integrated value of $f(x)$ over the region $(-0.5, m+0.5)$ is unity, is needed to find out the value of the constant c_0 .

The approximate formula L_1 of (2.6), for the mean number of customers in the system, is then modified as

$$(3.5) \quad L_2 = \sum_{n=1}^m n P_n = \sum_{n=1}^m n \int_{n-0.5}^{n+0.5} f(x) dx$$

where $f(x)$ is shown in Fig. 6. D_1 and N_1 can be modified in a similar manner.

4. Numerical Examples

The proposed approximation method has been tested for eleven different $E_k/E_k/s(m)$ systems, namely $M/M/s(m)$, $M/E_2/s(m)$, $E_2/M/s(m)$, $E_2/E_2/s(m)$, $M/D/s(m)$, $D/M/s(m)$, $E_2/D/s(m)$, $D/E_2/s(m)$, $E_{10}/E_{10}/s(m)$,

$E_{10}/D/s(m)$ and $D/E_{10}/s(m)$, with the following different values of s , m and ρ .

$$s = 2, 5, 10$$

$$m = s, 2s, 5s, 10s$$

$$\rho = 0.3, 0.5, 0.7, 1.0, 1.1, 1.5, 2.0$$

Some of the calculated values are shown in Table 1 ~ 4. In Table 1, values of both L_1 and L_2 are compared with exact values for $M/M/s(m)$ system [6]. From this table, it is seen that L_2 gives better approximation in all cases. In Tables 2 ~ 4, L_2 values are compared with simulation or exact results. Since the simulated values are unstable at and in the vicinity of $\rho = 1$, those values are not included in tables except in Table 1. In the tables 'E' stands for exact values and 'Sim' for simulated ones, but '*' in the rows of 'Sim' stands for exact values.

Numerical integration is generally needed for the calculation of approximate formulas. We used Gauss's formula for numerical integration which is available in FACOM M200 computer. It will be worth to note the following point in the calculation of approximate values. Function $g_i(x)$ is of the form

$$(4.1) \quad g_i(x) = (A_i x + B_i) \sum_i W_i e^{U_i x},$$

whose value occasionally becomes very large giving overflow problem in numerical integration. So, instead of $g_i(x)$ we used the following exponential function.

$$(4.2) \quad g_i^*(x) = \text{Exp} [W_i \log_e \{ (A_i x + B_i) / (A_i x_1 + B_i) \} + U_i (x - x_1)]$$

where x_1 is such a fixed value of x that overflow does not occur in the calculation of $g_i^*(x)$ in the considered region of x . The proposed approximate formulas can easily be rewritten to contain this particular form.

There is another general method for the numerical calculation which is also described below. Since our calculation is based on the numerical integration of positive functions, we can use a method based on the logarithmic principles, where values of integrated functions are transformed into logarithms and summations of function values are performed in a manner such as: if $\log_e a = \alpha$, $\log_e b = \beta$ and $a \geq b > 0$, then $\alpha + \log_e (1 + e^{\beta - \alpha}) = \log_e (a+b)$, where the underflow of $e^{\beta - \alpha}$ (≤ 1) can be detected before its calculation and if so, it is regarded as zero.

For the deterministic distributions (D-type, i.e., l or $k = \infty$), the

equations (1.4), (2.3), (3.1) and (3.2) can not directly be used in the calculation. In such cases, the corresponding equations can easily be derived using (1.3).

Table 5 shows the upper and lower bounds of relative errors of L_2 compared with exact or simulated values for all of the eleven systems described above. Each upper bound is the maximum value and each lower bound is the minimum value among all the relative error values for all values of m and ρ except for $\rho = 1$, which are listed above, for a particular value of s .

5. Conclusion

A diffusion approximation method for the solution of multi-server finite queueing problems has been proposed. Since, the diffusion equation depends only on the means and variances of the arrival and service processes, this method is easily applicable to problems having general independent inter-arrival time and general service time distributions.

The approximate formulas are simple and values of the system parameters can be easily calculated using a small computer. The calculation time is extremely short in comparison with the simulation time to calculate the same values. Moreover, from Table 5, it is seen that the relative errors of the approximate values are within the tolerable range especially for greater values of s . So, we conclude that the proposed approximation method is useful, rather than time consuming simulation procedures, to solve practical finite queueing problems where exact results are not available.

For single server queues, approximate values L_2 and N_2 can be also calculated by our method with small modifications, but, guessing from our numerical results, the errors will be comparatively larger.

Table 1. Mean number of customers in the system in $M/M/s(m)$ system

ρ	Type of result	s = 2			s = 5		
		m = 2	4	10	5	10	25
0.3	E	0.539	0.641	0.659	1.479	1.509	1.509
	L_2	0.555	0.687	0.718	1.516	1.560	1.560
	L_1	0.662	0.848	0.897	1.608	1.668	1.669
0.7	E	0.994	1.614	2.493	2.961	3.965	4.377
	L_2	0.993	1.631	2.558	2.929	3.972	4.414
	L_1	1.002	1.670	2.677	2.870	3.971	4.447
1.0	E	1.200	2.222	5.238	3.576	6.175	13.720
	L_2	1.188	2.207	5.222	3.520	6.127	13.675
	L_1	1.149	2.148	5.152	3.395	5.997	13.539
1.1	E	1.253	2.385	6.101	3.718	6.782	17.802
	L_2	1.238	2.362	6.059	3.659	6.721	17.715
	L_1	1.188	2.278	5.924	3.520	6.554	17.485
1.5	E	1.412	2.867	8.153	4.102	8.291	23.002
	L_2	1.393	2.826	8.069	4.040	8.211	22.902
	L_1	1.308	2.672	7.824	3.867	7.971	22.632
2.0	E	1.539	3.213	9.008	4.361	9.028	24.000
	L_2	1.520	3.168	8.930	4.304	8.954	23.920
	L_1	1.412	2.977	8.678	4.114	8.706	23.665

Table 2. Mean number of customers in the system in $E_2/E_2/s(m)$ system

ρ	Type of result	s = 2			s = 5		
		m = 2	4	10	5	10	25
0.3	Sim	0.575	0.615	0.611	1.484	1.508	1.504
	L_2	0.541	0.591	0.592	1.498	1.502	1.502
0.7	Sim	1.084	1.619	1.972	3.129	3.825	3.869
	L_2	1.047	1.569	1.901	3.143	3.722	3.765
1.1	Sim	1.354	2.577	6.849	3.912	7.269	19.488
	L_2	1.317	2.496	6.645	3.928	7.231	20.082
1.5	Sim	1.505	3.125	8.861	4.286	8.837	23.713
	L_2	1.483	3.046	8.802	4.280	8.824	23.796
2.0	Sim	1.630	3.431	9.378	4.499	9.336	24.278
	L_2	1.615	3.388	9.348	4.504	9.349	24.348

Table 3. Mean number of customers in the system in M/D/s (m) system

ρ	Type of result	s = 2			s = 5		
		m = 2	4	10	5	10	25
0.3	Sim	0.539*	0.633	0.630	1.479*	1.512	1.511
	L ₂	0.589	0.631	0.631	1.524	1.525	1.525
0.7	Sim	0.994*	1.602	2.023	2.961*	3.922	4.009
	L ₂	1.075	1.597	1.843	3.164	3.691	3.710
1.1	Sim	1.253*	2.481	6.561	3.718*	7.158	20.090
	L ₂	1.361	2.549	6.698	3.933	7.277	20.032
1.5	Sim	1.412*	3.020	8.796	4.102*	8.737	23.563
	L ₂	1.528	3.079	8.757	4.288	8.795	23.741
2.0	Sim	1.539*	3.356	9.213	4.361*	9.227	24.168
	L ₂	1.653	3.406	9.329	4.515	9.332	24.328

Table 4. Mean number of customers in the system in D/M/s (m) system

ρ	Type of result	s = 2			s = 5		
		m = 2	4	10	5	10	25
0.3	Sim	0.599	0.602	0.604	1.495	1.491	1.507
	L ₂	0.486	0.556	0.564	1.495	1.507	1.507
0.7	Sim	1.187	1.624	1.844	3.273	3.773	3.746
	L ₂	1.048	1.549	1.968	3.121	3.746	3.824
1.1	Sim	1.473	2.681	6.789	4.111	7.388	20.183
	L ₂	1.305	2.445	6.595	3.929	7.182	20.138
1.5	Sim	1.618	3.241	8.946	4.454	8.995	23.926
	L ₂	1.447	3.024	8.857	4.277	8.866	23.855
2.0	Sim	1.720	3.525	9.465	4.638	9.497	24.429
	L ₂	1.567	3.376	9.363	4.485	9.363	24.363

Table 5. Bounds of relative errors for different systems for L_2

Type of the system	s = 2		s = 5		s = 10	
	Lower	Upper	Lower	Upper	Lower	Upper
M/M/s (m)	-0.03	0.09	-0.02	0.04	-0.01	0.01
M/E ₂ /s (m)	-0.04	0.08	-0.04	0.04	-0.03	0.03
E ₂ /M/s (m)	-0.11	0.07	-0.04	0.02	-0.03	0.02
E ₂ /E ₂ /s (m)	-0.06	0.00	-0.03	0.04	-0.02	0.02
M/D/s (m)	-0.11	0.10	-0.09	0.07	-0.05	0.05
D/M/s (m)	-0.19	0.07	-0.05	0.03	-0.03	0.02
E ₂ /D/s (m)	-0.09	0.04	-0.04	0.02	-0.02	0.04
D/E ₂ /s (m)	-0.13	0.02	-0.03	0.01	-0.01	0.02
E ₁₀ /E ₁₀ /s (m)	-0.16	0.08	-0.02	0.02	-0.01	0.03
E ₁₀ /D/s (m)	-0.25	0.20	-0.08	0.01	-0.04	0.02
D/E ₁₀ /s (m)	-0.23	0.17	-0.05	0.02	-0.02	0.03

$$\text{Relative error of } L_2 = (L_2 - L)/L$$

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References

- [1] Bhat, U. N., Shalaby, M. and Fischer, M. J.: Approximation Techniques in the Solution of Queueing Problems, *Naval Research Logistics Quarterly*, Vol.26, No.2 (1979), 311-326.
- [2] Feller, W.: The Parabolic Differential Equations and the Associated Semi-Groups of Transformations. *Annals of Mathematics*, Vol.55 (1952), 468-519.
- [3] Feller, W.: Diffusion Processes in One Dimension. *Transactions of American Mathematical Society*, Vol.77 (1954), 1-31.
- [4] Gaver, D. P. and Shedler, G. S.: Processor Utilization in Multiprogramming Systems via Diffusion Approximations. *Operations Research*, Vol.21 (1973), 569-576.

- [5] Gelenbe, E.: On Approximate Computer System Models. *Journal of the Association of Computing Machinery*, Vol.22 (1975), 261-269.
- [6] Gross, D. and Harris, M.: *Fundamentals of Queueing Theory*, John Wiley and Sons, 1975, 64-67, 109-112, 275.
- [7] Halachmi, B. and Franta, W. R.: A Diffusion Approximate Solution to the G/G/k Queueing Systems. *Computers and Operations Research*, Vol.4 (1977), 37-46.
- [8] Halachmi, B. and Franta, W. R.: A Diffusion Approximation to the Multi-Server Queue. *Management Science*, Vol.24, No.5 (1978), 522-529.
- [9] Kobayashi, H.: Application of the Diffusion Approximation to Queueing Networks I. *Journal of the Association of Computing Machinery*, Vol.21 (1974), 316-328.
- [10] Newell, G. F.: *Applications of Queueing Theory*, Chapman and Hall, London, 1971, 68-74, 101-140.
- [11] Sunaga, T., Kondo, E. and Biswas, S. K.: An Approximation Method Using Continuous Models for Queueing Problems. *Journal of the Operations Research Society of Japan*, Vol.21, No.1 (1978), 29-42.
- [12] Sunaga, T. and Biswas, S. K.: An Approximation Method Using Continuous Models for Finite Queueing Problems. *Abstracts of Spring Research Conference of the Operations Research Society of Japan in Hiroshima*, October, 1977 (In Japanese).

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