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AN EPQ MODEL FOR DETERIORATING ITEMS UNDER LIFO POLICY

Hark Hwang
Korea Advanced Institute of Science and Technology

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Abstract Inventory level in production quantity model has been developed for items that deteriorate continuously in accordance with a general probability distribution under Last In First Out (LIFO) issuing policy. From the result developed, the earlier model by Misra for constant rate of deterioration can be obtained as a particular case. An approximate formula is derived using perturbation techniques. To illustrate the use of the formula an example problem is solved and an approximate optimum production quantity is found.

1. Introduction

A deteriorating product is one whose gradual loss of potential or utility is associated with the passage of time such as grain, photographic film, electronic components and radio-active material. Recently many inventory models have been considered in which inventory is depleted not only by physical depletion but also by deterioration. The first attempt on this subject was made by Ghare and Schrader [3] who determined economic order quantity and this model was extended to weibull distribution deterioration by Covert and Philip [2] and was further extended to the case of general distribution deterioration by Shah [8]. Cohen [1] considered the problem of simultaneously setting price and ordering quantity.

Nahmias and Wang [5] considered (Q,r) inventory system for item which is subject to continuous exponential decay and Nose, Ishii, Nishida and Hamada [6] extended it to the situation in which procurement lead time is treated as a finite varying stochastic variable. For a production quantity model, Shah and Jaiswal [9] considered a model in which backlogging is permissible with a constant deterioration rate and Misra [4] used a variable rate of deterioration by assuming a two parameter Weibull distribution. But the solution obtained by Misra is only correct for the case of constant rate of deterioration. When the rate of deterioration is variable, the items which have entered inventory at different times have a different rate of deterioration since the amount deteriorated during a given time interval depends on how long an item has been in stock. With an EOQ model with weibull distribution deterioration considered by Covert and Philip [2] and Philip [7], it is assumed that all items in one order enter the inventory at same time, so the above complexity does not occur.

In this article, inventory level for an Economic Production Quantity model with Last In First Out (LIFO) issuing policy for demand with items that deteriorate continuously in accordance with a general probability distribution for the lifetime of an item is developed. From the general model developed here, the result by Misra can be obtained as a particular case by taking an exponential distribution for the time to deterioration of an item.

2. Development of the Model

Following is the list of assumptions we make to develop the model;

(1) demand rate λ is known and constant

(2) production rate P governing supply is finite and constant

(3) units are available for satisfying demand after their production

(4) a deteriorated unit is not repaired or replaced

(5) the production rate P is greater than the demand rate

(6) shortages are not allowed

(7) the number of units is treated as a continuous variable

(8) the time for an item to deteriorate follows *probability density function* (*p.d.f.*) $f(t)$ $(t \geq 0)$ and *cumulative distribution function* (*c.d.f.*) $F(t)=1-R(t)$; so that, the instantaneous deterioration rate of an item is $D(t)=f(t)/(1-F(t)) = f(t)/R(t)$, $t \geq 0$.

(9) Last In First Out (LIFO) principle is applied in satisfying demand.

Figure 1. shows an inventory cycle for a finite production rate where T is a cycle time.

During time interval $(t,t+\Delta t)$ where $t \leq T_1$ production occurs at a constant rate of P units per unit time and demand occurs at a constant rate of λ units per unit time, leaving $(P-\lambda)\ \Delta t$ to enter the inventory system due to LIFO policy.

At time t_1 where $t \leq t_1 \leq T_1$, the quantity $(P-\lambda)\ \Delta t$ which entered the inventory during $(t,t+\Delta t)$ reduced to $(P-\lambda)R(t_1-t)\ \Delta t$ due to deterioration process shown in Fig. 1.

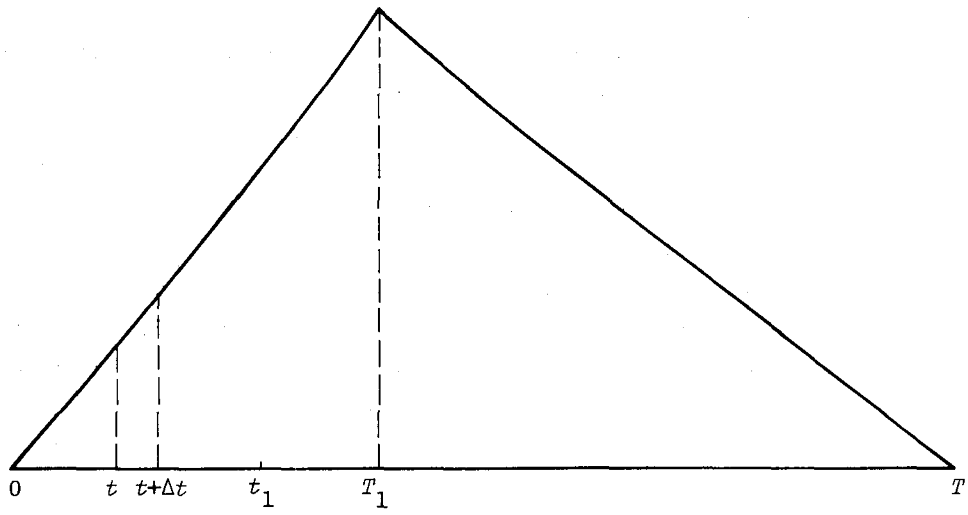

Figure 1. Economic Production Quantity Model with Deterioration

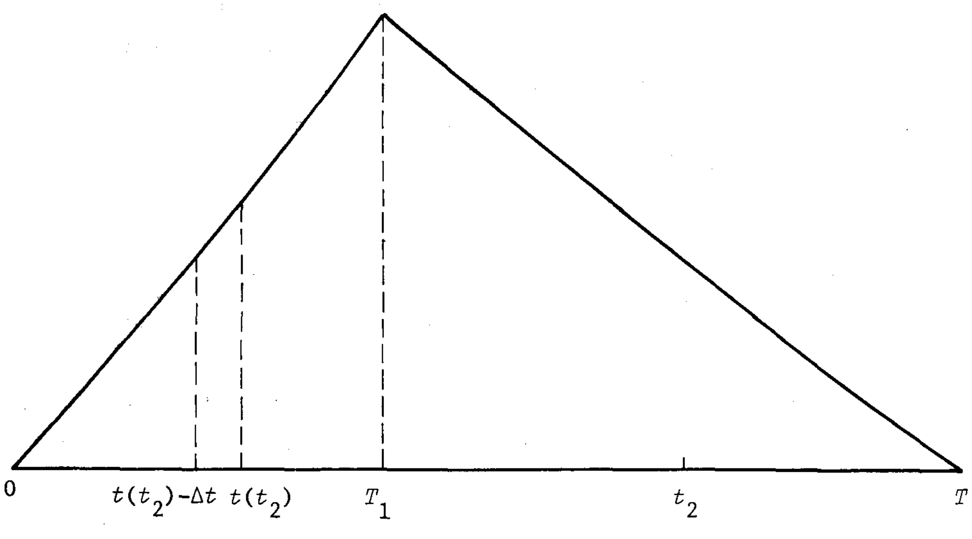

Figure 2. Economic Production Quantity Model with Deterioration

This gives the inventory level at time t_1, I_{t_1} as follows

(1) $$I_{t_1} = \int_0^{t_1} (P-\lambda)R(t_1-t)\,dt.$$

. During time interval $(t_2, t_2+\Delta t_2)$ where $T_1 \leq t_2 \leq T$, there is no production and demand during this interval is $\lambda\Delta t_2$ which is satisfied from the inventory accumulated during $(0, T_1)$. Assume that the demand $\lambda\Delta t_2$ is satisfied with the items produced during $(t(t_2) - \Delta t,\ t(t_2))$ shown in Fig. 2.

Notice that at time t_2 the item produced during $(t(t_2), T_1)$ is not in the inventory system because they already satisfied the demand occured during $(t(t_2), t_2)$ due to LIFO principle.

The above argument gives

(2) $$\lambda\Delta t_2 = (P-\lambda)R(t_2-t)(-\Delta t).$$

Therefore,

(3) $$\lambda \frac{dt_2}{dt} = -(P-\lambda)R(t_2-t).$$

If $R(t)$ is known, $t(t_2)$ can be found from eq. (3) with initial condition, $t=T_1$ at $t_2=T_1$ due to LIFO policy and I_{t_2} becomes

(4) $$I_{t_2} = \int_0^{t(t_2)} (P-\lambda)R(t_2-y)\,dy.$$

3. Case

From the general inventory level developed above, two particular cases are considered by taking the exponential distribution and the Weibull distribution for the time to deterioration of an item.

3.1. Exponential distribution for the time to deterioration of an item

Let the p.d.f. of the time to deterioration of an item be

$$f(t) = \alpha \exp(-\alpha t), \qquad t \geq 0,\ \alpha > 0$$
$$= 0 \qquad \text{otherwise.}$$

For this distribution $R(t)$, $D(t)$ are $R(t) = \exp(-\alpha t)$ and $D(t) = \alpha$. With substitution of $\exp(-\alpha(t_1-t))$ into $R(t_1-t)$ of eq. (1) and integration of the resultant equation, we obtain

(5) $I_{t_1} = \frac{P-\lambda}{\alpha}(1-\exp(-\alpha t_1)) \quad , \quad 0 \le t_1 \le T_1.$

Similarly, from eqs. (3) and (4)

(6) $I_{t_2} = \frac{P-\lambda}{\alpha}\exp(-\alpha t_2)\ (\exp(\alpha t)-1),$

and

(7) $\lambda \frac{dt_2}{dt} = -(P-\lambda)\ \exp(-\alpha(t_2-t)).$

The solution of eq. (7) with the boundary condition, $t=T_1$ at $t_2=T_1$ is

(8) $(P-\lambda)\ \exp(\alpha t) = P\ \exp(\alpha T_1) - \lambda\ \exp(\alpha t_2).$

Substituting eq. (8) into eq. (6) yields

(9) $I_{t_2} = \frac{1}{\alpha}[P \cdot \exp(\alpha(T_1-t_2)) - \lambda - (P-\lambda)\ \exp(-\alpha t_2)].$

We can show that our results are the same as those obtained by Misra but need to recognize that the origin of t_2 in his formulation is at the end of t_1, T_1 while that of t_1 and t_2 in this article have a common origin at the beginning of a cycle, that is t_2 is equivalent to (T_1+t_2) of Misra. In case of a constant rate of deterioration, the LIFO issuing policy does not have any effect on the inventory level.

3.2. Weibull distribution for the time to deterioration of an item

Assume that the p.d.f. of the time to deterioration of an item is

$$f(t) = \alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta}) \quad , \quad t \ge 0 \quad , \quad \alpha > 0, \quad \beta > 0$$
$$= 0 \qquad \text{otherwise,}$$

where α,β are some constants determined by the deterioration process. For this distribution $R(t)$, $D(t)$ are

$$R(t) = \exp(-\alpha t^{\beta}) \text{ and } D(t) = \alpha\beta t^{\beta-1}.$$

Then, I_{t_1} and I_{t_2} become

(10) $I_{t_1} = \int_0^{t_1} (P-\lambda)\ \exp(-\alpha(t_1-t)^{\beta})dt,$

and

(11) $I_{t_2} = \int_0^{t(t_2)} (P-\lambda)\ \exp(\alpha(t_2-y)^{\beta})dy,$

where

$$\lambda \frac{dt_2}{dt} = -(P-\lambda)\ \exp(-\alpha(t_2-t)^{\beta}). \tag{12}$$

To solve eq. (12), let

$$\alpha\ (t_2 - t)^{\beta} = x. \tag{13}$$

Substituting eq. (12) into the result obtained from differentiation of eq. (13) yields

$$\alpha^{\frac{1}{\beta}}\ \beta X^{\frac{\beta-1}{\beta}}(\frac{\lambda-P}{\lambda}\ \exp(-X)-1) = \frac{dX}{dt}, \tag{14}$$

and with the boundary condition, at t_2-T_1, t=T_1 which in turn implies, at t=T_1, X=0, the solution of eq. (14) is

$$t(t_2) = -\int_0^x [\alpha^{\frac{1}{\beta}}\beta y^{\frac{\beta-1}{\beta}}(1+\frac{P-\lambda}{\lambda}\exp(-y))]^{-1}\ dy + T_1. \tag{15}$$

3.3. Approximation

We notice that solving eq. (15) with respect to $t(t_2)$ seems to be very difficult because it is a transcendental equation. One way to obtain an approximate solution of $t(t_2)$ is to solve eq. (12) under the assumption that $\alpha \ll 1$. The validity of this assumption and physical meaning of α when β=1 is explained in detail by Ghare and Schrader [3].

Let u=t_2-t and v=P-λ. Then eq. (12) becomes

$$\lambda(\frac{du}{dt} + 1) = -v\ \exp\ (-\alpha u^{\beta}), \tag{16}$$

and the solution of eq. (16) with the boundary condition, at u=0, t=T_1, is

$$\int_0^u \frac{-\lambda\ du'}{\lambda + \exp(-\alpha u'^{\beta})v} = \int_{T_1}^t dt' = t - T_1. \tag{17}$$

Using the series form of the exponential and ignoring terms with third and higher order powers of α, eq. (17) becomes

$$\frac{-\lambda u}{\lambda+v}\ [\ 1 + \frac{v}{\lambda+v}\frac{\alpha u^{\beta}}{\beta+1} + \frac{v(v-\lambda)\alpha^2 u^{2\beta}}{2(\lambda+v)^2(2\beta+1)} + O(\alpha^3)] = t-T_1. \tag{18}$$

Let t=g_0+ $\alpha g_1(t_2)$ + $\alpha^2 g_2(t_2)$ + $O(\alpha^3)$, then u becomes

$$u = t_2 - g_0 - \alpha g_1 - \alpha^2 g_2 + O(\alpha^3). \tag{19}$$

Substituting eq. (19) into eq. (18) and using approximation formula, $(1-x)^{\beta} = 1-\beta x$, eq. (18) becomes

$$\frac{-\lambda}{\lambda+v}(t_2-g_0-\alpha g_1-\alpha^2 g_2)\ [\ 1+\frac{v}{\lambda+v}\frac{\alpha}{\beta+1}(t_2-g_0)^{\beta}(1-\frac{\alpha\beta g_1}{t_2-g_0})$$

$$+\frac{v(v-\lambda)\alpha^2}{2(\lambda+v)^2(2\beta+1)}(t_2-g_0)^{2\beta}+0(\alpha^3)]$$

(20) $$= g_0+\alpha g_1+\alpha^2 g_2 - T_1 + 0(\alpha^3).$$

Equating terms with the same power of α,

(21) $$g_0 = \frac{1}{P-\lambda}(PT_1-\lambda t_2),$$

(22) $$g_1 = \frac{\lambda}{P}\frac{1}{\beta+1}(t_2-g_0)^{\beta+1},$$

and

(23) $$g_2 = \frac{(t_2-g_0)^{\beta}}{P}\ [\ \lambda\beta g_1 + \frac{\lambda g_1}{\beta+1} - \frac{\lambda(P-2\lambda)(t_2-g_0)^{\beta+1}}{2P(2\beta+1)}\ ]\ .$$

With this perturbation technique, it is theoretically possible to obtain an approximate value of t to any desired accuracy using higher powers of α. Table 1 is the tabulated results of $t(t_2)$ from example problems to compare the approximate formula of $t(t_2)$, $t(t_2) = g_0 + \alpha g_1(t_2) + \alpha^2 g_2(t_2)$ where $g_i(t_2)$ is defined as eqs.(21), (22) and (23) respectively with the exact values calculated from eq. (8) when $\beta = 1$.

cases / t_2	α=0.1	β=0.1	α=0.1 β=0.5	α=0.1 β=1.5
	E	A	A	A
5.0	5.0000	5.0000	5.0000	5.0000
5.5	4.4737	4.4731	4.4647	4.4781
6.0	3.8888	3.8850	3.8979	3.8565
6.5	3.2346	3.2244	3.3093	3.0343
7.0	2.4974	2.4800	2.7022	1,8736
7.5	1.6589	1.6406	2.0787	0.1945
8.0	0.6943	0.6950	1.4401	-
8.5	-	-	0.7874	-
9.0	-	-	0.1213	-
cycle time	8.3180	8.3333	9.0900	7.5476

Table 1. Calculated Values of $t(t_2)$ with P=8, λ=4 and T_1=5

E : Exact Value from Eq. (8)

A : Value from $t = g_0 + \alpha g_1(t_2) + \alpha^2 g_2(t_2)$

The results from the approximate formula are in good agreement with those by exact values. We notice that the size of error depends on the term $\max_{t_2,t} \alpha(t_2-t)^\beta = \alpha T^\beta$. Once $t(t_2)$ is found, the inventory level can be calculated with eqs. (10) and (11).

3.4. Application

To illustrate the use of the formula an approximate optimum production quantity is found in the E.P.Q. system where no shortage is permitted.

Let C = a unit cost

C_1 = inventory carrying cost/unit/unit time

C_3 = set up cost/cycle

Then *total cost* (*TC*) during a cycle time, T, consists of set up cost, production cost and holding cost.

Thus

$$(24) \qquad TC(T_1) = C_3 + CPT_1 + C_1 \left(\int_0^{T_1} I_{t_1} dt_1 + \int_{T_1}^{T} I_{t_2} dt_2 \right),$$

and then *total cost per unit time* (*TCU*) becomes $(TC)/T$. T_1^* which minimizes (*TCU*) cannot be derived in a closed form as we notice but an approximate optimal solution can be found by a numerical calculation using the formula developed.

A hypothetical problem is solved here with the following parameter values;

P = 7500 units/year

λ = 2500 units/year

α = 0.2

β = 1.2

C = \$3.00/unit

C_1 = \$0.60/unit/year

C_3 = \$50.00/set up.

For each value of T_1 chosen, cycle time, annual set up cost, annual production cost, annual holding cost and annual total cost (*TCU*) are calculated and Table 2, shows part of the results.

Table 2. T_1 and its corresponding costs

T_1 (year)	cycle time (year)	annual set up cost($)	annual prod. cost ($)	annual holding cost($)	annual total cost ($)
0.020	0.0597	837.521	7537.688	127.967	8503.176
0.060	0.1785	280.112	7563.025	129.566	7972.703
0.070	0.2079	240.500	7575.758	128.690	7944.948
0.080	0.2372	210.793	7588.533	144.271	7943.597
0.090	0.2665	187.617	7598.499	160.414	7946.530
0.100	0.2956	169.147	7611.637	189.830	7970.615
0.150	0.4396	113.740	7677.434	250.174	8041.348

An approximate optimal value of T_1 is 0.960 month, the cycle time, 2.846 month and the corresponding annual minimum cost is $7943.597. In this case the number of units that deteriorate during a cycle is 7.0.

If deterioration were disregarded the optimum value of T_1 and total cost per year would have been 1.264/month and $7816.2 respectively.

4. Conclusion

Inventory level in a production quantity model for items that deteriorate continuously in accordance with a general probability distribution has been developed. In case of a variable rate of deterioration, complexity arises due to the fact that the amount deteriorated during a given time interval depends on the time item has entered the inventory. To solve this difficulty we assume the inventory manager select Last In First Out (LIFO) as an issuing policy. Weibull and exponential distribution for the time to deterioration of an item are considered. With constant rate of deterioration, the result developed is shown to be consistent with the Misra's.

Due to difficulty in solving $t(t_2)$ in eq. (15), an approximation formula using perturbation techniques is developed and is illustrated its usage through a example problem.

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Hark HWANG: Associate Professor
Dept. of IE, Korea Advanced
Institute of Science and Technology
P.O. Box 150, Chongryang
Seoul, Korea.