

## A PROPERTY OF TWO PALM MEASURES IN QUEUEING NETWORKS AND ITS APPLICATIONS

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*Abstract* In this paper, we consider two Palm measures  $P_a, P_d$  which are strictly stationary for the time sequences  $\{a_n\}_{n=-\infty}^{\infty}, \{d_n\}_{n=-\infty}^{\infty}$  respectively, where  $\{a_n\}$  and  $\{d_n\}$  are the time sequences of arrival times and of departure times respectively at a certain station in the given network. And we give a sufficient condition for two events  $B'$  and  $B''$  to hold  $P_a(B') = P_d(B'')$ . This result means that some statistics of a customer who arrives at the station in an equilibrium state can be evaluated by the measure  $P_d$ , and we apply this result to obtain a ramification of the theory of Burke [3] and Reich [11] about tandem queues, and new properties of residual sojourn times in Jackson type networks etc.

### 1. Introduction

When we study queueing networks, it is needed sometimes that the future behaviour of a customer who arrives at the given system in a state of statistical equilibrium must be analyzed. For example, there may be a case where we need evaluate his residual sojourn time after the completion of his first service. In such a case, we want to know the distribution of the system state at this time epoch. It may be thought that this distribution is equal to that of the system state at his arrival epoch for some networks. But, it needs a rigorous proof.

In this paper, in order to contribute to such problems, we consider two Palm measures  $P_a, P_d$  which are strictly stationary for the time sequences  $\{a_n\}_{n=-\infty}^{\infty}, \{d_n\}_{n=-\infty}^{\infty}$  respectively, where  $\{a_n\}$  and  $\{d_n\}$  are the time sequences of arrival and departure times respectively at an arbitrarily fixed station in the given system. And we will give a theorem which asserts that  $P_a(B') = P_d(B'')$  if  $B'$  and  $B''$  are defined as certain time shifts of an arbitrary event  $B$ . Usually, we appreciate  $B$  and  $B''$  as an event which is concerned with the customer

$C'_1$  who departs at time  $d_1$ , and  $B'$  as a corresponding event to  $B''$  concerning the customer  $C_1$  who arrives at time  $a_1$ . For example, if  $B''$  is  $\{W'_1 \leq x\}$  then  $B'$  is  $\{W_1 \leq x\}$ , where  $W_1(W'_1)$  is the waiting time of  $C_1(C'_1)$ . So, this result means that some statistics of a customer who arrives at the system in an equilibrium state can be evaluated by the measure  $P_d$ , and we apply this result to obtain a ramification of the theory of Burke [3] and Reich [11] about tandem queues, and new properties of residual sojourn times in Jackson type networks, etc.

## 2. Fundamental Results

We consider here a general queueing network, where every customer behaves always alone, that is, we exclude a system which has batch arrivals or batch services. Now, let  $\xi(t)$  ( $-\infty < t < \infty$ ) be a stochastic process or a sample path which describes the behaviour of the given system for the time interval  $(-\infty, \infty)$ . We use a same notation for a stochastic process and its sample path (realization), this will make no confusions.  $\xi(t)$  is defined on a probability space  $(\Omega, \mathcal{E}, P)$ , and it is assumed that  $\xi(t)$  has sufficient information, so that each of random variables considered in the followings can be expressed as a measurable function of  $\xi(t)$  for  $\mathcal{E}$ . That is, we consider  $\omega = \xi(t)$  for  $\omega \in \Omega$ , and  $(\Omega, \mathcal{E}, P)$  is the induced space by  $\xi(t)$  from an original probability space. Moreover, we assume that the process  $\xi(t)$  is time continuous strictly stationary and ergodic for  $P$ , in other words,  $P$  satisfies that (cf. Billingsley [2]),

$$P(B) = P(T(s)B) \text{ for any } B \in \mathcal{E} \text{ and any real number } s,$$

$$P(B) = 1 \text{ or } 0 \text{ for any } B \text{ such that } T(s)B = B \text{ for all } s,$$

where a shift operator  $T(s)$  is described as follows

$$(T(s)\xi)(t) = \xi(t+s) \quad (-\infty < t < \infty),$$

$$T(s)B = \{\xi: T(-s)\xi \in B\},$$

( $\xi \in B$  is equivalent to  $T(s)\xi \in T(s)B$ ).

In the probability space  $(\Omega, \mathcal{E}, P)$ , it is assumed that we can observe two time sequences  $\{a_n\}_{n=-\infty}^{\infty}$ ,  $\{d_n\}_{n=-\infty}^{\infty}$  ( $\dots < a_{-1} < a_0 < 0 \leq a_1 < \dots$ ,  $\dots < d_{-1} < d_0 < 0 \leq d_1 < \dots$ ), where  $\{a_n\}(\{d_n\})$  is a sequence of arrival (departure) times of an arbitrarily specified station in the system. We assume

$$(2.1) \quad a_n \rightarrow \infty, d_n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ w.p.1 for } P.$$

Let  $C'_n(C_n)$  be the customer who arrives at  $a_n$  (departs at  $d_n$ ), and  $d_{v(n)}$  be the departure time of  $C_n$  ( $n=0, \pm 1, \pm 2, \dots$ ), that is,

$$C'_{\nu(n)} = C_n, \quad d_{\nu(n)} - a_n \geq 0 \quad (n=0, \pm 1, \pm 2, \dots).$$

We can say that, from assumptions, for any  $n$  there exists  $n'$  such that  $n = \nu(n')$ . Note that  $a_n, d_n, \nu(n)$  ( $n=0, \pm 1, \pm 2, \dots$ ) are random variables.

Remark. On a sample path  $\xi(t)$  ( $-\infty < t < \infty$ ), if a time epoch  $t_0$  is an arrival (departure) time, then, on the sample path  $T(s)\xi$ ,  $t_0 - s$  must be also an arrival (departure) time, that is, if  $\xi$  has  $k$  arrival (departure) points in a time interval  $[a+s, b+s]$ , then  $T(s)\xi$  has  $k$  arrival (departure) points in  $[a, b]$ . This means that, an arrival sequence  $\{a_n\}$  or a departure sequence  $\{d_n\}$  becomes necessarily a stationary point process for  $P$ . Formally, we can define  $\{a_n\}$  and  $\{d_n\}$  respectively by sequences of random variables which satisfy the followings.

$$\begin{aligned} \dots < a_0 < 0 \leq a_1 < \dots, \quad \dots < d_0 < 0 \leq d_1 < \dots, \\ a_n(T(s)\xi) &= a_{m+n}(\xi) - s, \\ d_n(T(s)\xi) &= d_{m'+n}(\xi) - s \quad \text{for all } n, \end{aligned}$$

where  $m = \max\{n: a_n(\xi) < s\}$ ,  $m' = \max\{n: d_n(\xi) < s\}$ . Moreover, there is an one to one correspondence between  $\{a_n\}$  and  $\{d_n\}$  such that  $a_n = d_{\nu(n)}$ ,  $d_{\nu(n)} - a_n \geq 0$ , which is invariant for a shift operator  $T(s)$  in the sense that

$$(2.2) \quad \nu(n)(T(s)\xi) = \nu(m+n)(\xi) - m' \quad \text{for all } n.$$

Let  $N(t)$  ( $N'(t)$ ) be the number of  $a_n$ 's ( $d_n$ 's) in a time interval  $[0, t)$  ( $(0, t]$ ), and  $q(t) = \sum_{n=-\infty}^{\infty} \mathbb{1}\{a_n < t < d_{\nu(n)}\}$ , the number of customers in the specified station at time  $t$ , where  $\mathbb{1}\{\}$  denotes a indicator function. Note that  $q(t)$  is stationary for  $P$  and  $q(t) = q(t-0)$  at  $t = a_n$ ,  $q(t) = q(t+0)$  at  $t = d_n$ . Thus, we can write

$$(2.3) \quad q(t) = q(0) + N(t) - N'(t) \quad \text{for } t > 0.$$

We assume  $E(q(t))$  and  $E(N(t))$  ( $t \neq 0$ ) to be positive finite, then, from the stationarity of  $q(t)$ , there exists a positive constant  $\lambda$ , say intensity, such that

$$(2.4) \quad E(N(1)) = E(N'(1)) = \lambda.$$

Further, we have the following results from the ergodicity of  $P$ ,

$$(2.5) \quad \lim_{t \rightarrow \infty} N(t)/t = \lim_{t \rightarrow \infty} N'(t)/t = \lambda,$$

$$(2.6) \quad \lim_{t \rightarrow \infty} q(t)/t = 0 \quad \text{w.p.1 for } P.$$

(2.6) is obtained easily from (2.3) and (2.5).

For a given system, if we can define a process  $\xi(t)$  which satisfies assumptions expressed above, we say  $\xi(t)$  is an elementary process of the given system in the sequel.

Now, let us consider two Palm measures,  $P_\alpha$  and  $P_d$ , defined with respect to the sequences  $\{\alpha_n\}$  and  $\{d_n\}$  respectively.  $P_\alpha(P_d)$  is a probability measure of the space  $(\Omega, \Xi)$ , which may be thought as a conditional distribution of  $P$  under the condition  $\{\alpha_1=0\}(\{d_1=0\})$ .  $P_\alpha$  and  $P_d$  are uniquely determined from the stationarity of  $P$  under the condition (2.4) (c.f. for example Miyazawa [9]). A formal definition of Palm measure is also given in [9].  $P_\alpha$  and  $P_d$  are strictly stationary for the sequences  $\{\alpha_n\}$  and  $\{d_n\}$  respectively. That is,

$$(2.7) \quad P_\alpha(B) = P_\alpha(B_{(n)}), \quad P_d(B) = P_d(B_{(n)}) \text{ for all integer } n,$$

where  $B_{(n)}$  is defined as  $\{\xi: T(\alpha_n)\xi \in B\}$  and  $B_n$  as  $\{\xi: T(d_n)\xi \in B\}$  for any  $B \in \Xi$ .  $T(\alpha_n)\xi$  should be written precisely as  $T(\alpha_n(\xi))\xi$ , etc., however, we use simple notations for convenience. Note that, we have

$$(2.8) \quad \begin{aligned} P_\alpha(B_{(1)}) &= P_\alpha(B, \alpha_1=0) = P_\alpha(B), \quad P_\alpha(\alpha_1=0)=1, \\ P_d(B_{(1)}) &= P_d(B, d_1=0) = P_d(B), \quad P_d(d_1=0)=1. \end{aligned}$$

If  $B(B')$  is an event in connection with  $C_1$  ( $C'_1$ ), such that the waiting time of  $C_1$  ( $C'_1$ ) is less than a certain value, then  $B_{(n)}$  ( $B'_n$ ) is a corresponding event for  $C_n$  ( $C'_n$ ). Therefore, (2.7) means that  $P_\alpha$  evaluates statistics of a customer who arrives at the specified station in equilibrium, and  $P_d$  evaluates of a customer who departs in equilibrium. By the way, in this case, we have  $B=B_{(1)}=B'_{(1)}$  since  $C_1$  in  $\xi$  and  $T(\alpha_1)\xi$  and  $C'_1$  in  $T(d_{(1)})\xi$  are identical. Of course,  $B_{(i)}$  is defined as  $\{\xi: T(d_{(i)})\xi \in B\}$  ( $i=0, \pm 1, \pm 2, \dots$ ). Similarly, we have  $B'=B'_1$ .

It is known that, if  $P$  is ergodic and stationary,  $P_\alpha$  and  $P_d$  are also determined by appropriate sample averages. That is, we have

Lemma 2.1 (Corollary 2.1 in Miyazawa [10])

$$(2.9) \quad P_\alpha(B) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1_{\{B_{(i)}\}} \quad \text{w.p.1 for } P,$$

$$(2.10) \quad P_d(B) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1_{\{B_{(i)}\}} \quad \text{w.p.1 for } P.$$

Remark. If we denote  $1_{\{T(-s)B\}}$  by  $f(\Phi_*(s))$  in Corollary 2.1 in Miyazawa [10], we have the same expression to it. Clearly, the process  $1_{\{T(-s)B\}} \cdot \xi$  ( $-\infty < s < \infty$ ) is stationary.

Consider a queuing process  $q(t)$  of  $M/M/1$  with traffic intensity  $\rho < 1$ . Then,  $P$  of  $q(t)$  is determined by the initial condition such that

$$P(q(0)=n) = (1-\rho)\rho^n \quad (n=0, 1, 2, \dots).$$

$P_\alpha$  and  $P_d$  satisfy, respectively,

$$\begin{aligned} P_\alpha(\alpha_1=0) &= P_d(d_1=0) = 1, \\ P_\alpha(q(0^-)=n) &= P_d(q(0^+)=n) = (1-\rho)\rho^n \quad (n=0,1,2,\dots), \end{aligned}$$

which are easily proved by Lemma 2.1 or well known queueing theories. Then we have

$$\begin{aligned} P_\alpha(q(\alpha_1-0)=0) &= 1-\rho, \\ P_d(q(\alpha_1-0)=0) &= P(q(\alpha_1-0)=0) \\ &= P(q(0)=0) + \sum_{n=1}^{\infty} P(q(0)=n, q(\alpha_1-0)=0) > 1-\rho. \end{aligned}$$

So we have  $P_\alpha(q(\alpha_1-0)=0) \neq P_d(q(\alpha_1-0)=0)$ , that is,  $P_\alpha$  and  $P_d$  do not coincide with each other.

In the following theorem, we give a relation between  $P_\alpha$  and  $P_d$ , which is the main result of this paper.

**Theorem 2.1** For any event  $B$ , we have

$$P_\alpha(B_{\vee(1)}) = P_d(B_1).$$

**Proof:** From (2.8) and (2.10), we have

$$(2.11) \quad P_d(B_1) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{B_i\} \quad \text{w.p.1 for } P.$$

While, if we denote  $B_{\vee(1)}$  by  $B'$ , we obtain  $B'(i) = B_{\vee(i)}$  ( $i=1,2,\dots$ ),

because, noting that  $T(d_{\vee(1)})(T(\alpha_i)\xi) = T(d_{\vee(i)})(T(\alpha_i)\xi)$ ,

$$\begin{aligned} B'(i) &= \{\xi: T(\alpha_i)\xi \in B_{\vee(1)}\} = \{\xi: T(d_{\vee(1)})(T(\alpha_i)\xi) \in B\} \\ &= \{\xi: T(d_{\vee(i)})(T(\alpha_i)\xi) \in B\} = B_{\vee(i)}. \end{aligned}$$

Thus, from (2.9), we have

$$(2.12) \quad P_\alpha(B_{\vee(1)}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{B_{\vee(i)}\} \quad \text{w.p.1 for } P.$$

Further, considering customers in the specified station at time 0 and  $\alpha_n$ , we obtain

$$\left| \sum_{i=1}^n \mathbb{1}\{B_{\vee(i)}\} - \sum_{i=1}^{N'(\alpha_n)} \mathbb{1}\{B_i\} \right| \leq q(0) + q(\alpha_n) + 1.$$

Dividing the both sides by  $n$ , and using (2.1), (2.5), (2.6) and the fact that  $N(\alpha_n)+1 = n$ , we have

$$(2.13) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{B_{\vee(i)}\} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{N'(\alpha_n)} \mathbb{1}\{B_i\} \\ &= \lim_{n \rightarrow \infty} \frac{\alpha_n}{N(\alpha_n)+1} \frac{N'(\alpha_n)}{\alpha_n} \frac{1}{N'(\alpha_n)} \sum_{i=1}^{N'(\alpha_n)} \mathbb{1}\{B_i\} \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1\{B_i\} \quad \text{w.p.1 for P.}$$

By (2.11), (2.12), and (2.13), we get the desired result.

Q.E.D.

In the above, we assumed that the stationary measure  $P$  is given at first, from which  $P_a$  and  $P_d$  are defined. For the case where  $P_a$  or  $P_d$  is given at first, we have the same result, since  $P$  is also determined uniquely by  $P_a$  or  $P_d$ . These kinds of definition problems of queueing processes are discussed in Miyazawa [9].

### 3. An Application to Jackson Type Networks

Here, we consider an open or closed Jackson type network with product form solutions (that is, which satisfies an equilibrium condition) (Jackson [6]), consisting of  $M$  service stations labeled  $1, 2, \dots, M$ . For a closed system, we only consider one with triggered arrivals where the number of customers in the system is always equal to  $N$ , and a departing customer triggers immediately another arrival. Therefore, both in an open system and in a closed system, each arriving customer departs eventually from the system. Further, in order to be able to pursue each customer's behaviour, we assume that there are infinite  $(N+1)$  tickets labeled  $1, 2, \dots, (1, 2, \dots, N+1)$  in a box for an open (closed) case. Each customer receives a ticket immediately before his arrival and returns it to the box immediately after his departure. Tickets in the box are always ordered according to their numbers, and an arriving customer receives the ticket which has the minimum number in the box. Then, every customer in the system has a differently labeled ticket one another. Referring tickets, we can distinguish customers with a same history, and a departing customer and the triggered customer by him in a closed case.

The state of the system is represented by a vector  $(x_1, x_2, \dots, x_n)$  where  $n$  represents the number of customers in the system (which may vary time to time in an open case), and  $x_m$  represents the history of the  $m$ -th customer which is expressed as follows ( $1 \leq m \leq n$ );

$$x_m = (r_m, i_m, s_m, z_{m1}, \dots, z_{mM}),$$

which means that  $m$ -th customer has the ticket numbered  $r_m$  ( $r_1 < r_2 < \dots < r_n$ ) occupying the position  $s_m$  in the queue of the  $i_m$ -th station, and he has completed the  $j$ -th station service just  $z_{mj}$  times ( $j=1, 2, \dots, M$ ). If  $n=0$ , then we say the system is null. Let  $S$  be the set of possible states.  $S$  is clearly discrete. Let  $\xi(t)$  ( $-\infty < t < \infty$ ) be a stochastic process with the state space  $S$ ,

which describes the behaviour of the system for the time interval  $(-\infty, \infty)$ . Clearly,  $\xi(t)$  is a time-homogeneous irreducible Markov process. Then, we can deduce that  $\xi(t)$  has a stationary probability measure  $P$  over the space  $(\Omega, \Xi)$ , where  $\Omega$  is the set of all  $\xi(t)$ ,  $\Xi$  is the  $\sigma$ -field generated by cylinder sets of  $\xi(t)$ . This fact can be proved by the following arguments.

Let  $\{a_n\}, \{a_{in}\}, \{d_{in}\}$  ( $\dots < a_0 < 0 \leq a_1 < \dots$ , and so forth) be the sequences of arrival times from the external, arrival and departure times at the  $i$ -th station respectively ( $i=1, 2, \dots, M$ ). Obviously, these sequences are defined from  $\xi(t)$  ( $-\infty < t < \infty$ ). We allow that  $p_{ii} > 0$ , where  $p_{ii}$  is the routing probability that a customer proceeds to the  $i$ -th station immediately after his service completion of the  $i$ -th station. For each pair of the sequences  $\{a_{in}\}$  and  $\{d_{in}\}$ , we can define an one to one correspondence such that  $a_{in} \rightarrow d_{i, \nu i(n)}$ , where  $d_{i, \nu i(n)}$  is the service completion time of the customer at the  $i$ -th station who has arrived there at time  $a_{in}$ . Let  $q_i(t)$  define

$$(3.1) \quad q_i(t) = \sum_{n=-\infty}^{\infty} I\{a_{in} < t < d_{i, \nu i(n)}\}, \quad (i=1, 2, \dots, M),$$

then  $q_i(t)$  represents the queue length at the  $i$ -th station at time  $t$ . From the assumption that the system satisfies an equilibrium condition, the queuing process  $Q(t) = (q_1(t), q_2(t), \dots, q_M(t))$  has an equilibrium probability distribution with product form solutions. So there exists elements  $s, s'$  in the state space  $S$ , such that

$$\lim_{t \rightarrow \infty} \Pr\{\xi(t)=s \mid \xi(0)=s'\} > 0,$$

where  $\Pr\{\xi(t)=s \mid \xi(0)=s'\}$  is a transition probability of  $\xi(t)$ . Thus,  $\xi(t)$  has a positive recurrent state, that is  $\xi(t)$  has a stationary probability measure  $P$  (cf. Chung [4]).

From the above, we can say that  $\{a_n\}, \{a_{in}\}, \{d_{in}\}$  are stationary point processes each of which has a finite intensity, for it is less than the summation of service rates and the arrival rate from the external if any. Thus  $\xi(t)$  becomes an elementary process of the given system.

Note that  $Q(t)$  is not continuous at  $t=a_n, a_{in}, d_{in}$  ( $n=0, \pm 1, \pm 2, \dots$ ). From (3.1), a causing customer of each discontinuous point does not counted in  $Q(t)$  at its discontinuous point. That is, for example, if a customer proceeds from the  $j$ -th station to the  $i$ -th at  $t=d_{jn}=a_{in}$  and  $q(t) = (q_1, \dots, q_i, \dots, q_j, \dots, q_M)$ , then we have

$$Q(t-0) = (q_1, \dots, q_i, \dots, q_j+1, \dots, q_M),$$

$$Q(t+0) = (q_1, \dots, q_i+1, \dots, q_j, \dots, q_M).$$

So, in a closed case, we have  $\sum_{i=1}^M q_i(t) = N-1$  at  $t=a_n$ , etc.

Let  $P_a, P_{ia}, P_{id}$  be Palm measures defined from  $\{a_n\}, \{a_{in}\}, \{d_{in}\}$  respectively. Then, from the product form solutions of  $Q(t)$ , we have

$$(3.2) \quad P_a(Q(a_n)=Q) = P_{ia}(Q(a_{in})=Q) = P_{id}(Q(d_{in})=Q) \\ = f(Q) = f(q_1, q_2, \dots, q_M) \quad (n=0, \pm 1, \pm 2, \dots),$$

where  $f(Q)$  is an product form, which does not depend on  $i$ .  $f(Q)$  coincides with the equilibrium distribution of  $Q(t)$  in an open case, and with that of a closed system with  $N-1$  customers in a closed case (cf. Kawashima [7]).

Now, let  $B$  in Theorem 2.1 be  $\{Q(d_{i1})=Q\}$ , considering the correspondence between  $\{a_{in}\}$  and  $\{d_{in}\}$ , we can write  $B_1$  and  $B_{\vee(1)}$  in Theorem 2.1 as  $B$  and  $\{Q(d_{i, \vee i(1)})=Q\}$  respectively. Thus, we have

$$(3.3) \quad P_{ia}(Q(d_{i, \vee i(1)})=Q) = P_{id}(Q(d_{i1})=Q) = f(Q).$$

Let  $p_i(Q|Q')$  be  $P(Q(d_{i, \vee i(n)})=Q|Q(a_{in})=Q')$ , which does not depend on  $n$  and underlying measure  $P$ , since  $Q(t)$  is also a time-homogeneous Markov process. Then, from (3.3), we have

Lemma 3.1

$$f(Q) = \sum_Q f(Q') p_i(Q|Q') \quad \text{for } i=1, 2, \dots, M,$$

where  $\sum_Q$  means the summation for possible states.

Proof:

$$P_{ia}(Q(d_{i, \vee i(1)})=Q) = \sum_Q P_{ia}(Q(a_{i1})=Q', Q(d_{i, \vee i(1)})=Q) \\ = \sum_Q P_{ia}(Q(a_{i1})=Q') p_i(Q|Q')$$

This means the desired result.

Q.E.D.

From this, we have the following theorem.

Theorem 3.1 Let  $I_n$  ( $n=1, 2, \dots$ ) be the station which  $C_1$  (the customer who arrives at the system at time  $a_1$ ) visits at the  $n$ -th time along his route, and  $t_n$  be the time epoch at which  $C_1$  arrives at  $I_n$ , then we have

$$P_a(Q(t_n)=Q|I_1=j, I_2=k, \dots, I_n=i) = f(Q).$$

Proof: At time  $a_1=t_1$ ,  $C_1$  joins the  $j$ -th queue with a fixed probability independently of any others, so we have from (3.2)

$$P_a(Q(t_1)=Q|I_1=j) = P_a(Q(t_1)=Q) = f(Q).$$

Similarly,  $I_2$  and  $Q(t_2)$  are conditional independent each other under the condition  $\{I_1=j\}$ , so, using Lemma 3.1, we have

$$P_a(Q(t_2)=Q|I_1=j, I_2=k) = P_a(Q(t_2)=Q|I_1=j)$$



$$= \sum_Q, P_\alpha(Q(t_1)=Q' | I_1=j) p_j(Q|Q') = f(Q).$$

Repeating this, we have the desired result.

Q.E.D.

Theorem 3.1 means also that, a customer, who arrives at the system in equilibrium, finds the system state to be  $Q$  with probability  $f(Q)$ , whenever he completes a service. Now, let us consider  $C_1$ 's waiting time at  $I_n (=t_{n+1}-t_n)$ , and the residual sojourn time ( $=d-t_n$ :  $d$  is the time spock at which  $C_1$  departs from the system). Then, we have

Corollary 3.1

$$E_\alpha(t_{n+1}-t_n | I_1=j, I_2=k, \dots, I_n=i) = E_{i\alpha}(w_{i1}),$$

$$E_\alpha(d-t_n | I_1=j, I_2=k, \dots, I_n=i) = E_{i\alpha}(W_{i1}'),$$

where  $E_\alpha$  ( $E_{i\alpha}$ ) denotes the expectation for  $P_\alpha$  ( $P_{i\alpha}$ ),  $w_{i1}$  ( $W_{i1}$ ) is the waiting time (residual sojourn time) of the customer who arrives the  $i$ -th station at  $t=\alpha_{i1}$ .

Proof: The conditional distribution of  $t_{n+1}-t_n$  under  $Q(t_n)=Q$  is determined independently of the past because of Jackson type assumptions. Noting that  $P_{i\alpha}(\alpha_{i1}=0) = 1$ , we have

$$\begin{aligned} E_\alpha(t_{n+1}-t_n | I_1=j, I_2=k, \dots, I_n=i) \\ &= \sum_Q E_\alpha(t_{n+1}-t_n | I_n=i, Q(t_n)=Q) P_\alpha(Q(t_n)=Q | I_1=j, \dots, I_n=i) \\ &= \sum_Q E_{i\alpha}(w_{i1} | Q(0)=Q) P_{i\alpha}(Q(0)=Q) = E_{i\alpha}(w_{i1}). \end{aligned}$$

The second equation is deduced similarly.

Q.E.D.

Remark: We can calculate  $E_{i\alpha}(w_{i1})$  ( $i=1, 2, \dots, M$ ) from the product form solutions, and  $E_{i\alpha}(W_{i1}')$ 's are given by the turnaround time equations in Kawashima [7], however in [7], Lemma 3.1 given above has been assumed tacitly.

In the above discussion, we have not assumed service discipline at each station precisely. Indeed, any type, such as FCFS, LCFS, Processor sharing, etc., goes in as much as  $Q(t)$  has product form solutions (cf. Baskett et al. [1]). Further, applying Theorem 2.1 to a general network introduced by Baskett et al. [1], we may be able to obtain similar results for this network, but we will not go into details.

#### 4. Ramifications of Theorems by Burke and Reich

We consider a tandem queue  $M/M/s_1 \rightarrow M/s_2$  with FCFS discipline, which is a special one of Jackson network, so we use same notations as the previous

section. Note that, we have  $a_n = a_{1n}$ ,  $d_{1n} = a_{2n}$  ( $n=0, \pm 1, \pm 2, \dots$ ). Let  $w_{i1}(w'_{i1})$  be the waiting time at the  $i$ -th station ( $i=1, 2$ ) of  $C_1(C'_1)$  who arrives at (departs from) at time  $a_{11}(d_{11})$ .

Theorem 4.1 (Burke [3])  $w'_{11}$  and  $w'_{21}$  are independent each other for  $P_{1d}$ , that is,

$$P_{1d}(w'_{11} \leq x_1, w'_{21} \leq x_2) = P_{1d}(w'_{11} \leq x_1) P_{1d}(w'_{21} \leq x_2) \text{ for all } x_1, x_2.$$

Now, let  $B$  in Theorem 2.1 be  $\{w'_{11} \leq x_1, w'_{21} \leq x_2\}$ , then, considering the correspondence between  $\{a_{1n}\}$  and  $\{d_{1n}\}$ , we can write as  $B_1=B$  and  $B_{\vee(1)} = \{w_{11} \leq x_1, w_{21} \leq x_2\}$  in Theorem 2.1. Thus, we have

$$(4.1) \quad P_a(w_{11} \leq x_1, w_{21} \leq x_2) = P_{1d}(w'_{11} \leq x_1, w'_{21} \leq x_2) \text{ for all } x_1, x_2.$$

Similarly, we have

$$(4.2) \quad P_a(w_{11} \leq x_1) = P_{1d}(w'_{11} \leq x_1), \quad P_a(w_{21} \leq x_2) = P_{1d}(w'_{21} \leq x_2).$$

Consequently, imposing (4.1) and (4.2) to Theorem 4.1, we have the following corollary.

Corollary 4.1  $w_{11}$  and  $w_{21}$  are also independent for  $P_a$ , that is,

$$P_a(w_{11} \leq x_1, w_{21} \leq x_2) = P_a(w_{11} \leq x_1) P_a(w_{21} \leq x_2).$$

When we estimate the behaviour of a customer, we usually use the measure  $P_a$ , so Corollary 4.1 may be practically important. In Burke [3], it is shown that, using our notations,

$$P_a(w_{11} \leq x | q_1(a_1) = k) = P_{1d}(w'_{11} \leq x | q_1(d_{11}) = k).$$

From this and (3.1), we can easily obtain the first equation of (4.2), however, he mentioned scarcely the difference between  $P_a$  and  $P_{1d}$ .

Clearly,  $P_a(w_{11} \leq x)$  and  $P_{1d}(w'_{21} \leq x)$  coincide with the so called stationary distribution of the waiting time of  $M/M/s$ , and so the distribution of the sojourn time,  $w_{11} + w_{21}$ , for  $P_a$  can be expressed as these convolution.

Further, for a tandem queuing system  $M/M/1 \rightarrow M/1 \rightarrow \dots \rightarrow M/1$  consisting of  $M$  stations, Reich showed the next.

Theorem 4.2 (Reich [11])

$$P_{Ma}(w'_{11} \leq x_1, \dots, w'_{M1} \leq x_M) = \prod_{i=1}^M P_{Ma}(w'_{i1} \leq x_i),$$

where  $w'_{i1}$  is the waiting time at the  $i$ -th station of the customer who arrives at the  $M$ -th station at time  $a_{M1} = d_{M-1,1}$ .

Then, similarly to Corollary 4.1, we have

## Corollary 4.2

$$P_{\alpha}(w_{11} \leq x_1, \dots, w_{M1} \leq x_M) = \prod_{i=1}^M P_{\alpha}(w_{i1} \leq x_i),$$

where  $w_{i1}$  is the waiting time at the  $i$ -th station of the customer who arrives at the system at time  $a_1$ .

Remark: In this system, we have an one to one correspondence between  $\{a_n\}$  and  $\{a_{Mn}\}$  which satisfies (2.2).

For an  $M/M/1$  system, Lemoine [8] showed that, the output flow of customers over the time interval  $(a_n, a_n + w_{1n})$  under  $P_{\alpha}$  is also a Poisson process with same rate to the input process. And, using this result, he proved Corollary 4.2 for a more general system, a tree type network consisting of single server stations.

## 5. Another Application

Consider a  $G/G/s$  with a stationary input with a traffic intensity less than 1. For this system, an elementary process can be defined (cf. Miyazawa [10]). It is known (cf. Franken [5], for example) that

$$(5.1) \quad P_{\alpha}(q(a_1)=k) = P_d(q(d_1)=k) \quad \text{for } k=0,1,\dots$$

While, from Theorem 2.1, we have

$$(5.2) \quad P_{\alpha}(q(d_{\vee(1)})=k) = P_d(q(d_1)=k) \quad \text{for } k=0,1,\dots$$

Accordingly,

$$(5.3) \quad P_{\alpha}(q(d_{\vee(1)})=k) = P_{\alpha}(q(a_1)=k) \quad \text{for } k=0,1,\dots$$

(5.3) means that, for an arbitrarily chosen customer from arrival time epochs, the queue length distribution at his arrival epoch and that at his departure epoch are identical, while (5.1) means that, in equilibrium, the queue length distribution at an arbitrary arrival epoch is identical to that at an arbitrary departure epoch. Further, two terms in (5.2) are defined as limits, if they exists, as follows.

$$\lim_{n \rightarrow \infty} P(q(d_{\vee(n)})=k), \quad \lim_{n \rightarrow \infty} P(q(d_n)=k)$$

It is not necessarily trivial that these are identical. The existence of these limits are discussed in Miyazawa [9].

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