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AN OPTIMAL INSPECTION AND MAINTENANCE POLICY OF A DETERIORATING SYSTEM

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Abstract This paper treats an optimal inspection and maintenance policy of a continuous time Markovian deteriorating system, which minimizes the total discounted expected time in which the system is not operating. The problem is formulated by a semi-Markov decision process and some properties of an optimal policy are obtained, that is, a control limit rule holds and as the system degrades, an optimal inspection time interval becomes shorter. For a discrete time Markovian deterioration case, almost the same discussion is made.

1. Introduction

This paper is concerned with a deteriorating system in which the level of deterioration is quantized in many discrete states, 0,1,...,s,s+1 in the order of increasing deterioration, where state 0 is a good state, i.e., the system is like new, states 1,...,s are deteriorating states and state s+1 is a failed state. In a normal operation, these states are assumed to constitute a continuous time Markov process with an absorbing state s+1 and as the deterioration increases, the system is more likely to fail. For such a system, Flehinger [3] has derived some operating characteristics of the policy called a Control Limit Rule (CLR) when the system is inspected at a constant time interval. CLR is of a simple form; Maintenance is performed if and only if the system is in any states k,k+1,...,s,s+1. The set of the states k,...,s+1 is called a marginal set. Barlow and Proschan [1] considered an inspection policy which minimizes the expected cost of one period (the time interval between two successive replacements) when the marginal set is specified. Mine and Kawai [5] discussed an inspection and replacement problem of minimizing the expected cost per unit time in an infinite time span without predetermining either inspection time interval or a marginal set, and obtained some properties of an optimal policy.

Another important characteristic of the system is the system availability or unavailability. In this paper, we consider an optimal inspection and maintenance problem of minimizing the expected total discounted time in which the system is not operating (hereafter we shall use the word "discounted system unavailability). The problem is formulated by a semi-Markov decision process and some properties of an optimal policy are presented, which are also very useful to determine the optimal policy. Furthermore, we refer to a discrete time Markovian deteriorating system. Rosenfield [6] considered an inspection and replacement problem and showed the optimality of a monotonic four-region policy. He formulated the problem by a Markov decision process. But his procedure to derive an optimal policy is not practical since the state space in his decision process is infinite. For the discrete time case, we make a similar discussion as the continuous time case by formulating the problem by a semi-Markov decision process with a finite state space.

2. System Description

The continuous time Markovian deteriorating system is assumed to have the following properties.

i) From state i, a random transition is only possible to state $i+1$ or s+1. For example, consider a parallel system consisting of s components each failure time of which has an exponential distribution and the system subjects to random failure at any state.

ii) The transition rate from state i to s+1 increases as i increases.

iii) The failure of the system can be detected at any time without inspection and then it is immediately maintained: Corrective Maintenance (CM). CM time has a distribution function R(t).

iv) It is impossible to determine the state of the system (except s+1) without inspection. Inspection time is nonnegligible and has a distribution function Q(t). During inspection, the system is not operating.

v) When the system is observed by inspection to be in state i ($=0,\ldots,s$), only one of the following actions can be made:

- PM — the system is preventively maintained: Preventive Maintenance (PM). PM time has a distribution function M(t).
- I(t) — No PM; the system is inspected t hours after.
- I(∞) — No PM; the system is operated without inspection until it fails. This action is a special case of I(t) with an infinite inspection time interval.

vi) Immediately after PM or CM completion, we can take one of the above actions.

We let E_i denote the time instant at which inspection or maintenance has just been completed and the system is in state i, i=0,1,...,s. Our problem is to find an optimal action at each E_i to minimize the discounted system unavailability. All of E_i are easily seen to be regeneration points, which implies that our problem can be solved by a semi-Markov decision process. A sample path of the system behavior is given in Fig.1.

3. Transition Probability

In this section, we treat the transition probability of the states with neither inspection nor maintenance, which describes the behavior of the system between two successive inspection and/or maintenance. These probabilities play an important role in our discussion. We first introduce the following notations and definitions.

S	$\{ i \mid i=0,1,\ldots,s \}$.
S+1	$\{ i \mid i=0,1,\ldots,s+1 \}$.
α_i	transition rate from state i to state s+1.
(3.1)	$\alpha_i < \alpha_j$ for $i < j$.
β_i	transition rate from state i to state i+1, $\beta_s=0$.
λ_i	$\alpha_i+\beta_i$
$P_{ij}(t)$	Pr{ the system is in state j at time t | it was in state i at time 0 }.

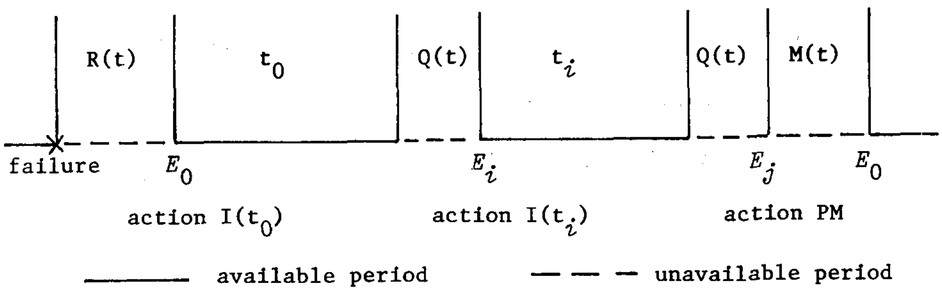

Fig. 1 System behavior

$p_{ij}(t)$ $\mathrm{d}P_{ij}(t)/\mathrm{d}t$

$\bar{F}_i(t)$ $1 - F_i(t)$, that is

$$\bar{F}_i(t) = \sum_{j=i}^{s} P_{ij}(t). \tag{3.2}$$

$f_i(t)$ $\mathrm{d}F_i(t)/\mathrm{d}t = \sum_{j=i}^{s} P_{ij}(t)\alpha_j$

$P_{ij}(t)$ are given by the following equations.

$$p_{ij}(t) = -\lambda_i P_{ij}(t) + \beta_i P_{i+1,j}(t) \tag{3.3}$$

$$= -\lambda_j P_{ij}(t) + \beta_{j-1} P_{i,j-1}(t), \tag{3.3'}$$

$$f_i(t) = \lambda_i \bar{F}_i(t) - \beta_i \bar{F}_{i+1}(t), \quad i,j \in S. \tag{3.4}$$

Explicit expressions of $P_{ij}(t)$ are given by [5].
Concerning $P_{ij}(t)$, we have the following lemmas.

Lemma 3.1. $P_{ij}(t)$ are totally positive of order 2 (TP_2) in $i,j \in S$, that is,

$$\begin{vmatrix} P_{im}(t) & P_{in}(t) \\ P_{jm}(t) & P_{jn}(t) \end{vmatrix} \geqq 0, \qquad \text{for } i < j,\ m < n. \tag{3.5}$$

Proof. We let $A_{ij}(t)$ denote the left hand side of inequality (3.5) and let $a_{ij}(t)$ denote $\mathrm{d}A_{ij}(t)/\mathrm{d}t$. It is easily seen that

$$A_{ij}(t) \geqq 0 \text{ for } j > m, \qquad A_{ii}(t) = 0, \quad A_{ij}(0) \geqq 0. \tag{3.6}$$

From equation (3.3), we have

$$a_{ij}(t) = -(\lambda_i+\lambda_j)A_{ij}(t) + \beta_i A_{i+1,j}(t) + \beta_j A_{i,j+1}(t). \tag{3.7}$$

Solving the above equation, we have

$$A_{ij}(t) = \mathrm{e}^{-(\lambda_i+\lambda_j)t}\Big[\int_0^t \mathrm{e}^{(\lambda_i+\lambda_j)x}\{\beta_i A_{i+1,j}(x) + \beta_j A_{i,j+1}(x)\}\mathrm{d}x + A_{ij}(0)\Big]. \tag{3.8}$$

By equations (3.6) and (3.8), we can show that $A_{ij}(t) \geqq 0$ for $i < j$, $m < n$, through mathematical induction method in i,j for arbitrary fixed m,n.

Lemma 3.2. $P_{ij}(t)$ are TP_2 in j,t ($j \in S$), that is

$$(3.9)\qquad \begin{vmatrix} P_{ij}(u) & P_{ij}(v) \\ P_{ik}(u) & P_{ik}(v) \end{vmatrix} \geqq 0 \qquad \text{for } j < k,\ u < v.$$

Proof. It is sufficient to show that $P_{ik}(t)/P_{ij}(t)$ is nondecreasing in t for $j < k$. Differentiating $P_{ik}(t)/P_{ij}(t)$ by t, we have

$$(3.10)\qquad \mathrm{d}\{P_{ik}(t)/P_{ij}(t)\}/\mathrm{d}t = \beta_i\{P_{i+1,k}(t)P_{ij}(t) - P_{i+1,j}(t)P_{ik}(t)\}/\{P_{ij}(t)\}^2$$

$$\geqq 0 \quad \text{from lemma 3.1.}$$

Lemma 3.3.

$$(3.11)\qquad \bar{F}_i(t) \geqq \bar{F}_j(t) \qquad \text{for } i < j.$$

Proof. We let $z_i(t)$ denote the failure rate function of $F_i(t)$, that is

$$(3.12)\qquad z_i(t) = f_i(t)/\bar{F}_i(t), \quad \bar{F}_i(t) = \exp\{-\int_0^t z_i(x)\mathrm{d}x\}.$$

$$(3.13)\qquad f_i(t) - \mathrm{c}\bar{F}_i(t) = \sum_{j=i}^{s} P_{ij}(t)(\alpha_j - \mathrm{c}),$$

where c is a positive real number. Since α_i is increasing in i, α_i-c changes its sign at most once and if a change occurs, it is from negative for any c. By a variation diminishing property of TP_2 [2, p.93], $f_i(t)-\mathrm{c}\bar{F}_i(t)$ change its sign at most once in i and t, and if a change occurs, it is from negative for any c. This implies that $z_i(t)$ is nondecreasing in i and t. Hence, from equation (3.12), equation (3.11) holds.

Lemma 3.4.

$$(3.14)\qquad \sum_{j=k}^{s+1} P_{ij}(t) \text{ is nondecreasing in } i \text{ for any } k.$$

Proof. We let B_{ik} denote $\sum_{j=k}^{s+1} P_{ij}(t)$. We show that $B_{i+1,k} \geqq B_{ik}$

From lemma 3.1, we have

$$(3.15)\qquad P_{i+1,j}(t)(1-B_{ik}) \geqq P_{ij}(t)(1-B_{i+1,k}), \ j \geqq k$$

From the above equation, we have

$$(3.16)\qquad (B_{i+1,k} - F_{i+1}(t))(1-B_{ik}) \geqq (B_{ik}-F_i(t))(1-B_{i+1,k}).$$

From lemma 3.3 and equation (3.16), we have

$$(3.17)\qquad B_{i+1,k} - B_{ik} \geqq F_{i+1}(t)(1-B_{ik}) - F_i(t)(1-B_{i+1,k}) \geqq F_i(t)(B_{i+1,k}-B_{ik}),$$

which implies that $B_{i+1,k} \geqq B_{ik}$.

The following lemma is equivalent to lemma 3.4.

Lemma 3.5. For any nondecreasing function g_i,

(3.18) $$\sum_{j=0}^{s+1} P_{ij}(t) g_j \text{ is nondecreasing in } i.$$

For proof, see Derman [3].

4. Semi-Markov Decision Process Formulation

In this section, we formulate our problem by a semi-Markov decision process and derive an optimal policy. For the purpose, the following notations are introduced.

- v_i the optimal discounted system unavailability when the system starts with E_i, $i \in S+1$.
- α discount factor.
- Q,R,M the discounted expected inspection, CM and PM time, respectively, e.g., $Q = \int_0^\infty e^{-\alpha t}\bar{Q}(t)dt$.
- q,r,m $1-\alpha Q$ ($= \int_0^\infty e^{-\alpha t} dQ(t)$), $1-\alpha R$ and $1-\alpha M$, respectively.
- μ_i $\alpha+\lambda_i$

By an elementary probabilistic consideration, we can express v_i as

(4.1) $$v_i = \min\{ \min_{0 \leqq t \leqq \infty} H_i(t),\ M + mv_0 \}, \quad i \in S$$

where

(4.2) $$H_i(t) = v_{s+1}\int_0^t e^{-\alpha x} f_i(x)dx + e^{-\alpha t}\sum_{j=i}^{s} P_{ij}(t)(Q+qv_j),$$

(4.3) $$v_{s+1} = R + rv_0.$$

$H_i(t)$ corresponds to action I(t) and $M+mv_0$ does to action PM. Equations (4.1)-(4.3) can be solved by well-known sequential approximation method. That is, an optimal policy and the optimal v_i are obtained by the following procedure.

Step 1.

$$v_i^0 = M+mv_0^0,\ i \in S, \quad v_{s+1}^0 = R+rv_0^0.$$

This implies that $v_i^0 = 1/\alpha$, $i \in S+1$.

$n = 0$ and go to step 2.

Step 2.

$$(4.4)\quad \begin{aligned} v_i^{n+1} &= \min\{ \min_{0 \leqq t \leqq \infty} H_i^{n+1}(t),\ M+mv_0^n \} \quad i \in S, \\ v_{s+1}^{n+1} &= R+rv_0^n, \end{aligned}$$

$$(4.5)\quad H_i^{n+1}(t) = v_{s+1}^n \int_0^t e^{-\alpha x} f_i(x)\,dx + e^{-\alpha t} \sum_{j=i}^{s} P_{ij}(t)(Q+qv_j^n).$$

Go to step 3.

Step 3. If $v_i^n - v_i^{n+1} \leqq \varepsilon$ for all $i \in S+1$, then stop. Otherwise, $n=n+1$ and go to step 2.

Optimal v_i and an optimal policy can be determined to the degree of accurancy required by choosing suitable ε.

In the following, we show that this procedure gives an optimal policy. For the purpose, the following notations are introduced.

D_i^n	the action which gives v_i^n,
$v^n - v^{n+1}$	$\max_{i \in S+1}\{ v_i^n - v_i^{n+1} \}$,
β	$\max\{ q, r, m, \lambda_s/\mu_s \} < 1$,
$A_i(t)$	$e^{-\alpha t} q\bar{F}_i(t) + \int_0^t e^{-\alpha x} f_i(x)\,dx$.

Lemma 4.1.

$$(4.6)\quad A_i(t) \leqq \max\{ q, \lambda_s/\mu_s \}.$$

Proof.

$$dA_i(t)/dt = \alpha Q e^{-\alpha t} \bar{F}_i(t) \{ z_i(t) - q/Q \}.$$

Since $z_i(t)$ is nondecreasing in t (see the proof of lemma 3.3), $dA_i(t)/dt$ changes its sign at most once and if a change occurs, it is from negative. This implies $A_i(t)$ is unimodular in t. Hence

$$(4.7)\quad A_i(t) \leqq \max\{ A_i(0), A_i(\infty) \}.$$

$$(4.8)\quad A_i(0) = q,\ A_i(\infty) = 1-\alpha\int_0^\infty e^{-\alpha x}\bar{F}_i(x)\,dx \leqq 1-\alpha\int_0^\infty e^{-\alpha x}\bar{F}_s(x)\,dx = \lambda_s/\mu_s,$$

from lemma 3.3. This completes the proof.

Lemma 4.2. v_i^n is nonincresing in n.

Proof is easily done and is omitted.

Lemma 4.3.

(4.9) $v^n - v^{n+1} \leqq \beta(v^{n-1} - v^n)$.

Proof.

(4.10) $v^n_{s+1} - v^{n+1}_{s+1} = r(v^{n-1}_0 - v^n_0) \leqq \beta(v^{n-1} - v^n)$.

If $D^{n+1}_i =$ PM, $i \in S$, then

(4.11) $v^n_i - v^{n+1}_i \leqq m(v^{n-1}_0 - v^n_0) \leqq \beta(v^{n-1} - v^n)$.

If $D^{n+1}_i = I(t)$, $i \in S$, then

(4.12) $$v^n_i - v^{n+1}_i \leqq H^n_i(t) - H^{n+1}_i(t) = (v^{n-1}_{s+1} - v^n_{s+1})\int_0^t e^{-\alpha x} f_i(x)dx + e^{-\alpha t} q \sum_{j=i}^{s} P_{ij}(t)(v^{n-1}_j - v^n_j) \leqq (v^{n-1} - v^n)A_i(t) \leqq \beta(v^{n-1} - v^n)$$

from lemma 4.1.

Equations (4.10)-(4.12) imply that equation (4.9) holds.

From lemma 4.2 and lemma 4.3, it holds that

$$v^n_i \to v_i \quad \text{as } n \to \infty \quad \text{for all } i \in S+1.$$

This implies that our procedure gives an optimal policy and the optimal v_i.

5. Some Properties of An Optimal Policy

In this section, we give some properties of an optimal policy. For PM to have meaning, we assume that

(5.1) $R \geqq Q + qM$,

where it is noted that PM can be made only after inspection.

Theorem 5.1.
If $D^n_i =$ PM, then $D^n_j =$ PM for $i < j \in S$.

This theorem implies that our procedure gives an optimal policy which is a CLR and that there exists a CLR which is optimal.

Proof. Rewriting equation (4.5), we have

(5.2) $$H^{n+1}_i(t) = v^n_{s+1}\alpha\int_0^t e^{-\alpha x} F_i(x)dx + e^{-\alpha t}\{ \sum_{j=i}^{s} P_{ij}(t)(Q+qv^n_j)+F_i(t)v^n_{s+1} \}$$

(5.3) $$v^{n+1}_{s+1} - (Q+qv^{n+1}_i) \geqq R+rv^n_0 - \{ Q+q(M+mv^n_0) \} = (R-Q-qM)(1-\alpha v^n_0) \geqq 0,$$

from equation (5.1) and that $v^n_0 \leqq v^0_0 = 1/\alpha$.

If we assume that v_i^n is nondecreasing in i, then we can easily see that $H_i^{n+1}(t)$ is nondecreasing in i for all t from lemma 3.3, lemma 3.4 and equation (5.2). Since $v_i^0 = 1/\alpha$, $i \in S+1$, we can show that $H_i^n(t)$ is nondecreasing i for all t. through mathematical induction method. This implies that theorem 5.1 holds.

From here, we discuss the behavior of $H_i^{n+1}(t)$ with respect to t. We let $h_i^{n+1}(t)$ denote $dH_i^{n+1}(t)/dt$. Using equation (3.3'), we have

$$h_i^{n+1}(t) = e^{-\alpha t} \sum_{j=i}^{s} P_{ij}(t)(q\xi_j^n - \zeta_j^n), \tag{5.4}$$

where

$$\xi_i^n = -\mu_i v_i^n + \beta_i v_{i+1}^n + \alpha_i v_{s+1}^n, \tag{5.5}$$

$$\zeta_i^n = Q\{ \alpha + \alpha_i(1-\alpha v_{s+1}^n) \} > 0. \tag{5.6}$$

Furthermore, we have by using equation (3.3),

$$h_i^{n+1}(t) = -\mu_i H_i^{n+1}(t) + \beta_i H_{i+1}^{n+1}(t) + \alpha_i v_{s+1}^n. \tag{5.7}$$

Lemma 5.2.

$$v_i^n < 1/\alpha, \qquad i \in S, \qquad n \geqq 1. \tag{5.8}$$

Proof.

$$v_i^n \leqq H_i^n(\infty) = v_{s+1}^n \int_0^\infty e^{-\alpha x} f_i(x)dx \leqq \frac{1}{\alpha}\int_0^\infty e^{-\alpha x} f_i(x)dx < 1/\alpha.$$

Lemma 5.3.

$$D_i^n \neq I(0),\ D_0^n \neq PM,\ n \geqq 1.$$

Proof. If $D_i^n = I(0)$, then

$$v_i^n = H_i^n(0) = Q+qv_i^{n-1} = v_i^{n-1} + Q(1-\alpha v_i^{n-1}) \geqq v_i^{n-1}. \tag{5.9}$$

Equality holds when $v_i^n = v_i^{n-1} = 1/\alpha$. This contradicts to lemma 5.2. In a similar way, it is easily shown that $D_0^n \neq PM$.

Lemma 5.4. If $D_i^n = I(t)$, then $\xi_i^n \leqq 0$.

Proof. If $D_i^n = I(t)$, then $h_i^n(t) = 0$ and $H_i^n(t) = v_i^n$. From equation (5.7), we have

$$\mu_i v_i^n = \mu_i H_i^n(t) = \beta_i H_{i+1}^n(t) + \alpha_i v_{s+1}^{n-1} \geqq \beta_i v_{i+1}^n + \alpha_i v_{s+1}^n. \tag{5.10}$$

Hence, $\xi_i^n \leqq 0$.

Lemma 5.5. If $q\xi_i^n - \zeta_i^n > 0$, then $q\xi_j^n - \zeta_j^n > 0$ for $j > i$.

Proof. Since $\zeta_i^n > 0$, we have $\xi_i^n > 0$, which denotes that $D_i^n = PM$. Hence,

D_j^n = PM for all $j > i$ from theorem 5.1. From equations (5.5) and (5.6), we have for $j > i$,

$$(5.11) \qquad q\xi_j^n - \zeta_j^n = \alpha_j(R-Q-qM)(1-\alpha v_0^{n-1}) - \alpha(Q+qM+qmv_0^{n-1}).$$

Hence, $q\xi_j^n - \zeta_j^n$ is nondecreasing in j, since α_j is nondecreasing and R-Q-qM is nonnegative. This completes the proof.

Theorem 5.1, lemma 5.4 and lemma 5.5 imply that $q\xi_i^n - \zeta_i^n$ changes its sign at most once and if a change occurs, it is from negative. Therefore, since $P_{ij}(t)$ is TP_2 in i,j and j,t for $i,j \in S$, we can conclude that $h_i^n(t)$ changes its sign at most once in both i and t, and if a change occurs, it is from negative, by using variation diminishing property of TP_2. That is, $H_i^n(t)$ behaves as in Fig 2. From the above discussion, we have the following theorem.

Theorem 5.6. $H_i^n(t)$ is unimodular in t and has at most one minimum.

This theorem is very useful to find the optimal inspection time.

Theorem 5.7. If $D_i^n = I(t_i)$, $D_{i+1}^n = I(t_{i+1})$, then $t_i \geqq t_{i+1}$.

Proof. If $D_i^n = I(t_i)$, then $h_i^n(t_i) = 0$. From the above discussion with respect to the behavior of $h_i^n(t)$ in t and i, we have $h_{i+1}^n(t_i) \geqq 0$, which shows that $t_{i+1} \leqq t_i$.

This theorem implies that as the system degrades, optimal inspection time interval becomes shorter.

Theorems 5.1, 5.6 and 5.7 imply that an optimal policy in our problem is of the form, where D_i is an optimal action at E_i,

for $i = 0,1,\ldots,h-1$, $\quad D_i = I(\infty)$,
for $i = h,\ldots\ldots,k-1$, $\quad D_i = I(t_i)$ and t_i is nonincreasing in i,
for $i = k,\ldots\ldots,s$, $\quad D_i$ = PM,
where $1 \leqq h \leqq k \leqq s+1$.

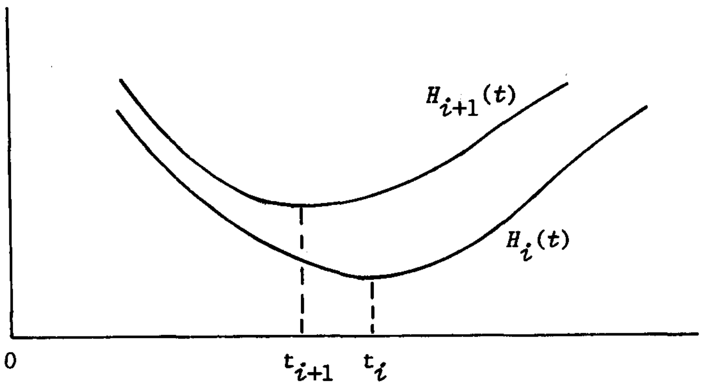

Fig.2 Behavior of $H_i(t)$

6. Discrete Time Case

For a discrete time Markovian deteriorating system, we consider the same problem as the one in the previous sections. The following quantities are introduced.

p_{ij}	one step transition probability.
α_i	$p_{i,s+1}$.
$P_{ij}(t)$	t-step transition probability.
$\bar{F}_i(t)$	$\sum_{j=i}^{s} P_{ij}(t)$.
$f_i(t)$	$\bar{F}_i(t-1) - \bar{F}_i(t)$, $f_i(0) = 0$.
q_t, r_t, m_t	probabilities that inspection, CM and PM are completed at time t, respectively, $q_0 = r_0 = m_0 = 0$.
$\bar{Q}(t)$, $\bar{R}(t)$, $\bar{M}(t)$	$\sum_{x=t+1}^{\infty} q_x$, $\sum_{x=t+1}^{\infty} r_x$ and $\sum_{x=t+1}^{\infty} m_x$, respectively.
β	discount factor.
Q,R,M	discounted expected inspection, CM and PM time, respectively, e.g., $Q = \sum_{t=0}^{\infty} \beta^t \bar{Q}(t)$,
q,r,m	$\sum_{t=0}^{\infty} \beta^t q_t$ ($= 1 - (1-\beta)Q$), $\sum_{t=0}^{\infty} \beta^t r_t$ and $\sum_{t=0}^{\infty} \beta^t m_t$, repectively.

We assume that

i) $p_{ij} = 0$ for $i > j$,

ii) p_{ij} is TP_2 in i,j, i.e.,

$$\begin{vmatrix} p_{im} & p_{in} \\ p_{jm} & p_{jn} \end{vmatrix} \geqq 0, \text{ for } \begin{matrix} i < j \\ m < n, \end{matrix} \quad i,j,m,n \in S+1, \tag{6.1}$$

iii) $R \geqq Q+qM$.

Equation (6.1) implies that the system is more likely to make transition to higher states with increasing state number. This is thought to be one definition for the system to be a deteriorating system.

Under these assumptions, we have

Lemma 6.1. $P_{ij}(t)$ is TP_2 in i,j and in j,t.

Lemma 6.2. $\sum_{j=k}^{s+1} P_{ij}(t)$ is nondecreasing in i for any k.

For these lemmas, see Rosenfield [6].

In a similar way as the continuous time case, the optimal system discounted unavailability can be expressed by the following equations, where v_i are defined in the same fashion as the continuous time case.

(6.2) $$v_i = \min\{ \min_{0 \leqq t \leqq \infty} H_i(t),\ M+mv_0 \},\quad i \in S,$$

where

(6.3) $$H_i(t) = v_{s+1} \sum_{x=1}^{t} \beta^x f_i(x) + \beta^t \sum_{j=i}^{s+1} P_{ij}(t)(Q+qv_j),$$

(6.4) $$v_{s+1} = R+rv_0.$$

Equations (6.2)-(6.4) can be solved by the same procedure as the continuous time case and the discussion about convergence to an optimal policy and the optimal discounted system unavailability is omitted. In the following, we discuss the properties of an optimal policy. We have the following lemmas, where v_i^n, $H_i^n(t)$ ans D_i^n are defined in the same fashion as the continuous time case.

Lemma 6.3.

(6.5) $$v_{s+1}^n \geqq Q+qv_i^n,\quad i \in S.$$

Lemma 6.4. v_i^n is nondecreasing in i.

Lemma 6.5.

(6.6) $$v_i^n < 1/(1-\beta),\quad n \geqq 1.$$

Lemma 6.6. $D_i^n \neq I(0)$, $D_0^n \neq PM$, $n \geqq 1$.

Lemmas 6.3-6.6 can be proved in a similar way as in the continuous time case. It is noted that lemma 6.4 implies that a CLR is optimal for each n and that there exists a CLR which is optimal.

We define

(6.7) $$h_i^n(t) = H_i^n(t+1) - H_i^n(t),$$

then we have

(6.8) $$h_i^{n+1}(t) = \beta^t \sum_{j=i}^{s} P_{ij}(t) h_j^{n+1}(0),$$

where

(6.9) $$h_i^{n+1}(0) = q\xi_i^n - \zeta_i^n,$$

(6.10) $$\xi_i^n = -v_i^n + \beta \sum_{j=i}^{s} p_{ij} v_j^n + \beta\alpha_i v_{s+1}^n,$$

(6.11) $$\zeta_i^n = Q[\ 1-\beta+\beta\alpha_i\{1-(1-\beta)v_{s+1}^n\}\ ] > 0.$$

Lemma 6.7. If $D_i^n = I(t)$, then $\xi_i^n \leqq 0$.

Proof. For $t \geqq 1$, we have

$$(6.12) \qquad v_i^n = H_i^n(t) = \beta\alpha_i v_{s+1}^{n-1} + \beta \sum_{j=i}^{s} p_{ij} H_j^n(t-1) \geqq \beta\alpha_i v_{s+1}^n + \beta \sum_{j=i}^{s} p_{ij} v_j^n$$

In a similar way as in the proof of lemma 5.5, we have the following lemma.

Lemma 6.8. If $q\xi_i^n - \zeta_i^n > 0$, then $q\xi_j^n - \zeta_j^n > 0$ for $j > i$.

Lemma 6.7 and lemma 6.8 imply that $h_i^n(0)$ changes its sign at most once and if a change occurs, it is from negative. From these lemmas and that $P_{ij}(t)$ is TP_2 in i,j and in j,t, we can conclude that an optimal policy has the same properties as those in the continuous time Markovian deterioration case.

7. Numerical Example

A system has only two degraded states (s = 2). As a numerical example, we let $\lambda_0=0.001$, $\lambda_1=0.003$, $\lambda_2=0.005$, $\alpha_0=0$, $\alpha_1=0$, $\alpha=0.001$, R=500, Q=10 and consider five cases for M: M=50, 100, 200, 300, 400. For each case, an optimal policy and the optimal discounted unavailability are shown below (when $D_i = I(t_i)$, only t_i is written).

M	D_0	D_1	D_2	v_0	v_1	v_2
50	273	PM	PM	102.6	147.5	147.5
100	285	PM	PM	131.0	217.9	217.9
200	369	82	PM	161.5	295.8	329.2
300	626	153	PM	181.8	351.7	423.7
400	∞	∞	∞	182.5	370.4	493.8

8. Conclusion

In this paper, we have mainly discussed an optimal inspection and maintenance problem of a continuous time Markovian deteriorating system. An optimal policy minimizes the total discounted system unavailability. We have formulate the problem by a semi-Markov decision process and have shown that a sequential approximation procedure gives an optimal policy and the optimal system unavailability. Some interesting properties of the optimal policy have been obtained, that is, in the optmal policy, a control limit rule holds and the optimal inspection time intervals become shorter as the system degrades. These properties are very useful to determine the optimal policy. Furthermore, for a discrete time Markovian deteriorating system, almost the same discussion has been made.

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