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THE GI/E_k/1 QUEUE WITH FINITE WAITING ROOM

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Abstract The GI/E_k/1 queue with finite waiting room is investigated by using a combination of the supplementary variable and phase techniques. We start by studying the steady-state joint distribution of the number of phases present and the elapsed time since the last arrival. From this distribution, we obtain the distributions of the number of customers both at an arbitrary moment and just after an arrival. Furthermore, we show some interesting properties on the mean number of customers from numerical examples.

1. Introduction

Many researchers have analysed queueing systems with finite waiting room. Keilson [5] derived several results on the *M/G/1/N* and the *GI/M/1/N* queues, where N denotes the maximum number of customers allowed in the system. The *M/G/1/N* queue was also studied by Cohen [1] and he gave some further references. In recent years, Truslove [7] obtained the asymptotic queue length distribution just after a departure for the $E_k/G/1/N$ queue by using a phase technique. Also he [8] studied the busy period for the $E_k/G/1/N$ queue. Hokstad [4] studied the joint distribution of the number of phases present and the remaining service time of a customer in service for the $E_k/G/1/N$ queue. He obtained not only the queue length but also the waiting time and the idle period distributions and the mean length of the idle and busy periods. He [3] also treated the *GI/M/m/N* queue.

In this paper, we shall study the joint distribution of the number of phases present and the elapsed time since the last arrival for the $GI/E_k/1/N$ queue in the steady-state. The distributions of the number of customers both at an arbitrary moment and just after an arrival are also derived. Finally, we show some interesting properties on the mean number of customers from numerical examples.

2. Assumptions and Notation

We consider a $GI/E_k/1$ queue with finite waiting room. We assume that the maximum number of customers allowed in the system is N, i.e., the number of waiting places is $N - 1$. Due to Kendall's notation [6], this system will be denoted by $GI/E_k/1/N$.

For the $GI/E_k/1/N$ queue, it is assumed that interarrival times are independent and identically distributed nonnegative random variables having a distribution function $A(x)$ $(x \geq 0)$ with a probability density function (p.d.f.) $a(x)$ $(x \geq 0)$ and mean $1/\lambda$. It is perhaps possible to proceed our analysis without assuming that interarrival times have the p.d.f. $a(x)$ (see Gnedenko and Kovalenko [2]), but this assumption of arriving customers fairly simplifies the following argument. For the notational convenience, we define

$$\bar{A}(x) = 1 - A(x), \quad x \geq 0,$$

and

$$\Lambda(x) = \frac{a(x)}{\bar{A}(x)}, \quad x \geq 0.$$

Further let T be the elapsed time since the last arrival at an arbitrary moment. Service times of arriving customers are random variables distributed according to the k-phase Erlang distribution with mean $1/\mu$. We assume that the maximum number of customers allowed in the system is N including a customer in service. A customer who arrives at the moment when the number of customers present is N is not admitted to the system. He disappears and never returns.

Since the service time distribution is the k-phase Erlang, each service time of arriving customers can be decomposed into k independent phases which are distributed exponentially. Let Q be the steady-state number of customers in the system, X the number of phases to be served for customers in the system and J the number of phases remaining in the service facility for a customer in service. Then we have the relation

$$X = \begin{cases} (Q - 1)k + J, & \text{if } N \geq Q \geq 1 \\ 0, & \text{if } Q = 0. \end{cases}$$

Thus the maximum number of phases in the system is Nk and X can take the values $0,1,\ldots,Nk$.

3. The Steady-State Probabilities

Define the steady-state joint distribution of the number of phases present and the elapsed time since the last arrival as

$$p_i(x)dx = P\{X = i,\ x \le T < x + dx\},$$

for $x \ge 0$ and $i = 0,1,\ldots,Nk$. Then, by the usually argument, we have for $x > 0$ that

$$\frac{d}{dx} p_0(x) + \Lambda(x)p_0(x) = k\mu p_1(x),$$

(3.1)
$$\frac{d}{dx} p_i(x) + \{\Lambda(x) + k\mu\}p_i(x) = k\mu p_{i+1}(x), \quad \text{for } i = 1,\ldots,Nk-1$$

$$\frac{d}{dx} p_{Nk}(x) + \{\Lambda(x) + k\mu\}p_{Nk}(x) = 0.$$

Also since $p_i(0)$ is the probability of i phases in the system immediately just after an arrival, we have, for $N = 1$,

(3.2)
$$p_i(0) = \begin{cases} 0, & \text{for } i = 0 \\ \int_0^\infty p_i(x)\Lambda(x)dx, & \text{for } i = 1,\ldots,k-1 \\ \int_0^\infty \{p_0(x) + p_k(x)\}\Lambda(x)dx, & \text{for } i = k, \end{cases}$$

and for $N \ge 2$,

(3.3)
$$p_i(0) = \begin{cases} 0, & \text{for } i = 0,\ldots,k-1 \\ \int_0^\infty p_{i-k}(x)\Lambda(x)dx, & \text{for } i = k,\ldots,Nk-k \\ \int_0^\infty \{p_{i-k}(x) + p_i(x)\}\Lambda(x)dx, & \text{for } i = Nk-k+1,\ldots,Nk. \end{cases}$$

The normalizing condition

(3.4)
$$\int_0^\infty \sum_{i=0}^{Nk} p_i(x)dx = 1$$

should be imposed on these probabilities $\{p_i(x),\ x \ge 0\}$. It is shown inductively that the general solutions of the system of linear differential equations (3.1) are given by

$$(3.5)\qquad p_i(x) = \begin{cases} \bar{A}(x)\exp(-k\mu x)\sum_{j=0}^{Nk-i} c_{i+j}\dfrac{(k\mu x)^j}{j!}, & \text{for } i = 1,\ldots,Nk \\ \bar{A}(x)\{c_0 - \exp(-k\mu x)\sum_{j=0}^{Nk-1} c_{j+1}\sum_{l=0}^{j}\dfrac{(k\mu x)^l}{l!}\}, & \text{for } i = 0, \end{cases}$$

where c_i $(i = 0,\ldots,Nk)$ are constants to be determined. From (3.5) we get easily that

$$(3.6)\qquad p_i(0) = \begin{cases} c_i, & \text{for } i = 1,\ldots,Nk \\ c_0 - \sum_{j=1}^{Nk} c_j, & \text{for } i = 0. \end{cases}$$

Also let us define that for $\mathrm{Re}(s) \geq 0$ and $j = 0,\ldots,Nk-1$,

$$A_j^*(s) = \int_0^\infty x^j a(x)\exp(-sx)dx$$

$$= (-1)^j \frac{d^j}{ds^j}\int_0^\infty a(x)\exp(-sx)dx,$$

then from (3.5) we find that

$$(3.7)\qquad \int_0^\infty p_i(x)\Lambda(x)dx = \begin{cases} \sum_{j=0}^{Nk-i} c_{i+j}\dfrac{(k\mu)^j}{j!}A_j^*(k\mu), & \text{for } i = 1,\ldots,Nk \\ c_0 - \sum_{l=0}^{Nk-1} c_{l+1}\sum_{j=0}^{l}\dfrac{(k\mu)^j}{j!}A_j^*(k\mu), & \text{for } i = 0. \end{cases}$$

Noticing the equality for arbitrary sequences $\{\xi_i\}$ and $\{\zeta_i\}$

$$\sum_{i=1}^{n}\sum_{j=0}^{n-i}\xi_{i+j}\zeta_j = \sum_{j=0}^{n-1}\xi_{j+1}\sum_{i=0}^{j}\zeta_i,$$

and using (3.5), we have that

$$\sum_{i=0}^{Nk} p_i(x) = c_0\bar{A}(x).$$

From (3.4) and (3.7), we have the equation

$$1 = \int_0^\infty \sum_{i=0}^{Nk} p_i(x)dx = \int_0^\infty c_0\bar{A}(x)dx = c_0/\lambda.$$

Hence we obtain

(3.8) $$c_0 = \lambda.$$

Now we are able to derive the following system of equations to determine c_i, $i = 0,\ldots,Nk$. From (3.2),(3.3),(3.6),(3.7) and (3.8), we have, for $N = 1$,

$$c_0 = \sum_{i=1}^{k} c_i,$$

(3.9) $$c_i = \sum_{j=0}^{k-i} c_{i+j} \frac{(k\mu)^j}{j!} A_j^*(k\mu), \quad \text{for } i = 1,\ldots,k-1$$

$$c_0 = \lambda,$$

and for $N \geq 2$,

$$c_0 = \sum_{i=1}^{Nk} c_i,$$

$$c_i = 0, \qquad \text{for } i = 1,\ldots,k-1$$

(3.10) $$c_i = \sum_{j=0}^{Nk+k-i} c_{i-k+j} \frac{(k\mu)^j}{j!} A_j^*(k\mu), \quad \text{for } i = k+1,\ldots,Nk-k$$

$$c_i = \sum_{j=0}^{Nk+k-i} c_{i-k+j} \frac{(k\mu)^j}{j!} A_j^*(k\mu) + \sum_{j=0}^{Nk-i} c_{i+j} \frac{(k\mu)^j}{j!} A_j^*(k\mu),$$

$$\text{for } i = Nk-k+1,\ldots,Nk$$

$$c_0 = \lambda.$$

Thus $p_i(x)$ $(i = 0,\ldots,Nk)$ can be completely determined from (3.5) and (3.9) or (3.5) and (3.10).

Now we determine

$$p_i^* = \int_0^\infty p_i(x)dx,$$

which is the probability of i phases in the system at an arbitrary moment. Using the relation

$$\int_0^\infty x^j \overline{A}(x)\exp(-sx)dx = \frac{j!}{s^{j+1}} - \sum_{i=1}^{j+1} \frac{j!}{s^i (j-i+1)!} A_{j-i+1}^*(s),$$

we have

$$(3.11)\qquad p_i^* = \begin{cases} \frac{1}{k\mu}\sum_{j=0}^{Nk-i} c_{i+j}\{1 - \sum_{m=0}^{j} \frac{(k\mu)^m}{m!} A_m^*(k\mu)\}, & \text{for } i = 1,\dots,Nk \\ 1 - \frac{1}{k\mu}\sum_{j=0}^{Nk-1} c_{j+1} \sum_{n=0}^{j}\{1 - \sum_{m=0}^{n} \frac{(k\mu)^m}{m!} A_m^*(k\mu)\}, & \text{for } i = 0. \end{cases}$$

Furthermore the probability of i phases in the system just after an arrival is defined by

$$p_i^\circ = \frac{p_i(0)}{\sum_{j=1}^{Nk} p_j(0)} .$$

We obtain from (3.6) and (3.8)

$$(3.12)\qquad p_i^\circ = \begin{cases} \frac{c_i}{\lambda}, & \text{for } i = 1,\dots,Nk \\ 0 & \text{for } i = 0. \end{cases}$$

Hence, if we let q_n^* and q_n° , respectively, be the probabilities of n customers in the system at an arbitrary moment and just after an arrival, then we obtain

$$q_n^* = \sum_{i=(n-1)k+1}^{nk} p_i^* ,$$

and

$$q_n^\circ = \sum_{i=(n-1)k+1}^{nk} p_i^\circ .$$

The mean waiting time and the number of customers can be derived from these probabilities.

4. Some Examples

(i) The $M/M/1/1$ queue

It is clear that $A(x) = 1 - \exp(-\lambda x)$ and $A_0^*(\mu) = \lambda/(\lambda+\mu)$ in this system. From (3.5), (3.9) and (3.11) we have

$$c_0 = \lambda,$$
$$c_1 = \lambda,$$
$$p_1(x) = \exp\{-(\lambda+\mu)x\}c_1$$
$$= \lambda\exp\{-(\lambda+\mu)x\},$$
$$p_0(x) = \exp(-\lambda x)\{\lambda-\exp(-\mu x)c_1\}$$
$$= \lambda\exp(-\lambda x)\{1-\exp(-\mu x)\},$$
$$p_1^* = c_1\{1 - A_0^*(\mu)\}/\mu$$
$$= \rho/(1+\rho)$$

and

$$p_0^* = 1 - c_1\{1 - A_0^*(\mu)\}/\mu$$
$$= 1/(1+\rho),$$

where $\rho = \lambda/\mu$. Clearly, we have

$$p_1^\circ = 1$$

and

$$p_0^\circ = 0.$$

(ii) The $M/M/1/2$ queue

For this system, we have that $A_1^*(\mu) = \lambda/(\lambda+\mu)^2$ and

$$c_0 = c_1 + c_2 ,$$
$$c_2 = c_1A_0^*(\mu) + c_2\mu A_1^*(\mu) + c_2A_0^*(\mu),$$
$$c_0 = \lambda.$$

Hence we have

$$c_0 = \lambda,$$
$$c_1 = \lambda\mu^2/(\lambda^2+\lambda\mu+\mu^2),$$
$$c_2 = \lambda^2(\lambda+\mu)/(\lambda^2+\lambda\mu+\mu^2).$$

From (3.5), (3.11) and (3.12) we obtain

$$p_2(x) = \exp\{-(\lambda+\mu)x\}c_2$$
$$= \frac{\lambda^2(\lambda+\mu)}{\lambda^2+\lambda\mu+\mu^2}\exp\{-(\lambda+\mu)x\},$$

$$p_1(x) = \exp\{-(\lambda+\mu)x\}\{c_1+c_2\mu x\}$$

$$= \frac{\lambda\mu}{\lambda^2+\lambda\mu+\mu^2} \exp\{-(\lambda+\mu)x\}\{\mu+\lambda(\lambda+\mu)x\},$$

$$p_0(x) = \exp(-\lambda x)\{\lambda-\exp(-\mu x)(c_1+c_2+c_2\mu x)\}$$

$$= \exp(-\lambda x)\{\lambda-\exp(-\mu x)(\lambda+ \frac{\lambda^2\mu(\lambda+\mu)}{\lambda^2+\lambda\mu+\mu^2} x) ,$$

$$p_2^* = c_2\{1 - A_0^*(\mu)\}/\mu$$

$$= \rho^2/(1+\rho+\rho^2),$$

$$p_1^* = [c_1\{1 - A_0^*(\mu) + c_2(1 - A_0^*(\mu) - \mu A_1^*(\mu))\}]/\mu$$

$$= \rho/(1+\rho+\rho^2),$$

$$p_0^* = 1/(1+\rho+\rho^2),$$

$$p_2^\circ = (1+\rho)/(1+\rho+\rho^2),$$

$$p_1^\circ = \rho^2/(1+\rho+\rho^2),$$

and

$$p_0^\circ = 0.$$

5. Numerical Results

Tables 1, 2, 3 and 4 give numerical values of q_n^* and $E[Q^*]$, the mean number of customers at an arbitrary moment, for the $M/M/1/5$, $M/E_3/1/5$, $E_3/M/1/5$ and $E_3/E_3/1/5$ queues, respectively. Tables 5, 6, 7 and 8 give those of q_n° and $E[Q^\circ]$, the mean number of customers just after an arrival for the $M/M/1/5$, $M/E_3/1/5$, $E_3/M/1/5$ and $E_3/E_3/1/5$ queues, respectively. From these tables we observe in light traffic that

$$\left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/M/1/5 \end{matrix} \right\} > \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_3/1/5 \end{matrix} \right\} > \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_3/M/1/5 \end{matrix} \right\} > \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_3/E_3/1/5 \end{matrix} \right\}.$$

On the other hand, in heavy traffic it holds that

$$\left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/M/1/5 \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_3/1/5 \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_3/M/1/5 \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_3/E_3/1/5 \end{matrix} \right\}.$$

For $E[Q°]$ we also have similar results as above.

Figures 1 ∿ 16 illustrate the mean number of customers $E[Q^*]$ at an arbitrary moment as a function of μ for various queueing systems with finite waiting room. Figures 1 ∿ 4 show $E[Q^*]$ for the $M/E_k/1/N$ queue with λ = 1.0 and k = 1, 2 and 3 for N = 1, 2, 3 and 5, respectively. From these figures it turns out that for $N \geq 2$ and $\mu > \mu^*$

$$\left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/M/1/N \end{matrix} \right\} > \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_2/1/N \end{matrix} \right\} > \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_3/1/N \end{matrix} \right\},$$

but for $\mu < \mu^*$ it holds that

$$\left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/M/1/N \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_2/1/N \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/E_3/1/N \end{matrix} \right\},$$

where μ^* is the abscissa of the point of intersection of graphs. Figures 5 ∿ 8 show the mean number of customers $E[Q^*]$ for the $E_m/M/1/N$ queue with λ = 1.0 and m = 1, 2 and 3 for N = 1, 2, 3 and 5, respectively. From these figures it turns out that for $N = 1$

$$\left\{ \begin{matrix} E[Q^*] \text{ for} \\ M/M/1/1 \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_2/M/1/1 \end{matrix} \right\} < \left\{ \begin{matrix} E[Q^*] \text{ for} \\ E_3/M/1/1 \end{matrix} \right\}.$$

For $N \geq 2$, we find that $E[Q^*]$ decreases (increases) as m increases for $\mu > \mu^*$ ($\mu < \mu^*$). Figures 9 ∿ 12 show $E[Q^*]$ for the $E_3/E_k/1/N$ queue with λ = 1.0 and k = 1, 2 and 3 for N = 1, 2, 3 and 5. Further in Figures 13 ∿ 16 we present $E[Q^*]$ for the $E_m/E_3/1/N$ queue with λ = 1.0 and m = 1, 2 and 3 for N = 1, 2, 3 and 5, respectively. The same assertion as above also holds for these systems.

Table 1. The probability q_n^* and the mean number of customers $E[Q^*]$ at an arbitrary moment for the $M/M/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0159	0.1667	0.5079
1	0.0001	0.0317	0.1667	0.2540
2	0.0009	0.0635	0.1667	0.1270
3	0.0090	0.1270	0.1667	0.0635
4	0.0900	0.2540	0.1667	0.0317
5	0.9000	0.5079	0.1667	0.0158
$E[Q^*]$	4.8889	4.0952	2.5000	0.9048

Table 2. The probability q_n^* and the mean number of customers $E[Q^*]$ at an arbitrary moment for the $M/E_3/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0033	0.1277	0.5028
1	0.0000	0.0118	0.1749	0.2956
2	0.0000	0.0366	0.1877	0.1273
3	0.0012	0.1110	0.1907	0.0498
4	0.0987	0.3358	0.1913	0.0189
5	0.9000	0.5016	0.1277	0.0056
$E[Q^*]$	4.8987	4.2691	2.5261	0.8031

Table 3. The probability q_n^* and the mean number of customers $E[Q^*]$ at an arbitrary moment for the $E_3/M/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0042	0.1200	0.5011
1	0.0000	0.0141	0.1800	0.3355
2	0.0002	0.0377	0.1799	0.1709
3	0.0039	0.0994	0.1793	0.0366
4	0.0655	0.2542	0.1765	0.0121
5	0.9303	0.5904	0.1644	0.0039
$E[Q^*]$	4.9259	4.3566	2.6055	0.7349

Table 4. The probability q_n^* and the mean number of customers $E[Q^*]$ at an arbitrary moment for the $E_3/E_3/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0001	0.0711	0.5000
1	0.0000	0.0008	0.1838	0.4092
2	0.0000	0.0064	0.2098	0.0784
3	0.0002	0.0489	0.2121	0.0108
4	0.0667	0.3175	0.2047	0.0014
5	0.9332	0.6263	0.1184	0.0002
$E[Q^*]$	4.9330	4.5619	2.6509	0.6048

Table 5. The probability q_n° and the mean number of customers $E[Q^\circ]$ just after an arrival for the $M/M/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0159	0.1667	0.5079
2	0.0001	0.0317	0.1667	0.2540
3	0.0009	0.0635	0.1667	0.1270
4	0.0090	0.1270	0.1667	0.0635
5	0.9900	0.7619	0.3333	0.0476
$E[Q^\circ]$	4.9889	4.5873	3.3333	1.8889

Table 6. The probability q_n° and the mean number of customers $E[Q^\circ]$ just after an arrival for the $M/E_3/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0033	0.1277	0.5028
2	0.0000	0.0118	0.1749	0.2956
3	0.0000	0.0366	0.1877	0.1273
4	0.0012	0.1110	0.1907	0.0498
5	0.9987	0.8374	0.3190	0.0245
$E[Q^\circ]$	4.9987	4.7675	3.3984	1.7975

Table 7. The probability q_n° and the mean number of customers $E[Q^\circ]$ just after an arrival for the $E_3/M/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0071	0.1800	0.6709
2	0.0000	0.0188	0.1799	0.2217
3	0.0004	0.0497	0.1793	0.0733
4	0.0066	0.1271	0.1765	0.0241
5	0.9930	0.7973	0.2844	0.0099
$E[Q^\circ]$	4.9926	4.6888	3.2055	1.4805

Table 8. The probability q_n° and the mean number of customers $E[Q^\circ]$ just after an arrival for the $E_3/E_3/1/5$ queue with $\lambda = 1.0$.

n \ μ	0.1	0.5	1.0	2.0
0	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0001	0.1232	0.7244
2	0.0000	0.0014	0.2005	0.2343
3	0.0000	0.0108	0.2117	0.0360
4	0.0003	0.0784	0.2098	0.0047
5	0.9997	0.9092	0.2549	0.0006
$E[Q^\circ]$	4.9997	4.8952	3.2728	1.3230

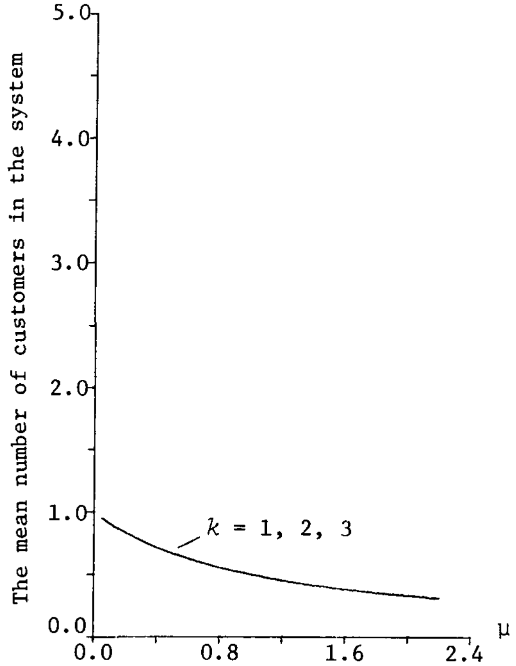

Figure 1. The mean number of customers in the system for the $M/E_k/1/1$ queue.

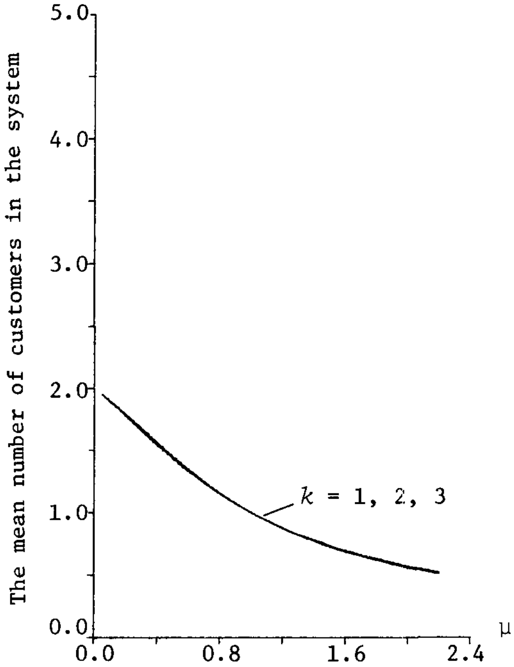

Figure 2. The mean number of customers in the system for the $M/E_k/1/2$ queue.

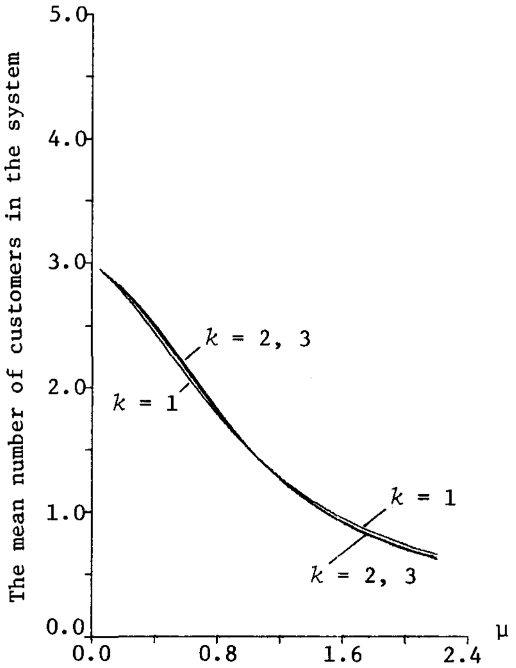

Figure 3. The mean number of customers in the system for the $M/E_k/1/3$ queue.

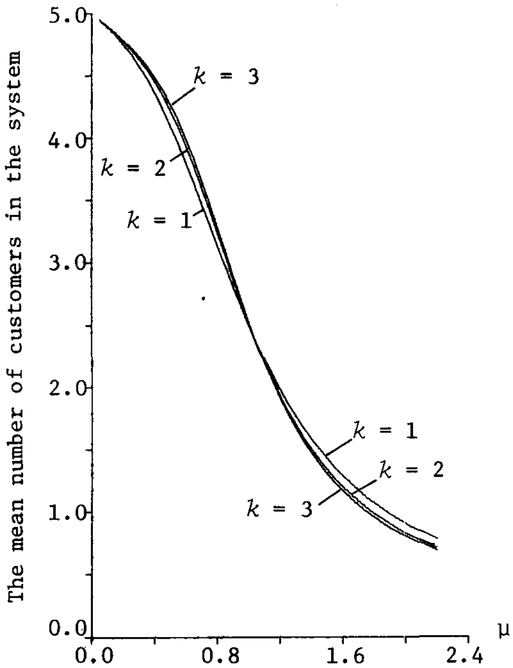

Figure 4. The mean number of customers in the system for the $M/E_k/1/5$ queue.

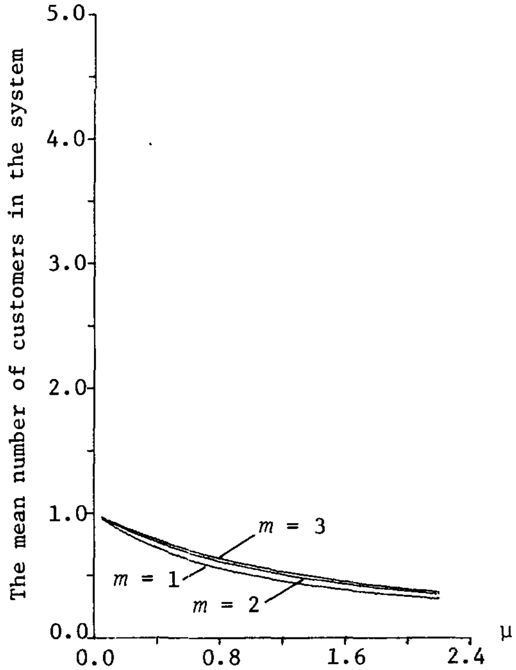

Figure 5. The mean number of customers in the system for the $E_m/M/1/1$ queue.

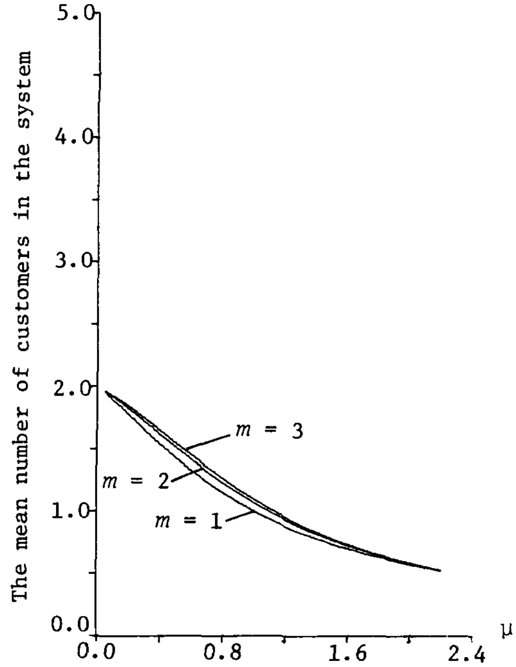

Figure 6. The mean number of customers in the system for the $E_m/M/1/2$ queue.

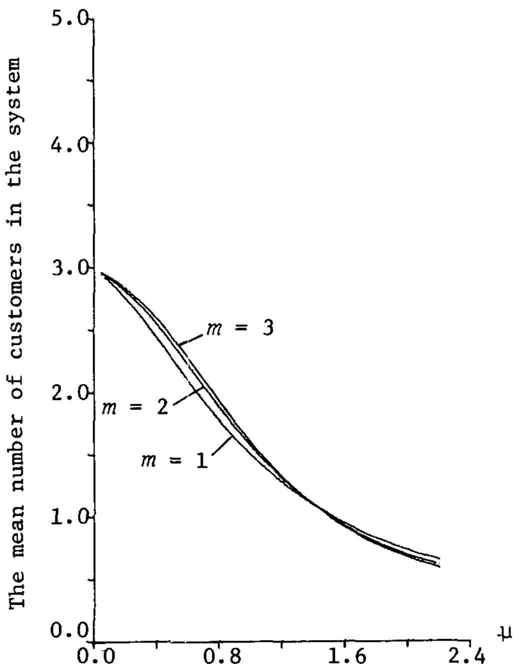

Figure 7. The mean number of customers in the system for the $E_m/M/1/3$ queue.

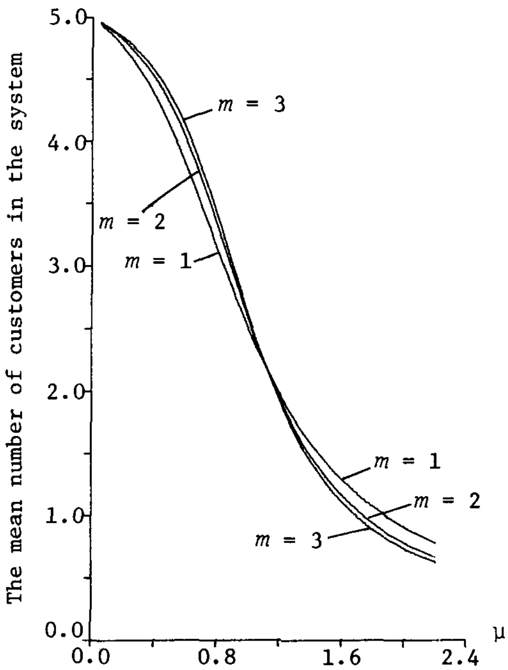

Figure 8. The mean number of customers in the system for the $E_m/M/1/5$ queue.

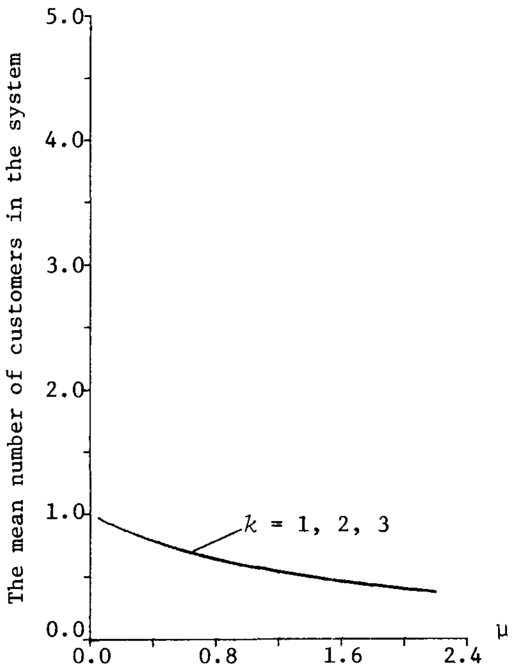

Figure 9. The mean number of customers in the system for the $E_3/E_k/1/1$ queue.

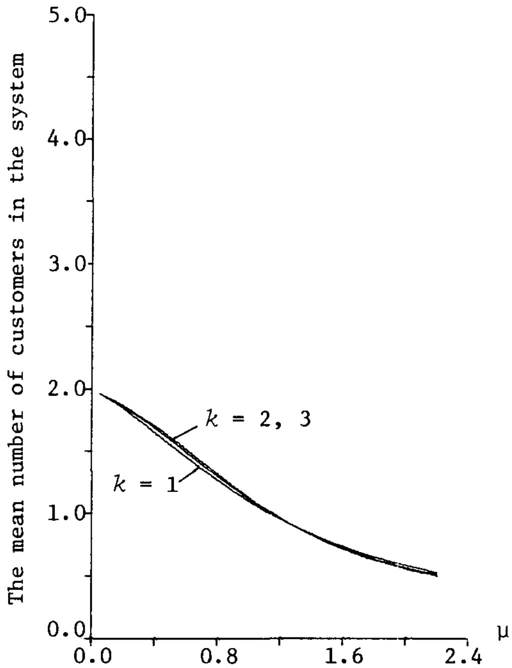

Figure 10. The mean number of customers in the system for the $E_3/E_k/1/2$ queue.

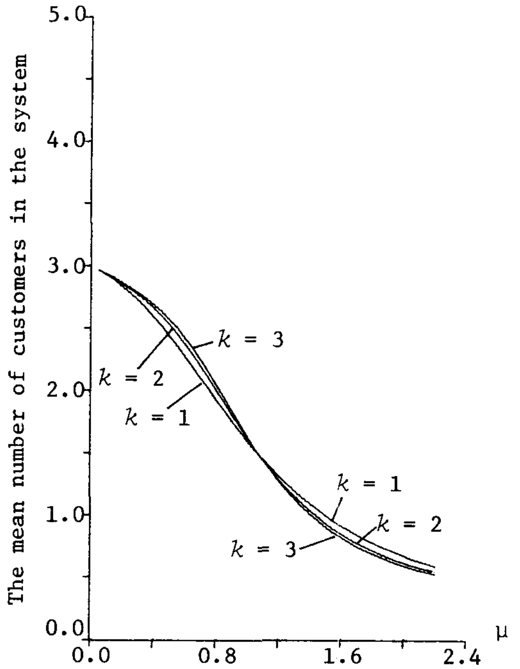

Figure 11. The mean number of customers in the system for the $E_3/E_k/1/3$ queue.

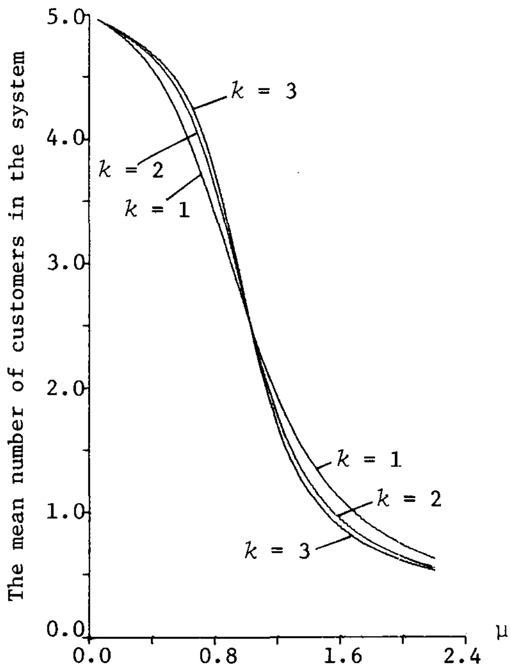

Figure 12. The mean number of customers in the system for the $E_3/E_k/1/5$ queue.

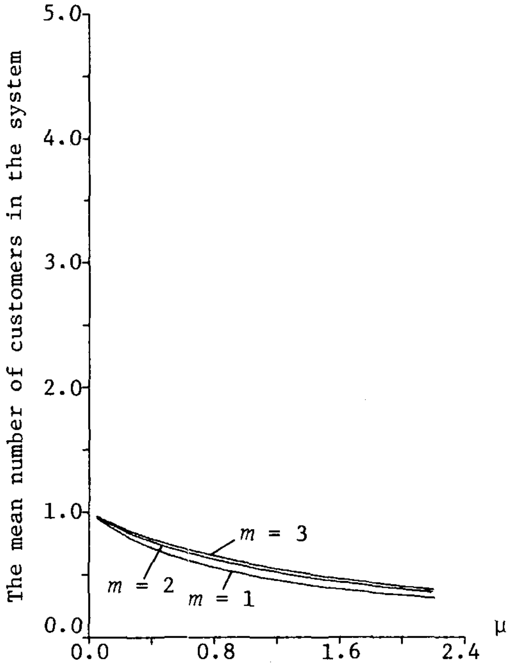

Figure 13. The mean number of customers in the system for the $E_m/E_3/1/1$ queue.

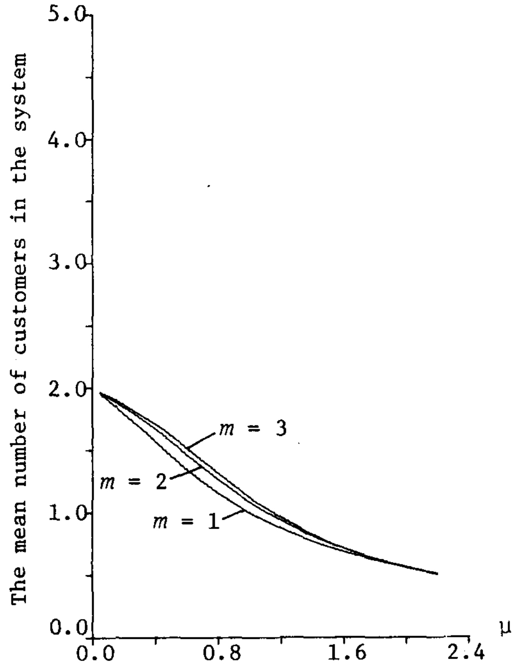

Figure 14. The mean number of customers in the system for the $E_m/E_3/1/2$ queue.

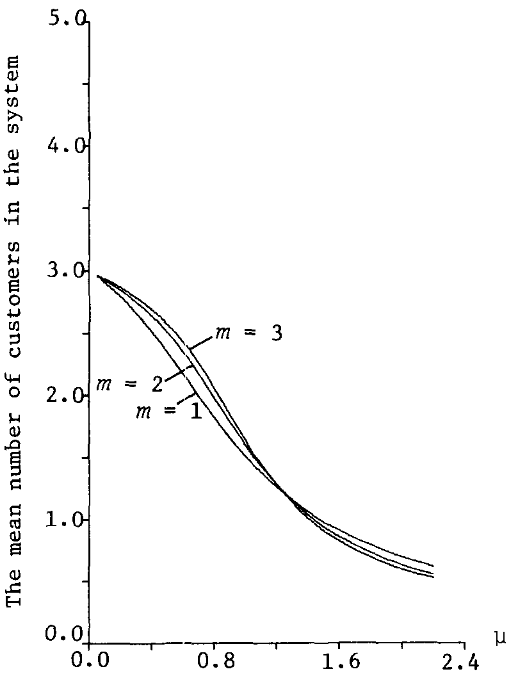

Figure 15. The mean number of customers in the system for the $E_m/E_3/1/3$ queue.

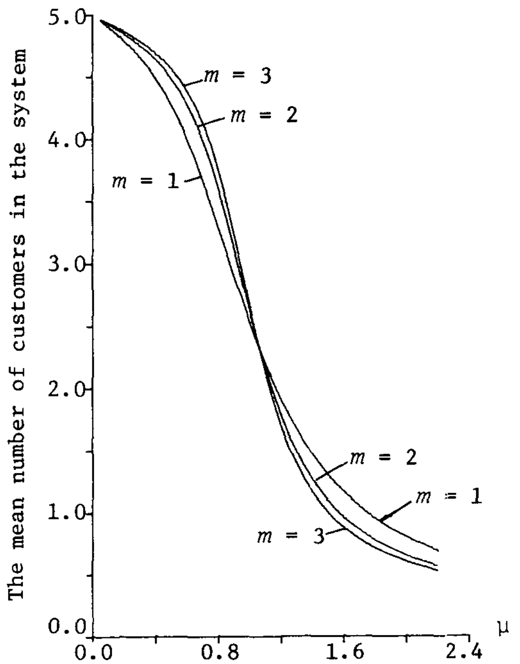

Figure 16. The mean number of customers in the system for the $E_m/E_3/1/5$ queue.

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