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THE SECRETARY PROBLEM IN A COMPETITIVE SITUATION

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Abstract This paper is concerned with two-person non-zero-sum game versions of the secretary problem. A remarkable feature of our models compared with previous ones is that Nash equilibrium strategies are different for two players, i.e. one player should behave more hastily, and the other less hastily, than in the secretary problem.

1. Introduction

This paper will be concerned with two-person non-zero-sum game versions of the so-called secretary problem. For the original secretary problem, the reader is referred to, e.g., Gilbert & Mosteller [1]. Our basic model (*Model A*) is as follows.

There are two companies each of which is faced with the problem of employing a secretary from one and the same set of n girls. If the companies could interview all the girls, they could rank the applicants absolutely with no ties, from best (rank 1) to worst (rank n), ranking being identical for both companies. However, the applicants present themselves one by one, in random order, and when the t-th applicant appears, the companies can observe only her rank relative to her (t-1) predecessors. We further assume that the recall is not permitted, i.e., each company must decide either to accept or to reject the t-th applicant to appear based upon her and the predecessors' relative ranks, without delay after the interview.

Two companies interview an identical applicant one by one every morning, independently of the other company, and the results of the interviews are communicated to that applicant in that afternoon. If only one of the companies decides to accept her, she agrees to this offer at once, although the other company is not informed of this fact and continues interviewing. If, on the other hand, both companies decide to accept her, she selects one of them with equal probabilities and the process stops.

The objective of each company is to maximize the probability of employing the best of n girls.

Game-theoretic versions of the secretary problem have been discussed by Presman & Sonin [3], Kurano et al. [2], and Sakaguchi [4]. A remarkable feature of our model compared with previous ones is as follows. The optimal strategy for the original secretary problem has a threshold character, i.e., there exists an integer $s(n)$ such that one must stop at the appearance of the first candidate (the object which is better than all the predecessors) after $(s(n)-1)$th, and shch a strategy is called $s(n)$-threshold strategy. In all of the previous models, Nash equilibrium solutions are such that players use the same threshold which are smaller than that for the original secretary problem (without a competitor). On the other hand, equilibrium strategies in our model are different for two players, i.e. one player should behave more hastily, and the other less hastily, than in the original secretary problem. This fact will be shown in the next section. We will also discuss about slightly different versions of our model in section 3.

2. Equilibrium Solutions

Suppose that companies #1 and #2 employ x- and y-threshold strategies respectively, and that the probabilities that each company succeeds in employing the best of n girls be denoted by $M_1(x,y)$ and $M_2(x,y)$. Then it is easy to see that $M_2(x,y)=M_1(y,x)$ for any x and y, and that

$$(2.1)\qquad M_1(x,y) = \begin{cases} \sum_{t=x}^{n} \frac{1}{n}\cdot\frac{x-y}{t-1} + \frac{1}{2}\sum_{t=x}^{n} \frac{1}{n}\cdot\frac{y-1}{t-1}, & \text{if } x \geq y \\ \sum_{t=x}^{y-1} \frac{1}{n}\cdot\frac{x-1}{t-1} + \frac{1}{2}\sum_{t=y}^{n} \frac{1}{n}\cdot\frac{x-1}{t-1}, & \text{if } x < y. \end{cases}$$

Similarly as in the original secretary problem and the related works, we are interested in the limiting form of the optimal strategy as n tends to infinity. Thus we let $n \to \infty$, and we will use the symbols x, y, t and $M_1(x,y)$, throughout the rest of this section, for the limiting values of x/n, y/n, t/n and $M_1(x,y)$ respectively in (2.1), since this convention will not make any confusion. Then we have, in the limit,

$$(2.2)\qquad M_1(x,y) = \int_x^1 \frac{x-y}{t}\,dt + \frac{1}{2}\int_x^1 \frac{y}{t}\,dt = -x\log x + \frac{y}{2}\log x, \quad \text{if } x \geq y$$

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$$= \int_x^y \frac{x}{t}\,dt + \frac{1}{2}\int_y^1 \frac{x}{t}\,dt = -x\log x + \frac{x}{2}\log y, \qquad \text{if} \quad x < y.$$

This is the payoff function for the company #1 in the non-zero-sum game on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

For any fixed y, $M_1(x,y)$ is a concave function of x in the two regions with $y \le x \le 1$ and with $0 \le x \le y$, and the x which locally maximizes $M_1(x,y)$ is the unique solution of the equation

(2.3) $$y = \begin{cases} y_1(x) \equiv (ex)^2, & \text{if} \quad x < y \\ y_2(x) \equiv 2x(1+\log x), & \text{if} \quad x \ge y. \end{cases}$$

The curves $y_1(x)$ and $y_2(x)$ are as shown in Fig. 1. We have $M_1(x,y_1(x)) = x$, and $M_1(x,y_2(x)) = x(\log x)^2$, which is decreasing in x $(e^{-1} \le x \le e^{-1/2})$. The equation

(2.4) $$M_1(y_1^{-1}(y),y) = M_1(y_2^{-1}(y),y)$$

has the unique solution $y^* \approx .363$, and according as $y > y^*$ or $y < y^*$ the left-hand

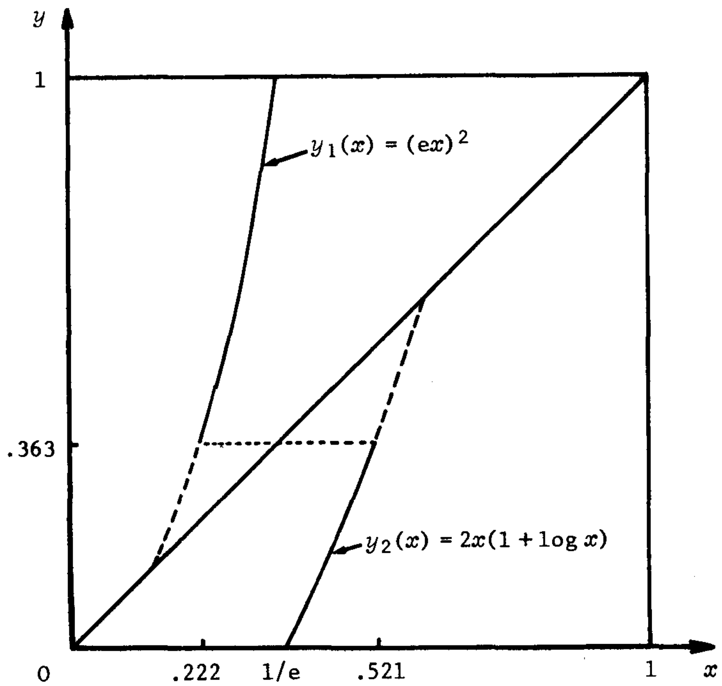

Fig. 1. Optimal x for fixed y in Model A

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or right-hand side of (2.4) is greater than the other side. Thus the x that globally maximizes $M_1(x,y)$ is obtained by those parts of the curves $y_1(x)$ and $y_2(x)$ which are shown by solid lines in Fig. 1.

Remember that the optimal strategy for the limiting version of the secretary problem is the 1/e-threshold strategy. Fig. 1 shows that the player in our game should acts more (less) hastily than in the secretary problem if the opponent's threshold is greater (smaller) than y^*.

A Nash equilibrium point for this game is a set of thresholds (x_o,y_o) for which both the inequalities

$$M_1(x_o,y_o) \geq M_1(x,y_o) \quad \text{and} \quad M_2(x_o,y_o) \geq M_2(x_o,y)$$

hold for any x and y. Thus there are two equilibrium points, which are intersections of the curves $y_1(x)$ and $y_2(x)$ with their reflections with respect to the line $y=x$:

$$(x_o,y_o) \simeq (.255,\ .480),\ \text{for which}\ M_1 \simeq .255\ \text{and}\ M_2 \simeq .259,$$

$$\text{and}\quad (x_o,y_o) \simeq (.480,\ .255),\ \text{for which}\ M_1 \simeq .259\ \text{and}\ M_2 \simeq .255.$$

The fact that each player's threshold in the equilibrium strategy is not identical is a remarkable feature of our model and forms a striking contrast to the strategies obtained in [2], [3] and [4].

Incidentally, if both companies should employ the strategy which is optimal when there is no opponent, the payoffs for both companies would be

$$M_1(1/e,1/e) = M_2(1/e,1/e) = 1/2e \simeq .184,$$

significantly smaller than .255 or .259.

Fig. 1 suggests that there may exist another equilibrium solution which involves a mixed strategy. If company #2 uses the threshold y^*, then company #1 can maximize its payoff by using the thresholds $x_1=y_1^{-1}(y^*)\simeq.222$ or $x_2=y_2^{-1}(y^*)\simeq.521$. Thus suppose company #2 uses the threshold y, where $x_1\leq y\leq x_2$, and company #1 uses x_1 and x_2 with probabilities p and $1-p$ respectively. Then the payoff for company #2, V, is given by

$$V = p(-y\log y + 0.5\,x_1\log y) + (1-p)(-y\log y + 0.5\,y\log x_2). \tag{2.5}$$

V is maximized by setting

$$\frac{\partial V}{\partial y} = -\log y - 1 + 0.5\,(\,px_1/y + (1-p)\log x_2) = 0. \tag{2.6}$$

The unique solution y of this equation is equal to y^* for the choice of p value

$$p^* = (2(\log y^* + 1) - \log x_2)/(x_1/y^* - \log x_2) \simeq .495.$$

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Hence an equilibrium solution is as follows: one company uses the threshold $y^* \simeq .363$, and the other uses $x_1 \simeq .222$ and $x_2 \simeq .521$ with probabilities $p^* \simeq .495$ and $1-p^* \simeq .505$ respectively. The payoffs are approximately .252 for the former and .222 for the latter. This solution is unattractive for both companies compared with the previous solutions involving pure strategies only.

3. Other Models

In this section we consider two variants of our basic model. *Model B* is different from model A in that when both companies want to employ the same applicant and she selects one of them, the other company can continue interviewing and employ another applicant. Let $g(t)$ designate the probability that the one company succeeds in employing the best girl using the optimal strategy given that the other company has employed the t-th applicant and left the game. Then $M_1(x,y)$ is given by the following:

$$(3.1) \qquad M_1(x,y) = \begin{cases} \sum_{t=x}^{n} \frac{1}{n}\cdot\frac{x-y}{t-1} + \frac{1}{2}\sum_{t=x}^{n} \{\frac{1}{n}\cdot\frac{y-1}{t-1} + \frac{1}{t}\cdot\frac{y-1}{t-1}\cdot g(t)\}, & \text{if } x \geq y \\ \sum_{t=x}^{y-1} \frac{1}{n}\cdot\frac{x-1}{t-1} + \frac{1}{2}\sum_{t=y}^{n} \{\frac{1}{n}\cdot\frac{x-1}{t-1} + \frac{1}{t}\cdot\frac{x-1}{t-1}\cdot g(t)\}, & \text{if } x < y. \end{cases}$$

Now we consider the limiting case as $n \to \infty$, and we use the same notational convention as in section 2. After a company knows that the opponent has employed an applicant and has left the game, the optimal strategy for the remaining company is the same one as in the original secretary problem. Thus we have

$$(3.2) \qquad g(t) = \begin{cases} 1/e, & \text{if } t \leq 1/e \\ -t\log t, & \text{if } t > 1/e \end{cases}$$

and

$$(3.3) \qquad M_1(x,y) = \begin{cases} \int_x^1 \frac{x-y}{t}\,dt + \frac{1}{2}\int_x^1 \{\frac{y}{t} + \frac{y}{t^2}\cdot g(t)\}dt, & \text{if } x \geq y \\ \int_x^y \frac{x}{t}\,dt + \frac{1}{2}\int_y^1 \{\frac{x}{t} + \frac{x}{t^2}\cdot g(t)\}dt, & \text{if } x < y. \end{cases}$$

After elementary and lengthy calculus we come to the conclusion that the x which maximizes $M_1(x,y)$ for fixed y is given by the following:

$$(3.4) \qquad x = 1/e, \qquad \text{if } y < y^{**}$$

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$= \exp[-1 + 0.5 \log y + 0.5\{1/(ey) - 0.5\}],$ if $y^{**} \le y \le 1/e$

$= \exp[-1 + 0.5 \log y + (\log y)^2/4],$ if $1/e < y.$

Fig. 2 shows the relationship between the optimal x and y. We have $M_1(x,y)=x$

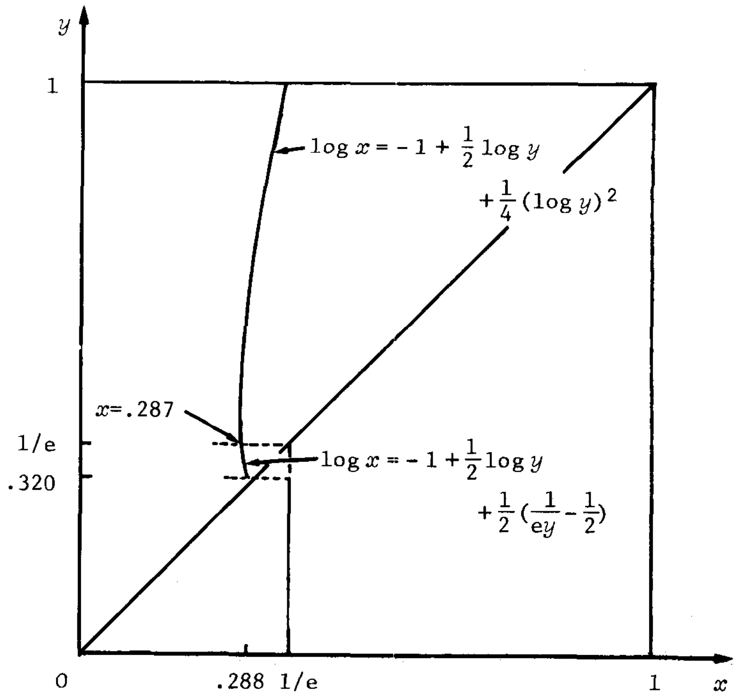

Fig. 2. Optimal x for fixed y in Model B

along the curve in the upper triangle and $M_1(x,y)=1/e - y/4$ along the line segment in the lower triangle, and $y^{**}\simeq.320$ is a root of the equation

(3.5) $\quad 1/e - y/4 = \exp[-1 + 0.5 \log y + 0.5\{1/(ey) - 0.5\}].$

The Nash equilibrium solutions are $(x,y)\simeq(.287,1/e)$, for which $(M_1,M_2)\simeq(.287, .296)$ and $(x,y)\simeq(1/e,.287)$, for which $(M_1,M_2)\simeq(.296,.287)$.

Another equilibrium solution which involves a mixed strategy is obtained in a similar way as in the previous section and it is as follows: one company uses the threshold $y^{**}\simeq.320$ and the other uses thresholds $x_3\simeq.288$ and $x_4=1/e$ with approximate probabilities .600 and .400. The payoffs are approximately .290 for the former and .288 for the latter.

Now we consider *Model C* which is different from Model B in that when the

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t-th applicant is employed by one company, this fact is immediately communicated to the other company whether this company wanted to employ this applicant or not. This is a time sequential game with perfect information.

$M_1(x,y)$ for this fodel is given by

$$(3.6)\quad M_1(x,y) = \begin{cases} \sum_{t=y}^{x-1} \frac{1}{t}\cdot\frac{y-1}{t-1}\cdot g(t) + \frac{1}{2}\sum_{t=x}^{n} \{\frac{1}{n}\cdot\frac{y-1}{t-1} + \frac{1}{t}\cdot\frac{y-1}{t-1}\cdot g(t)\}, & \text{if } x \geq y \\ \sum_{t=x}^{y-1} \frac{1}{n}\cdot\frac{x-1}{t-1} + \frac{1}{2}\sum_{t=y}^{n} \{\frac{1}{n}\cdot\frac{x-1}{t-1} + \frac{1}{t}\cdot\frac{x-1}{t-1}\cdot g(t)\}, & \text{if } x < y \end{cases}$$

and in the limit as $n \to \infty$,

$$(3.7)\quad M_1(x,y) = \begin{cases} \int_y^x \frac{y}{t^2}\cdot g(t)\,dt + \frac{1}{2}\int_x^1 \{\frac{y}{t} + \frac{y}{t^2}\cdot g(t)\}dt, & \text{if } x \geq y \\ \int_x^y \frac{x}{t}dt + \frac{1}{2}\int_y^1 \{\frac{x}{t} + \frac{x}{t^2}\cdot g(t)\}dt, & \text{if } x < y \end{cases}$$

where $g(t)$ is given by (3.2).

After elementary calculus it turns out that the x which maximizes $M_1(x,y)$ for fixed y is exactly same as that in Model B. Nash equilibrium points and the payoffs at these points are also identical in the two models. An equilibrium solution which involves a mixed strategy is as follows: one company uses the threshold $y^{**}\simeq.320$ and the other uses thresholds $x_3\simeq.288$ and $x_4=1/e$ with approximate probabilities .615 and .385. The payoffs are approximately .291 for the former and .288 for the latter.

It will be of some interest to compare our Model C with "the secretary problem with two choices" considered by Gilbert & Mosteller [1]. Let $S(x,y)$ be the probability that either of the two companies succeeds in employing the best girl when they use x- and y-threshold strategies respectively. Then

$$S(x,y) = M_1(x,y) + M_2(x,y) = x\int_x^1 \{\frac{1}{t} + \frac{g(t)}{t^2}\}dt.$$

This is independent of y and is maximized by setting $x=e^{-3/2}$, and the maximum S value is equal to $e^{-1}+e^{-3/2}\simeq.591$, which is equal to the value obtained by Gilbert & Mosteller, as is to be expected.

4. Concluding Remarks

We have adopted an elementary way of derivation in this paper following

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Gilbert & Mosteller [1], but the equilibrium points could have been obtained in a slightly different way as follows using the general theory of optimal stopping for a Markov process. As an example, we will briefly illustrate the key idea by deriving an equilibrium solution involving pure strategies only in Model A. It is well-known (see, e.g., [3]) that the limiting version of the secretary problem is an optimal stopping problem for a Markov process with a monotone transition density function

$$p(x,y) = \begin{cases} x/y^2, & \text{if } x < y \\ 0, & \text{if } x \geq y \end{cases}$$

and a continuous reward function $\phi(x)=x$. For such a problem, the optimal stopping time is such that the expected reward from stopping at that time is equal to the expected reward from continuing optimally. Thus, in our particular problem, if x_o and y_o, where $x_o < y_o$, are the levels of the equilibrium strategies of player I and II respectively, then following equations hold.

$$\phi(x_o) = \int_{x_o}^{y_o} p(x_o,t)\phi(t)dt + \frac{1}{2}\int_{y_o}^{1} p(x_o,t)\phi(t)dt,$$

$$\{\int_{x_o}^{y_o} p(x_o,t)dt\}\phi(y_o) + \{1 - \int_{x_o}^{y_o} p(x_o,t)dt\}\frac{\phi(y_o)}{2} = \int_{y_o}^{1} p(y_o,t)\phi(t)dt.$$

From these we obtain a set of equations which is equivalent to (2.3).

Many extensions of the present work suggest themselves. For example, one would be interested in the result for the problem with more than two employers. It is expected, and is confirmed by a preliminary computation, that the equilibrium solution for this problem is such that all the employers use the different threshold strategies. Problems with other reward functions are also of some interest and worth investigating. These problems will be discussed in a forthcoming paper.

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