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SOME REMARKS ON SEQUENTIAL ALLOCATION PROBLEM OVER UNKNOWN NUMBER OF PERIODS

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Abstract A decision maker will allocate his resources to appearing targets during a random number fo periods. There are finite types of targets. Each type of them appears with some fixed probability at the beginning of each period. When the target appears, he expends some of his resources to obtain the reward which depends on the number of resources expended and the type of appearing target. The objective is to find a sequence of number of resources to be expended which maximizes the total expected reward. The case of a given horizon was discussed in the previous paper [9]. It will be shown that the similar structure of an optimal policy also holds for the case of random horizon if a reasonable assumption is satisfied.

1. Introduction

A decision maker will allocate his resources to appearing targets during a random number of periods. The target appears one by one at the beginning of each period with some fixed probability. The decision maker acquires the expected reward which depends on the number of resources expended and the appearing target. How does he allocate his resources to maximize the total expected reward?

Sequential allocation problems described above are studied by many authors [4,6,7,9,10] ([6,10] dealt with the continuous-time version) if the planning horizon is fixed. In [2(Sections 5 and 6),8] optimal stopping problems are considered when the planning horizon is random. In [3] a renewal decision problem is considered for a fixed horizon case, and Derman and Smith [5] extends it to a random horizon case.

In the present paper, the sequential allocation problem described above is investigated. It is shown that the similar structure of an optimal policy to that of [9] also holds for a random horizon case under the assumption that Pr{planning horizon$\geq n$+1|planning horizon$\geq n$} is nonincreasing with respect to n. It is noted that the assumption is satisfied in the following cases:

(i) A finite horizon case considered in [9] such that $q_1=\cdots=q_L=1$, and $q_{L+1}=q_{L+2}=\cdots=0$ for some L.

(ii) A finite horizon case with a discount factor α ($0<\alpha<1$) such that $q_1=\cdots=q_L=\alpha$, and $q_{L+1}=q_{L+2}=\cdots=0$ for some L.

(iii) An infinite horizon case with a discount factor α ($0<\alpha<1$) such that $q_1=a_2=\cdots=\alpha$.

Thus, the model proposed in the paper is often found in practice. For example, as in [9] suppose that a whaler with some harpoons voyages to catch whales. The whales appear one by one at each day with a certain probability and are classified into finite types in terms of their size etc.. The events such as failure of a whaling gun force the decision maker to stop his mission time. Further suppose that the longer the elapsed periods, the larger the probability of forced stopping. Then, the model discussed in the paper may be applicable to this situation.

An outline of the paper is as follows: In Section 2 the problem is formulated into a dynamic programming one. In Section 3 the structure of an optimal policy is investigated. In Section 4 simple numerical examples are presented.

2. Model and Formulation

The decision maker will allocate M units of his resources to appearing targets during a random number N of periods. There are I types of targets. The target appears one by one at the beginning of each period, where the type i target appears with probability r_i (i=1,...,I), and no target with probability r_0 ($r_0+\cdots+r_I=1$). When he expends j units from the resources on hand to the type i target, he obtains the immediate expected reward $R_i(j)$ (i=1,...,I), where $R_i(j)$ is assumed to be a nondecreasing concave function of j and $R_i(0)=0$. The probability distribution of N is given by $\Pr\{N=n\}=p_n$, $\sum_{n=1}^{\infty}p_n=1$. It is assumed that $q_n=\Pr\{N\geq n+1|N\geq n\}$ (n=1,2,...) is nonincreasing with respect to n. For convenience we take $q_n=0$ if $\Pr\{N\geq n\}=0$. This assumption seems to correspond with that of IFR of the distribution of random horizon in [5]. We investigate the structure of an optimal policy which maximizes the total expected reward by allocating M units of resources to the appearing targets during a random number N of periods.

Define the following notations in order to formulate the problem into a dynamic programming one [1]:

$$S=\{(n,m;i);n=1,2,\ldots,\ m=0,\ldots,M,\ \text{and}\ i=0,\ldots,I\}\cup\{\delta\},$$

$$C=\{0,\ldots,M\},$$

the notation of $(n,m;i)$ denotes the state where the decision maker has m units of resources on hand and the type i target is appearing at the n th period given that $N\geq n$ (appearance of type 0 target means no appearance of target). The symbol δ denotes the termination of planning horizon, and $j\in C$ the expenditure of j units of resources. Then the reward function g and the transition probability p are derived as follows:

$$g(u,j,\cdot)=\begin{cases}0 & \text{for } u=(n,m;0) \text{ or } u=\delta\\ R_i(\min(j,m)) & \text{for } u=(n,m;i)(i\neq 0)\end{cases}$$

$$p_{uv}(j)=\begin{cases}1 & \text{for } u=v=\delta,\\ 1-q_n & \text{for } u\neq\delta \text{ and } v=\delta,\\ r_iq_n & \text{for } u=(n,m;0) \text{ and } v=(n+1,m;i),\\ r_iq_n & \text{for } u=(n,m;i)(i\neq 0) \text{ and } v=(n+1,(m-j)^+;i),\\ 0 & \text{otherwise.}\end{cases}$$

Let $V(s)(s\in S)$ denote the supremum of total expected reward. It is well known that V satisfies the following optimality equation since $g(\cdot,\cdot,\cdot)\geq 0$ [for example, 1 (we treat a maximizing problem instead of minimizing)]:

(1) $$V(n,m;i)=\max_{j=0,\ldots,M}\{R_i(\min(j,m))+(1-q_n)V(\delta)+r_0q_nV(n+1,(m-j)^+;0)+q_n\textstyle\sum_{i=1}^{I}r_iV(n+1,(m-j)^+;i)\}$$

for $n=1,2,\ldots,m=0,\ldots,M,\ i=1,\ldots,I,$

(2) $$V(n,m;0)=(1-q_n)V(\delta)+q_n\textstyle\sum_{i=0}^{I}r_iV(n+1,m;i)\quad \text{for } n=1,2,\ldots,\ m=0,\ldots,M.$$

Since $V(\delta)=0$, (1) and (2) are rewritten as

(3) $$V(n,m;i)=\max_{j=0,\ldots,m}\{R_i(j)+q_n(r_0V(n+1,m-j;0)+\textstyle\sum_{i=1}^{I}r_iV(n+1,m-j;i))\}$$

for $n=1,2,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I,$

(4) $$V(n,m;0)=q_n\textstyle\sum_{i=0}^{I}r_iV(n+1,m;i)\quad \text{for } n=1,2,\ldots,\ m=0,\ldots,M.$$

There may not exist an optimal policy in our undiscounted reward problem. But the problem can be reduced to a discounted reward one under our assumption that $\{q_n\}$ is a nonincreasing sequence. Since $\{q_n\}$ is a nonincreasing sequence, there exist n_0 and $\alpha(0<\alpha<1)$ such that $q_n\leq\alpha<1$ for each $n\geq n_0$. Consider the related problem with $S'=\{(n,m;i);n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ \text{and}\ i=0,\ldots,I\}\cup\{\delta\}$, $C'=C$, $g'=g$, and

$$p'_{uv}(j)=\begin{cases}1 & \text{for } u=v=\delta,\\ 1-q'_n & \text{for } u\neq\delta \text{ and } v=\delta,\\ r_i q'_n & \text{for } u=(n,m;0) \text{ and } v=(n+1,m;i),\\ r_i q'_n & \text{for } u=(n,m;i)(i\neq 0) \text{ and } v=(n+1,(m-j)^+;i),\\ 0 & \text{otherwise,}\end{cases}$$

where $q'_n=q_n/\alpha\leq 1$. Let $V'(s)(s\in S')$ denote the supremum of total expected discounted reward of the related problem, then V' satisfies the following optimality equation and $V(s)=V'(s)$ on $s\in S'$:

(5) $$V'(n,m;i)=\max_{j=0,\ldots,m}\{R_i(j)+\alpha q'_n(r_0V'(n+1,m-j;0)+\textstyle\sum_{i=1}^{I}r_iV'(n+1,m-j;i))\}$$
for $n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I,$

(6) $$V'(n,m;0)=\alpha q'_n\textstyle\sum_{i=0}^{I}r_iV'(n+1,m;i) \text{ for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M.$$

To solve the above optimality equation, the following successive approximation method with any bounded terminal reward is applicable from the presence of the discount factor. Define $V^L(n,m;i)$ for $L=1,2,\ldots$, $n=n_0,n_0+1,\ldots$, $m=0,\ldots,M$, $i=0,\ldots,I$ as follows:

(7) $$V^1(n,m;i)=\text{ any bounded function for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ i=0,\ldots,I.$$

For $L=1,2,\ldots$, we set

(8) $$V^{L+1}(n,m;i)=\max_{j=0,\ldots,m}\{R_i(j)+q_n(r_0V^L(n+1,m-j;0)+\textstyle\sum_{i=1}^{I}r_iV^L(n+1,m-j;i))\}$$
for $n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I,$

(9) $$V^{L+1}(n,m;0)=q_n\textstyle\sum_{i=0}^{I}r_iV^L(n+1,m;i) \text{ for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M.$$

For the notational convenience, let $\bar{V}(n,m)=r_0V(n,m;0)+\sum_{i=1}^{I}r_iV(n,m;i)$ and $\bar{V}^L(n,m)=r_0V^L(n,m;0)+\sum_{i=1}^{I}r_iV^L(n,m;i)$, then (3) and (4) are rewritten as

(3') $$V(n,m;i)=\max_{j=0,\ldots,m}\{R_i(j)+q_n\bar{V}(n+1,m-j)\} \text{ for } n=1,2,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I.$$

(4') $$\bar{V}(n,m)=r_0q_n\bar{V}(n+1,m)+\textstyle\sum_{i=1}^{I}r_iV(n,m;i) \text{ for } n=1,2,\ldots,\ m=0,\ldots,M,$$

and (7), (8), and (9) as

(7') $$\bar{V}^1(n,m)=\text{ any bounded function for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,$$

(8') $$V^{L+1}(n,m;i)=\max_{j=0,\ldots,m}\{R_i(j)+q_n\bar{V}^L(n+1,m-j)\} \text{ for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I,\ L=1,2,\ldots,$$

(9') $$\bar{V}^{L+1}(n,m)=r_0q_n\bar{V}^L(n+1,m)+\textstyle\sum_{i=1}^{I}r_iV^{L+1}(n,m;i) \text{ for } n=n_0,n_0+1,\ldots,\ m=0,\ldots,M,\ L=1,2,\ldots.$$

Lemma 1. $V(n,m;i)=\lim_{L\to\infty} V^L(n,m;i)$ and $\bar{V}(n,m)=\lim_{L\to\infty} \bar{V}^L(n,m)$ for $n=n_0,n_0+1,\ldots$, $m=0,\ldots,M$, $i=1,\ldots,I$. and $V(\bar{V})$ is the unique bounded solution of (3') and (4') for $n=1,2,\ldots$, $m=0,\ldots,M$, $i=1,\ldots,I$.

Proof: It is well known that $V(\bar{V})$ is the unique bounded solution of (3') and (4') for $n=n_0,n_0+1,\ldots$, $m=0,\ldots,M$, $i=1,\ldots,I$. Using $\bar{V}(n_0,\cdot)$, (3'), and (4'), $V(\bar{V})$ is uniquely computed for $n=1,\ldots,n_0-1$, $m=0,\ldots,M$, $i=1,\ldots,I$.

Q.E.D.

Since $V(\bar{V})$ is the unique bounded solution of (3') and (4'), it is optimal to select an action maximizing the right side of (3') by Corollary 9.2 of Section 6.4 in [1].

3. Structure of Optimal Policy

In this section the structure of an optimal policy is investigated. The values j that maximize the braces of the right side of (3) are the optimal numbers to be expended for the type i target. In order to determine one of the optimal policies we define $k(n,m;i)$ to be the smallest value of j that maximizes the braces of the right side of (3), that is,

$$k(n,m;i)=\min[t\,|\max_{j=0,\ldots,m}\{R_i(j)+q_n\bar{V}(n+1,m-j)\}=R_i(t)+q_n\bar{V}(n+1,m-t)]$$
$$\text{for } n=1,2,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I.$$

Then, it is the optimal policy that allocates $k(n,m;i)$ to the type i target when there are m units of resources on hand and the type i target is appearing at the n th period given that $N\geq n$.

First we show the monotonicity of $k(n,m;i)$ with respect to m. We need the following lemma.

Lemma 2. $V(n,m;i)$ and $\bar{V}(n,m)$ are nondecreasing and concave functions of m for $n=1,2,\ldots$, $i=1,\ldots,I$.

Proof: Let $\bar{V}^1(n,m)=0$ in (7'),(8'), and (9'). Then an induction argument shows that $V^L(n,m;i)$ and $\bar{V}^L(n,m)$ are nondecreasing and concave functions of m for $L=1,2,\ldots$, $n=n_0,n_0+1,\ldots$, $i=1,\ldots,I$. Therefore, so are $V(n,m;i)$ and $\bar{V}(n,m)$ for $n=1,2,\ldots$, $i=1,\ldots,I$ by Lemma 1. Q.E.D.

We can prove the following theorem (see Proposition 2 and Theorem 3 (ii) in [4] for proof), which shows the monotonicity of $k(n,m;i)$ with respect to m.

Theorem 1. $k(n,m;i)\leq k(n,m+1;i)\leq k(n,m;i)+1$ for $n=1,2,\ldots$, $m=0,\ldots,M-1$, $i=1,\ldots,I$.

Next we show the monotonicity of $k(n,m;i)$ with respect to n. To this end it is convenient to regard $\{q_n\}$ as variables. Let $\mathrm{q}_n=(q_n,q_{n+1},\ldots)$, then (3') and (4') are rewritten as

(3") $$V(n,m;i,\mathrm{q}_n)=\max_{j=0,\ldots,m}\{R_i(j)+q_n\bar{V}(n+1,m-j,\mathrm{q}_{n+1})\} \text{ for } n=1,2,\ldots,\ m=0,\ldots,M,\ i=1,\ldots,I,$$

(4") $$\bar{V}(n,m,\mathrm{q}_n)=r_0q_n\bar{V}(n+1,m,\mathrm{q}_{n+1})+\textstyle\sum_{i=1}^{I}r_iV(n,m;i,\mathrm{q}_n) \text{ for } n=1,2,\ldots,\ m=0,\ldots,M.$$

Theorem 2. $k(n,m;i,\mathrm{q}_n)\geq k(n,m;i,\mathrm{q}'_n)$ for $n=1,2,\ldots$, $m=0,\ldots,M$, $i=1,\ldots,I$, where $\mathrm{q}_n=(q_n,q_{n+1},\ldots)$ and $\mathrm{q}'_n=(q_n+\Delta_n,q_{n+1}+\Delta_{n+1},\ldots)$ with $\Delta_i\geq 0$ for $i=n,n+1,\ldots$.

To prove Theorem 2 the following two lemmas are needed.

Lemma 3. $\bar{V}(n+1,m+1,\mathrm{q}_{n+1})-\bar{V}(n+1,m,\mathrm{q}_{n+1})$ is a nondecreasing function of q_i $(i\geq n+1)$ for $n=1,2,\ldots$, $m=0,\ldots,M-1$.

Proof: $$\bar{V}(n+1,m+1,\mathrm{q}_{n+1})-\bar{V}(n+1,m,\mathrm{q}_{n+1})$$
$$=r_0q_{n+1}(\bar{V}(n+2,m+1,\mathrm{q}_{n+2})-\bar{V}(n+2,m,\mathrm{q}_{n+2}))$$
$$+\textstyle\sum_{i=1}^{I}r_i(V(n+1,m+1;i,\mathrm{q}_{n+1})-V(n+1,m;i,\mathrm{q}_{n+1})).$$

Let $j_0=k(n+1,m;i,\mathrm{q}_{n+1})$, then by Theorem 1 $k(n+1,m+1;i,\mathrm{q}_{n+1})=j_0$ or j_0+1. Therefore

$$V(n+1,m+1;i,\mathrm{q}_{n+1})-V(n+1,m;i,\mathrm{q}_{n+1})$$
$$=\max\{R_i(j_0+1)-R_i(j_0),q_{n+1}(\bar{V}(n+2,m+1-j_0,\mathrm{q}_{n+2})-\bar{V}(n+2,m-j_0,\mathrm{q}_{n+2}))\}.$$

Using an induction argument and the fact that $\bar{V}(\cdot,m,\cdot)$ is a nondecreasing function of m from Lemma 2, it is verified that $\bar{V}(n+1,m+1,\mathrm{q}_{n+1})-\bar{V}(n+1,m,\mathrm{q}_{n+1})$ is a nondecreasing function of q_i $(i\geq n+1)$. Q.E.D.

Let $\mathrm{q}^{(0)}_{n+1}=\mathrm{q}_{n+1}=(q_{n+1},q_{n+2},\ldots)$, $\mathrm{q}^{(1)}_{n+1}=(q_{n+1}+\Delta_{n+1},q_{n+2},q_{n+3},\ldots),\ldots,$ $\mathrm{q}^{(s)}_{n+1}=(q_{n+1}+\Delta_{n+1},\ldots,q_{n+s}+\Delta_{n+s},q_{n+s+1},\ldots),\ldots.$

Lemma 4. $\lim_{s\to\infty}\bar{V}(n+1,m,\mathrm{q}^{(s)}_{n+1})=\bar{V}(n+1,m,\mathrm{q}'_{n+1})$ for $n=1,2,\ldots$, $m=0,\ldots,M$.

Proof: It should be noted that for $n\geq n_0-1$ (n_0 is defined in Section 2) $\bar{V}(n+1,m,\mathrm{q}^{(s)}_{n+1})=\bar{V}^{s+1}(n+1,m,\mathrm{q}^{(s)}_{n+1})=\bar{V}^{s+1}(n+1,m,\mathrm{q}'_{n+1})$ with $\bar{V}^{1}(n+s+1,m)=$ $\bar{V}(n+s+1,m,\mathrm{q}_{n+s+1})$ in (7'),(8'), and (9'). Then

$$\lim_{s\to\infty} \bar{V}(n+1,m,\mathrm{q}_{n+1}^{(s)})= \lim_{s\to\infty} \bar{V}^{s+1}(n+1,m,\ \mathrm{q}_{n+1}')=\bar{V}(n+1,m,\mathrm{q}_{n+1}')$$

by Lemma 1 since $\bar{V}^{1}(n+s+1,m,\mathrm{q}_{n+s+1})=\bar{V}(n+s+1,m,\mathrm{q}_{n+s+1})\leqq M \max_{i=1,\ldots,I} R_i(1)$.

For the case of $1\leqq n\leqq n_0-1$, since

$$\bar{V}(n+1,m,\mathrm{q}_{n+1}^{(s)})=r_0(q_{n+1}+\Delta_{n+1})\bar{V}(n+2,m,\mathrm{q}_{n+2}^{(s-1)})+\textstyle\sum_{i=1}^{I} r_i V(n+1,m;i,\mathrm{q}_{n+1}^{(s)}),$$

$$V(n+1,m;i,\mathrm{q}_{n+1}^{(s)})= \max_{j=0,\ldots,m} \{R_i(j)+(q_{n+1}+\Delta_{n+1})\bar{V}(n+2,m-j,\mathrm{q}_{n+2}^{(s-1)})\},$$

$$\bar{V}(n_0-1,m,\mathrm{q}_{n_0-1}^{(s+n-n_0+2)})=r_0(q_{n_0-1}+\Delta_{n_0-1})\bar{V}(n_0,m,\mathrm{q}_{n_0}^{(s+n-n_0+1)})$$
$$+\textstyle\sum_{i=1}^{I} r_i V(n_0-1,m;i,\mathrm{q}_{n_0-1}^{(s+n-n_0+2)})$$

$$V(n_0-1,m;i,\mathrm{q}_{n_0-1}^{(s+n-n_0+2)})= \max_{j=0,\ldots,m} \{R_i(j)+(q_{n_0-1}+\Delta_{n_0-1})\cdot \bar{V}(n_0,m-j,\mathrm{q}_{n_0}^{(s+n-n_0+1)})\},$$

and $\lim_{s\to\infty} \bar{V}(n_0,m,\mathrm{q}_{n_0}^{(s+n-n_0+1)})=\bar{V}(n_0,m,\mathrm{q}_{n_0}')$, it follows that $\lim_{s\to\infty} \bar{V}(n+1,m,\mathrm{q}_{n+1}^{(s)})=$ $\bar{V}(n+1,m,\mathrm{q}_{n+1}')$, Q.E.D.

Proof of Theorem 2: Let $j_0=k(n,m;i,\mathrm{q}_n)$, then

$$R_i(j_0)+q_n\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1})\geqq R_i(j)+q_n\bar{V}(n+1,m-j,\mathrm{q}_{n+1}) \text{ for } j_0\leqq j\leqq m.$$

This yields

$$R_i(j)-R_i(j_0)\leqq q_n(\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1})-\bar{V}(n+1,m-j,\mathrm{q}_{n+1}))$$
$$\leqq(q_n+\Delta_n)(\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1})-\bar{V}(n+1,m-j,\mathrm{q}_{n+1}))$$
$$\leqq(q_n+\Delta_n)(\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1}^{(1)})-\bar{V}(n+1,m-j,\mathrm{q}_{n+1}^{(1)}))$$

$$\leqq(q_n+\Delta_n)(\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1}^{(s)})-\bar{V}(n+1,m-j,\mathrm{q}_{n+1}^{(s)})) \text{ for } j_0\leqq j\leqq m,$$

where we use the monotonicity of $\bar{V}(n+1,\cdot,\mathrm{q}_{n+1})$ and Lemma 3. Letting $s\to\infty$ in the above relation, we obtain, from Lemma 4,

$$R_i(j)-R_i(j_0)\leqq(q_n+\Delta_n)(\bar{V}(n+1,m-j_0,\mathrm{q}_{n+1}')-\bar{V}(n+1,m-j,\mathrm{q}_{n+1}')) \text{ for } j_0\leqq j\leqq m.$$

This implies $j_0=k(n,m;i,\mathrm{q}_n)\geqq k(n,m;i,\mathrm{q}_n')$. Q.E.D.

Corollary. Under the assumption that $q_1\geqq q_2\geqq\cdots$,

$$k(n,m;i)\leqq k(n+1,m;i) \text{ for } n=1,2,\ldots,\ m=0,\ldots,M,\ i=1,\ldots I.$$

Proof: It should be noted that $k(n+1,m;i)=k(n+1,m;i,\mathrm{q}_{n+1})=k(n,m;i,\mathrm{q}_{n+1})$ with $\mathrm{q}_{n+1}=(q_{n+1},q_{n+2},\ldots)$, and $k(n,m;i)=k(n,m;i,\mathrm{q}_n)$ with $\mathrm{q}_n=(q_n,q_{n+1},\ldots)$. Since $q_1\geqq q_2\geqq\cdots$, Corollary follows from Theorem 2. Q.E.D.

4. Numerical Examples

This section presents two simple numerical examples. One of them deals with the case of $q_1=q_2=\cdots=\alpha$, $0<\alpha<1$, and the other with the case of $\Pr\{N\leq n_1\}=1$ for some n_1.

4.1. The case of $q_1=q_2=\cdots=\alpha$, $0<\alpha<1$

Suppose that $q_1=q_2=\cdots=\alpha=0.8$, I=2, M=5, and $(r_0,r_1,r_2)=(0.4, 0.5, 0.1)$. Also $R_i(j)$ is defined in Table 1. Since q_n does not depend on n, it is unnecessary to keep the length of elapsed periods in mind. Then, the optimality equation becomes

$$V(m;i)=\max_{j=0,\ldots,m}\{R_i(j)+\alpha\bar{V}(m-j)\} \text{ for } m=0,\ldots,\text{M},\ i=1,2,$$

$$\bar{V}(m)=r_0\alpha\bar{V}(m)+\sum_{i=1}^{2}r_iV(m;i) \text{ for } m=0,\ldots,\text{M}.$$

A simple calculation yields the following optimal policy in Table 2. For example, if there are 4 units of resources on hand and the type 2 target is appearing, then it is optimal to allocate 3 units of resources. Note that $k(m;i)$ $(i=1,2)$ is nondecreasing with respect to m, which is in agreement with Theorem 1.

Table 1. $R_i(j)$

j	0	1	2	3	4	5
$R_1(j)$	0	1	1.8	2.4	2.8	3.0
$R_2(j)$	0	2	3.5	4.9	5.2	5.4

Table 2. $k(m;1)$ and $k(m;2)$

m	1	2	3	4	5
$k(m;1)$	1	1	2	2	2
$k(m;2)$	1	2	3	3	3

4.2. The case of $\Pr\{N\leq n_1\}=1$ for some n_1

Consider example 4.1 with $q_1=0.9$, $q_2=0.8$, $q_3=0.5$, $q_4=0.2$, $q_5=0.1$, and $q_6=0$ instead of $q_1=q_2=\cdots=\alpha$. Then a simple calculation gives the following optimal policy listed in Table 3 and 4. Note again that $k(n,m;i)$ $(i=1,2)$ is nondecreasing with respect to m and n, which is also in agreement with Theorem 1 and Corollary.

Table 3. $k(n,m;1)$

n \ m	1	2	3	4	5
1	1	1	2	2	3
2	1	2	2	3	3
3	1	2	3	4	4
4	1	2	3	4	5
5	1	2	3	4	5
6	1	2	3	4	5

Table 4. $k(n,m;2)$

n \ m	1	2	3	4	5
1	1	2	3	3	3
2	1	2	3	3	3
3	1	2	3	3	4
4	1	2	3	4	5
5	1	2	3	4	5
6	1	2	3	4	5

Conclusion

In this paper, we considered the discrete-time random horizon sequential allocation problem with discrete resource. In Section 3 we derived the structure of an optimal policy: The optimal number to be expended for the target is a nondecreasing function of the number of resources on hand and a nondecreasing function of the number of elapsed periods under a certain assumption. This result is a generalization of that of a fixed horizon case.

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