

GENERALIZED MODELS FOR DETERMINING OPTIMAL NUMBER OF MINIMAL REPAIRS BEFORE REPLACEMENT

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Abstract This paper summarizes the results of the basic model of Makabe and Morimura, in which a unit is replaced at k -th failure. The two generalized models are considered: A) A unit has two types of failures and is replaced at type 2 failure or k -th type 1 failure. B) A system has two types of units and is replaced at unit 2 failure or k -th unit 1 failure. The expected cost rates for each model are obtained, using the calculus method of probability. It is shown that the optimal numbers k^* which minimize the cost rates are given by unique solutions of equations under certain conditions.

1. Introduction

Makabe and Morimura [4, 5, 6] proposed the new replacement where a unit is replaced at k -th failure, and discussed the optimum policy. Morimura [7] also considered several extended models. This is a modification of periodic replacement model introduced by Barlow and Hunter [1], in which a unit is replaced at time T and undergoes only minimal repair at failures between periodic replacements. This policy is useful in the case where the total operating time of a unit is not recorded or it is much time and costly to replace a unit in operation. Therefore, this could be used in maintaining a complex system with many equipments of the same type.

This paper summarizes the known results of the basic model of Makabe and Morimura and makes additional discussions. Further, the following two generalized models are considered:

- A) A unit has two types of failures when it fails: Type 1 failure occurs with probability α and is corrected with minimal repair, and whereas type 2 failure occurs with probability $1 - \alpha$ and a unit has to be replaced. If the k -th type 1 failure occurs before type 2 failure, then a unit is replaced

preventively.

B) A system has two types of units: When unit 1 fails, it undergoes minimal repair, and when unit 2 fails, a system has to be replaced. If unit 1 fails at k times before unit 2 failure, then a system is replaced preventively.

We derive the expected cost rates for each model and obtain the optimal numbers k^* to minimize the cost rates, when the hazard rate is monotone increasing. Other useful discussions of results are further made.

2. Basic Model

The unit is replaced at the time of k -th failure ($k = 1, 2, \dots$) after its installation and undergoes only minimal repair at failures between replacements. Assume that the unit has a failure time distribution $F(t)$ with finite mean μ and has a density $f(t)$. Then, the hazard rate (or the failure rate) is $r(t) \equiv f(t)/\bar{F}(t)$ and the cumulative hazard is $R(t) \equiv \int_0^t r(u)du$, which has a relation $\bar{F}(t) = \exp[-R(t)]$, where $\bar{F}(t) \equiv 1 - F(t)$. It is further assumed that the hazard rate $r(t)$ is continuous, monotone increasing, and remains undisturbed by minimal repair. Thus, there exists the limit of $r(\infty) \equiv \lim_{t \rightarrow \infty} r(t)$, which may be possibly infinity.

Let c_1 be the cost of minimal repair and c_2 be the cost of replacement. Then, if the times for repair and replacement are negligible, then the expected cost rate is, from [4],

$$(1) \quad C(k) = \frac{(k-1)c_1 + c_2}{\sum_{j=0}^{\infty} \int_0^{\infty} p_j(t) dt} \quad (k = 1, 2, \dots),$$

where $p_j(t) \equiv \{[R(t)]^j / j!\} e^{-R(t)}$ ($j = 0, 1, \dots$), which represents the probability that j failures occur in the interval $[0, t]$.

We seek the optimal number k^* which minimizes $C(k)$ in (1) when $r(t)$ is continuous and monotone increasing.

Theorem 1. If there exists a solution k^* which satisfies

$$(2) \quad L(k) \geq c_2/c_1 \text{ and } L(k-1) < c_2/c_1 \quad (k = 1, 2, \dots),$$

it is unique and minimizes $C(k)$, where

$$(3) \quad L(k) \equiv \begin{cases} \frac{\sum_{j=0}^{k-1} \int_0^{\infty} p_j(t) dt}{\int_0^{\infty} p_k(t) dt} - (k-1) & (k = 1, 2, \dots), \\ 0 & (k = 0). \end{cases}$$

Proof: We suppose that $C(0) = \infty$ for simplicity. Then, a necessary condition that there exists a finite k^* minimizing $C(k)$ is that a k^* satisfies $C(k+1) \geq C(k)$ and $C(k) < C(k-1)$ ($k = 1, 2, \dots$). Thus, these inequalities give (2). Further,

$$L(k+1) - L(k) = \sum_{j=0}^k \int_0^\infty p_j(t) dt [1/\int_0^\infty p_{k+1}(t) dt - 1/\int_0^\infty p_k(t) dt] > 0,$$

since $\int_0^\infty p_k(t) dt$ is a decreasing function of k from Lemma 5 of [7, p. 103] or (i) of Appendix, when $r(t)$ is monotone increasing. Thus, a solution to (2) must be unique.

We may compute only a minimum k^* such that $L(k) \geq c_2/c_1$ if it exists and

$$(4) \quad \frac{c_1}{\int_0^\infty p_{k^*-1}(t) dt} < C(k^*) \leq \frac{c_1}{\int_0^\infty p_{k^*}(t) dt}.$$

Further, it is easily seen that $L(1) > 1$. Thus, if $c_1 \geq c_2$ then $k^* = 1$, viz., the unit should be replaced at the first failure.

From (i) of Appendix, we also have that $\lim_{k \rightarrow \infty} \int_0^\infty p_k(t) dt = 1/r(\infty)$, where $1/r(\infty) = 0$ whenever $r(\infty) = \infty$. Thus, we can give the following upper limit of k^* :

Theorem 2. Suppose that $r(\infty) > c_2/(\mu c_1)$. Then, there exists a unique minimum \bar{k} which satisfies

$$(5) \quad \int_0^\infty p_k(t) dt \leq \mu(c_1/c_2) \quad (k = 1, 2, \dots),$$

and $k^* \leq \bar{k}$.

Proof: We easily have

$$(6) \quad L(k) \geq \mu/\int_0^\infty p_k(t) dt \quad (k = 1, 2, \dots),$$

since $\int_0^\infty p_k(t) dt$ is decreasing in k . Further, from the assumption that $r(\infty) > c_2/(\mu c_1)$, there exists a unique minimum \bar{k} such that $\mu/\int_0^\infty p_k(t) dt \geq c_2/c_1$, i.e., $\int_0^\infty p_k(t) dt \leq \mu(c_1/c_2)$. Hence, from the inequality of (6), we have $k^* \leq \bar{k}$.

If $r(\infty) > c_2/(\mu c_1)$ then a solution to (2) also exists. For example, suppose that the failure time has a Weibull distribution, i.e., $\bar{F}(t) = \exp(-t^\beta)$ for $\beta > 1$. Then, $r(\infty) = \infty$ and from (6),

$$\int_0^{\infty} p_k(t) dt = \frac{1}{\beta} \frac{\Gamma(k+1/\beta)}{\Gamma(k+1)},$$

$$\sum_{j=0}^{k-1} \int_0^{\infty} p_j(t) dt = \frac{\Gamma(k+1/\beta)}{\Gamma(k)}.$$

Thus, there exists a unique k^* which minimizes $C(k)$ and it is, from (2),

$$(7) \quad k^* = \left[\frac{1}{\beta - 1} \left(\frac{c_2}{c_1} - 1 \right) \right] + 1,$$

where $[x]$ denotes the greatest integer contained in x .

Until now, we have assumed that the times for repair and replacement are negligible. In reality, it requires some time to make a repair or a replacement, and probably, it may be much smaller than an operating time of the unit. Let T_1 be the random variable denoting the repair time and T_2 denoting the replacement time. Further, let $c_1(T_1)$ be the repair cost which includes all costs incurred due to repair and unit failure, and $c_2(T_2)$ be the replacement cost. Then, in a similar way of obtaining (1), we easily have

$$(8) \quad C(k) = \frac{(k-1)E[c_1(T_1)] + E[c_2(T_2)]}{\sum_{j=0}^{k-1} \int_0^{\infty} p_j(t) dt + (k-1)\mu_1 + \mu_2} \quad (k = 1, 2, \dots),$$

where $\mu_i \equiv E[T_i]$ ($i = 1, 2$). When $c_1(T_1) = c_1 + c_0 T_1$ and $c_2(T_2) = c_2 + c_0 T_2$, $C(k)$ agrees with the equation (3.2) of [7]. Further, when $c_1(T_1) = T_1$ and $c_2(T_2) = T_2$, $1 - C(k)$ represents the limiting efficiency, i.e., the steady-state availability, which is given by (2.3) of [7].

In this case, Theorems 1 & 2 are rewritten as:

Theorem 1'. If there exists a solution k^* which satisfies

$$(9) \quad L(k) \geq E[c_2(T_2)]/E[c_1(T_1)] \text{ and } L(k-1) < E[c_2(T_2)]/E[c_1(T_1)] \\ (k = 1, 2, \dots),$$

it is unique and minimizes $C(k)$, where

$$(10) \quad L(k) \equiv \begin{cases} \frac{\sum_{j=0}^{k-1} \int_0^{\infty} p_j(t) dt + (k-1)\mu_1 + \mu_2}{\int_0^{\infty} p_k(t) dt + \mu_1} - (k-1) & (k = 1, 2, \dots) \\ 0 & (k = 0). \end{cases}$$

Theorem 2'. Suppose that $1/r(\infty) < (\mu + \mu_2)\{E[c_1(T_1)]/E[c_2(T_2)]\} - \mu_1$. Then, there exists a unique minimum \bar{k} which satisfies

$$(11) \quad \int_0^\infty p_k(t) dt \leq (\mu + \mu_2) \{E[c_1(T_1)]/E[c_2(T_2)]\} - \mu_1,$$

and $k^* \leq \bar{k}$.

Theorems 1' & 2' are easily proved because $L(k)$ is monotone increasing in k and

$$(12) \quad L(k) \geq \frac{\mu + \mu_2}{\int_0^\infty p_k(t) dt + \mu_1}.$$

3. Two Types of Failures

We consider the unit with two types of failures [3]. When the unit fails, type 1 failure occurs with probability α and is removed by minimal repair, and type 2 failure occurs with probability $1 - \alpha$ and is removed by replacement. Type 1 failure is a minor failure which is easily restored to the same operating state before failure by minimal repair, and whereas type 2 failure incurs a total breakdown.

The unit is replaced at the times of type 2 failure or k -th type 1 failure, whichever occurs first. Then, the expected number of minimal repairs (i.e., type 1 failures) before replacement is

$$(13) \quad (k - 1)\alpha^k + \sum_{j=1}^k (j - 1)\alpha^{j-1} (1 - \alpha) = (\alpha - \alpha^k)/(1 - \alpha).$$

Thus, the expected cost rate is easily given by

$$(14) \quad C(k; \alpha) = \frac{c_1 [(\alpha - \alpha^k)/(1 - \alpha)] + c_2}{\sum_{j=0}^{k-1} \alpha^j \int_0^\infty p_j(t) dt} \quad (k = 1, 2, \dots).$$

When $\alpha = 1$, $C(k; 1)$ is equal to (1) and the optimal policy is discussed in Section 2. When $\alpha = 0$, $C(k; 0) = c_2/\mu$, which is constant for all k , and hence, the unit is replaced only at type 2 failure. Therefore, we need only discuss the optimal policy in case of $0 < \alpha < 1$. To simplify equations, we denote $\mu_a \equiv \int_0^\infty [\bar{F}(t)]^a dt$. When $a = 1$, $\mu_\alpha = \mu$ which is the mean time to failure of the unit.

Theorem 3.

(i) Suppose that $0 < \alpha < 1$ and $r(\infty) > [\alpha c_1 + (1 - \alpha)c_2]/(\mu_{1-\alpha} c_1)$. Then, there exists a finite and unique $k^*(\alpha)$ which satisfies

$$(15) \quad L(k; \alpha) \geq c_2/c_1 \text{ and } L(k-1; \alpha) < c_2/c_1 \quad (k = 1, 2, \dots),$$

where

$$(16) \quad L(k; \alpha) \equiv \begin{cases} \frac{\sum_{j=0}^{k-1} \alpha^j \int_0^\infty P_j(t) dt}{\int_0^\infty P_k(t) dt} - \frac{\alpha - \alpha^k}{1 - \alpha} & (k = 1, 2, \dots), \\ 0 & (k = 0). \end{cases}$$

(ii) If $r(\infty) \leq [\alpha c_1 + (1 - \alpha)c_2]/(\mu_{1-\alpha} c_1)$ then $k^*(\alpha) \rightarrow \infty$, and the resulting expected cost is

$$(17) \quad \lim_{k \rightarrow \infty} C(k; \alpha) = \frac{c_1 [\alpha/(1 - \alpha)] + c_2}{\mu_{1-\alpha}}.$$

Proof: The inequalities $C(k+1; \alpha) \geq C(k; \alpha)$ and $C(k; \alpha) < C(k-1; \alpha)$ imply (15). Further, it is easily seen that $L(k; \alpha)$ is an increasing function of k , and hence, $\lim_{k \rightarrow \infty} L(k; \alpha) = \mu_{1-\alpha} r(\infty) - [\alpha/(1 - \alpha)]$. Thus, in a similar way of obtaining Theorem 1, if $r(\infty) > [\alpha c_1 + (1 - \alpha)c_2]/(\mu_{1-\alpha} c_1)$ then a solution to (15) exists and it is unique from the monotonicity of $L(k; \alpha)$. On the other hand, if $r(\infty) \leq [\alpha c_1 + (1 - \alpha)c_2]/(\mu_{1-\alpha} c_1)$ then $L(k; \alpha) < c_2/c_1$ for all k , and hence, $k^* \rightarrow \infty$, and from (14), we have (17).

If $c_1 \geq c_2$ then $k^*(\alpha) = 1$ since $L(1; \alpha) > c_2/c_1$. Further, note that if $r(t)$ is monotone increasing to infinity then a finite $k^*(\alpha)$ exists uniquely.

It is easily seen that $dL(k; \alpha)/d\alpha > 0$ for all k . Thus, we have the following theorem:

Theorem 4. If $r(\infty) > [\alpha c_1 + (1 - \alpha)c_2]/(\mu_{1-\alpha} c_1)$ for $0 \leq \alpha < 1$ then $k^*(\alpha)$ is decreasing in α , and hence, $\bar{k} \geq k^*(\alpha) \geq k^*$, where both k^* and \bar{k} exist and are given in (2) and (5), respectively.

4. Two Types of Units

Consider a system with two types of units which operate statistical independently. When unit 1 fails, it undergoes minimal repair instantaneously and begins to operate again. When unit 2 fails, the system is replaced without repairing unit 2. Unit 1 has a failure time distribution $F(t)$, the hazard rate $r(t)$ and the cumulative hazard $R(t)$, which have the same assumptions as the basic model. Unit 2 has a failure time distribution $G(t)$ with finite mean λ .

Suppose that the system is replaced at the times of unit 2 failure or k -th unit 1 failure, whichever occurs first. Then, the mean time to replacement is

$$(18) \quad \sum_{j=1}^k \int_0^\infty t p_{j-1}(t) dG(t) + \int_0^\infty t \bar{G}(t) p_{k-1}(t) r(t) dt = \sum_{j=0}^{k-1} \int_0^\infty \bar{G}(t) p_j(t) dt,$$

where $\bar{G}(t) \equiv 1 - G(t)$, and the expected number of minimal repairs before replacement is

$$(19) \quad \sum_{j=1}^k (j - 1) \int_0^\infty p_{j-1}(t) dG(t) + (k - 1) \int_0^\infty \bar{G}(t) p_{k-1}(t) r(t) dt \\ = \sum_{j=1}^{k-1} \int_0^\infty \bar{G}(t) p_{j-1}(t) r(t) dt.$$

Thus, the expected cost rate is

$$(20) \quad C(k;G) = \frac{c_1 \sum_{j=0}^{k-2} \int_0^\infty \bar{G}(t) p_j(t) r(t) dt + c_2}{\sum_{j=0}^{k-1} \int_0^\infty \bar{G}(t) p_j(t) dt} \quad (k = 1, 2, \dots).$$

When $\bar{G}(t) = 1$ for $t \geq 0$, $C(k;G)$ is equal to (1), and when $\bar{G}(t) = 1$ for $t \leq T$ and 0 for $t > T$, this corresponds to the model of "Policy IV" of Morimura [7].

Assume that the hazard rate of unit 2 is $h(t) \equiv g(t)/\bar{G}(t)$, where $g(t)$ is a density of $G(t)$.

Theorem 5. Suppose that $h(t)$ is continuous and increasing. Then, if there exists a solution k^* which satisfies

$$(21) \quad L(k;G) \geq c_2/c_1 \text{ and } L(k-1;G) < c_2/c_1 \quad (k = 1, 2, \dots),$$

it is unique and minimizes $C(k;G)$, where

$$(22) \quad L(k;G) \equiv \begin{cases} \frac{\int_0^\infty \bar{G}(t) p_{k-1}(t) r(t) dt}{\sum_{j=0}^{k-1} \int_0^\infty \bar{G}(t) p_j(t) dt} \\ - \sum_{j=0}^{k-2} \int_0^\infty \bar{G}(t) p_j(t) r(t) dt & (k = 1, 2, \dots), \\ 0 & (k = 0). \end{cases}$$

Proof: The inequalities $C(k+1;G) \geq C(k;G)$ and $C(k;G) < C(k-1;G)$ give (21). Further,

$$(23) \quad L(k+1;G) - L(k;G) = \sum_{j=0}^k \int_0^\infty \bar{G}(t) p_j(t) dt \left(\frac{\int_0^\infty \bar{G}(t) p_k(t) r(t) dt}{\int_0^\infty \bar{G}(t) p_{k+1}(t) dt} \right)$$

$$- \left. \frac{\int_0^\infty \bar{G}(t)p_{k-1}(t)r(t)dt}{\int_0^\infty \bar{G}(t)p_k(t)dt} \right\}$$

> 0,

since $\int_0^\infty \bar{G}(t)p_k(t)r(t)dt/\int_0^\infty \bar{G}(t)p_{k+1}(t)dt$ is an increasing function of k from (ii) of Appendix, when $h(t)$ is increasing. Thus, if a solution to (21) exists, it is unique.

Further, we have, from (ii) of Appendix,

$$(24) \quad \lim_{k \rightarrow \infty} L(k;G) = \lambda[r(\infty) + h(\infty)] - \int_0^\infty \bar{G}(t)r(t)dt.$$

Thus, if $r(t) + h(t)$ is monotone increasing and $r(\infty) + h(\infty) > (1/\lambda)[(c_2/c_1) + \int_0^\infty \bar{G}(t)r(t)dt]$ then there exists a finite and unique k^* which satisfies (21). For example, suppose that $h(t)$ is monotone increasing and $F(t)$ is exponential, i.e., $F(t) = 1 - \exp(-t/\mu)$. Then, if $h(\infty) > c_2/(\lambda c_1)$ then a solution to (21) exists uniquely.

In Sections 3 & 4, we have assumed that the replacement costs for the both cases of the k -th type 1 (unit 1) failure and type 2 (unit 2) failure are the same. In reality, it may be different from each other. We suppose that c_2 is the replacement cost of the k -th type 1 (unit 1) failure and c_3 is the replacement cost of the type 2 (unit 2) failure. Then, the expected cost rates in (14) and (20) are rewritten as, respectively,

$$(25) \quad C(k;\alpha) = \frac{c_1[(\alpha - \alpha^k)/(1 - \alpha)] + c_2\alpha^k + c_3(1 - \alpha^k)}{\sum_{j=0}^{k-1} \alpha^j \int_0^\infty p_j(t)dt},$$

$$(26) \quad C(k;G) = \frac{c_1 \sum_{j=0}^{k-2} \int_0^\infty \bar{G}(t)p_j(t)r(t)dt + c_2 \int_0^\infty \bar{G}(t)p_{k-1}(t)r(t)dt + c_3[1 - \int_0^\infty \bar{G}(t)p_{k-1}(t)r(t)dt]}{\sum_{j=0}^{k-1} \int_0^\infty \bar{G}(t)p_j(t)dt}$$

We can discuss the optimal numbers which minimize $C(k;\alpha)$ and $C(k;G)$ by a similar method, although we omit here.

5. Conclusions and Examples

We have considered the basic model, where the unit is replaced at k -th

failure, and the two generalized models which include the basic model as a special case. We have discussed the optimal numbers k^* which minimize the expected cost rates of each model. It has been shown in the theorems that k^* is given by a minimum such that $L(k; \cdot) \geq c_2/c_1$ if it exists.

Finally, we compute the optimal number $k^*(\alpha)$ which minimizes the expected cost $C(k; \alpha)$ in (25) when $\bar{F}(t) = \exp(-t^\beta)$ for $\beta > 1$. When $c_2 = c_3$, it is shown from Theorems 3 and 4 that $k^*(\alpha)$ exists uniquely and is decreasing in α for $0 < \alpha < 1$. Further, when $\alpha = 1$, $k^*(\alpha)$ is given in (7). If $c_1 + (c_3 - c_2)(1 - \alpha) > 0$ then $k^*(\alpha)$ is given by a minimum value such that

$$\frac{(1 - \alpha)\Gamma(k+1)}{\Gamma(k+1/\beta)} \sum_{j=0}^{k-1} \frac{\alpha^j \Gamma(j+1/\beta)}{\Gamma(j+1)} + \alpha^k > \frac{c_1\alpha + c_3(1 - \alpha)}{c_1 + (c_3 - c_2)(1 - \alpha)} .$$

It is easily seen that $k^*(\alpha)$ is small when c_1/c_2 or c_3/c_2 for $c_2 > c_1$ is large. Conversely, if $c_1 + (c_3 - c_2)(1 - \alpha) \leq 0$ then $k^*(\alpha) \rightarrow \infty$.

Table 1 shows the optimal number $k^*(\alpha)$ for the probability α of type 1 failure and the ratio of cost c_3 to cost c_2 when we assume $\beta = 2$ and $c_1/c_2 = 0.1$. It is of great interest that $k^*(\alpha)$ is increasing in α for $c_3 > c_2$, however, is decreasing for $c_3 \leq c_2$. We can explain the reason why $k^*(\alpha)$ is increasing in α for c_3/c_2 : If $c_3 > c_2$ then the cost of replacement for type 1 failure is cheaper than that for type 2 failure and the number of its failures increases with α , and hence, $k^*(\alpha)$ is large when α is large. This situation

Table 1. Variation in the optimal number $k^*(\alpha)$ for the probability α of type 1 failure and the ratio of c_3 to c_2 , where $\beta = 2$ and $c_1/c_2 = 0.1$.

$c_3/c_2 \backslash \alpha$	0.8	0.9	1.0	1.2	1.5	2.0	3.0
0.1	∞	∞	30	6	2	1	1
0.2	∞	∞	27	6	3	1	1
0.3	∞	220	24	6	3	2	1
0.4	∞	112	22	7	3	2	1
0.5	∞	64	20	7	4	2	1
0.6	288	39	17	7	4	2	1
0.7	64	25	15	8	5	3	2
0.8	26	17	13	8	6	4	2
0.9	14	12	11	9	7	5	4
1.0	10	10	10	10	10	10	10

would reflect a real world. On the other hand, if $c_3 \leq c_2$ then it is not good to replace the unit often before type 2 failure, however, the total cost of minimal repairs for type 1 failure increases as the number of its failures does with α . Thus, it may be better to replace the unit preventively at some number k when α is large. Evidently, $\hat{k}^*(\alpha)$ is rapidly increasing when c_1 is small enough.

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Appendix

We show the following two results which are useful for proving the theorems, when the hazard rate $r(t)$ is continuous and monotone increasing:

- (i) $\int_0^\infty p_k(t)dt$ is decreasing in k and $\lim_{k \rightarrow \infty} \int_0^\infty p_k(t)dt = 1/r(\infty)$, where $1/r(\infty) =$

0 whenever $r(\infty) = \infty$.

(ii) If $h(t)$ is continuous and increasing then

$$\frac{\int_0^\infty \frac{[R(t)]^{k-1}}{(k-1)!} \bar{G}(t) f(t) dt}{\int_0^\infty \frac{[R(t)]^k}{k!} \bar{G}(t) \bar{F}(t) dt}$$

is increasing in k and

$$\lim_{k \rightarrow \infty} \frac{\int_0^\infty \frac{[R(t)]^{k-1}}{(k-1)!} \bar{G}(t) f(t) dt}{\int_0^\infty \frac{[R(t)]^k}{k!} \bar{G}(t) \bar{F}(t) dt} = h(\infty) + r(\infty).$$

1. Proof of (i)

Using the relation

$$(A1) \quad \int_0^t \frac{[R(u)]^{k-1}}{(k-1)!} r(u) du = \frac{[R(t)]^k}{k!} \quad (k = 1, 2, \dots),$$

and the assumption that $r(t)$ is monotone increasing, we have

$$(A2) \quad \int_0^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt = \int_0^\infty \left[\int_0^t \frac{[R(u)]^{k-1}}{(k-1)!} r(u) du \right] e^{-R(t)} dt$$

$$< \int_0^\infty \left[\int_0^t \frac{[R(u)]^{k-1}}{(k-1)!} du \right] f(t) dt$$

$$= \int_0^\infty \frac{[R(t)]^{k-1}}{(k-1)!} e^{-R(t)} dt.$$

Thus, $\int_0^\infty p_k(t) dt$ is decreasing in k .

Further,

$$(A3) \quad \int_0^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt = \int_0^\infty \frac{[R(t)]^k}{k!} \frac{f(t)}{r(t)} dt$$

$$\geq \frac{1}{r(\infty)} \int_0^\infty \frac{[R(t)]^k}{k!} f(t) dt = \frac{1}{r(\infty)}.$$

On the other hand, for any $T \in (0, \infty)$,

$$\int_0^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt = \int_0^T \frac{[R(t)]^k}{k!} e^{-R(t)} dt + \int_T^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt$$

$$\leq \int_0^T \frac{[R(t)]^k}{k!} e^{-R(t)} dt + \frac{1}{r(T)}.$$

Thus,

$$(A4) \quad \lim_{k \rightarrow \infty} \int_0^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt \leq \frac{1}{r(T)}.$$

Therefore, combining (A3) and (A4), we have

$$(A5) \quad \lim_{k \rightarrow \infty} \int_0^\infty \frac{[R(t)]^k}{k!} e^{-R(t)} dt = \frac{1}{r(\infty)},$$

since T is arbitrary.

2. Proof of (ii)

Integrating by parts, we have

$$\int_0^\infty \frac{[R(t)]^{k-1}}{(k-1)!} \bar{G}(t) f(t) dt = \int_0^\infty \frac{[R(t)]^k}{k!} \bar{G}(t) f(t) dt + \int_0^\infty \frac{[R(t)]^k}{k!} \bar{F}(t) g(t) dt.$$

First, we show that

$$(A6) \quad \frac{\int_0^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt}$$

is increasing in k when $r(t)$ is monotone increasing. Let

$$q(T) \equiv \int_0^T [R(t)]^{k+1} \bar{G}(t) f(t) dt - \int_0^T [R(t)]^k \bar{G}(t) \bar{F}(t) dt \\ - \int_0^T [R(t)]^k \bar{G}(t) f(t) dt + \int_0^T [R(t)]^{k+1} \bar{G}(t) \bar{F}(t) dt.$$

Then, it is easily seen that

$$(A7) \quad q(0) = 0, \\ q'(T) > 0.$$

Thus, $q(T) > 0$ for all $T > 0$, and hence, (A6) is increasing in k . Similarly,

$$(A8) \quad \frac{\int_0^\infty [R(t)]^k \bar{F}(t) g(t) dt}{\int_0^\infty [R(t)]^k \bar{F}(t) \bar{G}(t) dt}$$

is also increasing, which may be possibly constant. Therefore, from (A6) and

(A8), we have that $\int_0^\infty \frac{[R(t)]^{k-1}}{(k-1)!} \bar{G}(t) f(t) dt / \int_0^\infty \frac{[R(t)]^k}{k!} \bar{G}(t) \bar{F}(t) dt$ is increasing in k .

Next, we show that

$$(A9) \quad \lim_{k \rightarrow \infty} \frac{\int_0^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt} = r(\infty),$$

Evidently,

$$(A10) \quad \frac{\int_0^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt} \leq r(\infty).$$

On the other hand, for any $T \in (0, \infty)$, we have

$$(A11) \quad \frac{\int_0^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt} = \frac{\int_0^T [R(t)]^k \bar{G}(t) f(t) dt + \int_T^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^T [R(t)]^k \bar{G}(t) \bar{F}(t) dt + \int_T^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt}$$

$$> \frac{r(T) \int_T^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt}{\int_0^T [R(t)]^k \bar{G}(t) \bar{F}(t) dt + \int_T^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt}$$

$$= \frac{r(T)}{1 + \{\int_0^T [R(t)]^k \bar{G}(t) \bar{F}(t) dt / \int_T^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt\}}.$$

Further, the bracket of the denominator is, for $T < T_1$,

$$(A12) \quad \frac{\int_0^T [R(t)]^k \bar{G}(t) \bar{F}(t) dt}{\int_T^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt} \leq \frac{[R(T)]^k \int_0^T \bar{G}(t) \bar{F}(t) dt}{\int_{T_1}^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt}$$

$$\leq \frac{\int_0^T \bar{G}(t) \bar{F}(t) dt}{[R(T_1)/R(T)]^k \int_{T_1}^\infty \bar{G}(t) \bar{F}(t) dt} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Thus, from (A10), (A11), and (A12), we have

$$(A13) \quad r(\infty) \geq \lim_{k \rightarrow \infty} \frac{\int_0^\infty [R(t)]^k \bar{G}(t) f(t) dt}{\int_0^\infty [R(t)]^k \bar{G}(t) \bar{F}(t) dt} \geq r(T),$$

which imply (A9) since T is arbitrary. In a similar way,

$$(A14) \quad \lim_{k \rightarrow \infty} \frac{\int_0^\infty [R(t)]^k \bar{F}(t) g(t) dt}{\int_0^\infty [R(t)]^k \bar{F}(t) \bar{G}(t) dt} = h(\infty).$$

Therefore, combining (A9) and (A14), we complete the proof.

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