Journal of the Operations Research
Society of Japan
Vol. 24, No. 4, December 1981

OPTIMAL REPLACEMENT POLICY FOR TWO-UNIT SYSTEM

Mamoru Ohashi
Anan Technical College

Toshio Nishida
Osaka University

(Received March 21, 1980; Revised April 21, 1981)

Abstract This paper deals with a system consisting of two units under Markovian deterioration. The transition probability of each unit is not independent each other, and the cost of replacing both units concurrently is less than the cost of replacing them at different times. Then we investigate a replacemant policy for units in a two-unit system possessing stochastic and economic dependence. The structural properties of optimal replacement policy that minimizes the expected total discounted cost are characterized. Also illustrative examples are presented.

1. Introduction

This paper considers a discrete time maintenance model for a two-unit system possessing stochastic and economic dependence. Most maintenance and replacement models for industrial equipment have been developed for independent single component machines. However most equipment consists of multiple components. If the transition probabilities of the several components are not stochastically independent or if the cost of replacing several components jointly is less than the cost of several separate replacements, then the replacement policy for each component may depend on the state of the other components. The system considered in this paper consists of two units under Markovian deterioration, and is not stochastically and economically independent. We investigate the structure of optimal replacement policy for units in such system.

The replacement policies for stochastically independent two-unit system were studied by several authors. Sethi [8] has dealt with the discrete time maintenance model for a series system consisting of two stochastically failing units, and shown that the optimal replacement policy has the form of a control limit policy. Berg [2] has considered a system consisting of two identical units with exponential life time distributions and linear running costs, and shown that the optimal policy is provided by the trigger-off replacement policy, introduced by Bansard et al. [1]. Radner and Jorgenson [6] have considered the replacement policy for a unit in series system with several monitored units having exponential life, and shown that the optimal policy is provided by an (n,N)-policy. When each unit is subject to preventive replacement i.e., at each unit of time any action (replacement of one unit or both) can be taken, Vergin [9] has derived recurrent functional equations by using the technique of dynamic programming and obtained numerical solution by using iterative methods for specific values of the parameters. However he did not mention the structure of optimal policy.

In this paper we consider a system consisting of two units under Markovian deterioration. The transition probability of each unit is not independent each other, and the cost of replacing both units concurrently is less than the cost of replacing them at different times. A decision is made at the beginning of each period to replace each unit or to keep it until the next decision epoch. The objective of this paper is to study the structure of optimal replacement policy minimizing the expected total discounted cost for units in a two-unit system possessing stochastic and economic dependence. We discuss the properties of the optimal region of the decision and show that the state space is divided into at most four regions. Then since these properties should enable us to decrease the number of policies that must be considered, better algorithms can be expected. Finally, to illustrate the optimal replacement policy, numerical examples are presented.

2. Model Formulation

Consider the following discrete time maintenance model. A system consists of two units, U_1 and U_2, under Markovian deterioration. Suppose each unit U_r (r=1,2) is inspected at the beginning of each period, and that after each inspection it is classified into one of L_r+1 states showing the degree of dererioration. Each unit is in state 0 (L_r) if and only if it is new (inoperative). Let $i=(i_1,i_2)$ denote the state of a two-unit system where $i_r \in I_r=\{0,1,\dots,L_r\}$ is the state of unit U_r. After observing the state of the system, a decision is made at the beginning of each period to replace each unit U_r or to keep it until the next decision epoch. Let $d=(d_1,d_2)$ represent the decision made for units in a two-unit system at each period, where $d_r \in D_r=\{0,1\}$ is a decision made for unit U_r. Here d_r=0 means doing nothing, while $d_r=1$ means replacing the unit U_r. We assume that the time for replacement is negligible compared with the length of a period.

If the current state of the system is $i=(i_1,i_2)$ and decision $d=(0,0)$ is made on the system, then the probability that the state of the system at the beginning of the next period is $j=(j_1,j_2)$ is given by P_{ij}. On the other hand, if the decision $d_r=1$ is made on unit U_r, it is immediately replaced by a new one, and it begins to operate in its best condition just after replacement. Thus if the state of the system is $i=(i_1,i_2)$ and decision $d=(1,0)$ is made, then the probability that the state of the system at the beginning of the next period is $j=(j_1,j_2)$ is given by $P_{(0,i_2)j}$. Similarly, decision $d=(0,1)$ ($d=(1,1)$) is made, then the probability that the state of the system at the beginning of the next period is $j=(j_1,j_2)$ is given by $P_{(i_1,0)j}$ ($P_{(0,0)j}$). As the costs associated with the two-unit system, we consider a replacement cost $C_r(i_r)$ of unit U_r, a set up cost K of replacement and an operating cost $A(i_1,i_2)$ per period when the system is in state $i=(i_1,i_2)$ at the beginning of the period. We assume that all costs and transition probabilities are known, and that all costs are bounded and non-negative.

For any real number x and y and any 2-vectors $X=(x_1,x_2)$ and $Y=(y_1,y_2)$, we define $x\vee y=\max(x,y)$, $x\wedge y=\min(x,y)$, $X\vee Y=(x_1\vee y_1,x_2\vee y_2)$ and $X\wedge Y=(x_1\wedge y_1,x_2\wedge y_2)$. If $X\vee Y=X$, we write $X\geq Y$. Then the state space $I=I_1\mathrm{x}I_2$ and action space $D=D_1\mathrm{x}D_2$ are partially ordered sets with relation $\geq$. Let $\delta(i_1,i_2)\in D^I$ be a decision made on the state $i=(i_1,i_2)$ and $\delta^*(i_1,i_2)$ be an optimal decision.

The objective is to investigate the strucutre of optimal replacement policy which minimizes the expected total discounted cost with discount factor $\alpha\in[0,1)$. Now let $V_\alpha(i_1,i_2)$ be the minimum expected total discounted cost when the state of the system is $i=(i_1,i_2)$ at the beginning. The $V_\alpha(i_1,i_2)$ obeys the functional equation:

$$
\begin{aligned}
V_\alpha(i_1,i_2)=\min\{&A(i_1,i_2)+\alpha\sum_{k\in I}P_{(i_1,i_2)k}V_\alpha(k),\\
&\mathrm{K}+C_1(i_1)+A(0,i_2)+\alpha\sum_{k\in I}P_{(0,i_2)k}V_\alpha(k),\\
&\mathrm{K}+C_2(i_2)+A(i_1,0)+\alpha\sum_{k\in I}P_{(i_1,0)k}V_\alpha(k),\\
&\mathrm{K}+C_1(i_1)+C_2(i_2)+A(0,0)+\alpha\sum_{k\in I}P_{(0,0)k}V_\alpha(k)\}\\
=&\min_{(d_1,d_2)\in D} R(i_1,i_2;d_1,d_2).
\end{aligned}
\tag{2.1}
$$

In the next section the structural properties of optimal replacement policy minimizing the expected total discounted cost are characterized under reasonable conditions. We discuss the properties of the optimal regions of the decision $d=(d_1,d_2)$ and show that the state space is divided into at most four regions. Here we notice that the existence of a stationary policy minimizing the expected total discounted cost is guaranteed since all costs are bounded and the action space is finite.

3. Structure of Optimal Replacement Policy

In this section we discuss an optimal replacement policy for units in a two-unit system. We can find the optimal replacement policy by solving the functional equation (2.1). However we cannot obtain a solution as a function of the parameters in the model. So the structural properties of optimal replacement policy for units in a two-unit system are characterized.

First we examine the structural property of optimal expected total discounted cost function $V_\alpha(i)$ under the following preliminary definitions and conditions introduced by White [10].

Definitions. (1) Let S be the family of all subsets S of the state space I such that if $i\in S\in\mathsf{S}$ then $i'\in S$ for all $i'\geq i$.
(2) Let $\mathsf{F}=\{f(i)\in R^I,\ i\in I;\ i'\geq i$ implies $f(i')\geq f(i)\}$, the set of all real-valued function on I which are increasing with respect to the partial ordering $\geq$.

Conditions. (1) Define $F(i,S)=\sum_{j\in I} P_{ij} I_S(j)$ for $i\in I$ and $S\in\mathsf{S}$, where I_S is the indicator function of the set S, i.e., $I_S(i)=0$ if $i\notin S$ and $I_S(i)=1$ if $i\in S$. Then for all $S\in\mathsf{S}$ the function $F(i,S)$ is a member of F.
(2) The operating cost $A(i)$ per period is a member of F and the replacement cost $C_r(i_r)$ of unit U_r is a non-decreasing function in i_r.
(3) The difference $A(i_1,i_2)-C_1(i_1)-C_2(i_2)$ is a member of F.

Condition (1) asserts a generalization of the condition, introduced by Derman [3], that the system has a trend for monotonically increasing expected deterioration. Conditions (2) and (3) state that an operating and replacement costs and their difference increase as a function of deterioration of the system state.

The following lemme is used in the proof of Lemma 2 which presents the structure of optimal expected total discounted cost function.

Lemma1. Assume the condition (1) holds. If a non-negative function f is a member of F, then $\sum_{j\in I} P_{ij} f(j)$ is a member of F.

Proof: From the definitions (1) and (2), for each $f\in\mathsf{F}$ there exist a non-negative sequence $\{a_s\}$ ($S\in\mathsf{S}$ and $S\neq I$) and real number a_I such that $f(i)=$

$\sum_{S \in \mathrm{S}} a_S I_S(i)$. Then we have

$$\sum_{j \in I} P_{ij} f(j) = \sum_{j \in I} P_{ij} \{ \sum_{S \in \mathrm{S}} a_S I_S(j) \}$$

$$= \sum_{S \in \mathrm{S}} a_S \sum_{j \in I} P_{ij} I_S(j).$$

Therefore the result follows directly from $\sum_{j \in I} P_{ij} I_S(j) \in \mathrm{F}$ for all $S \in \mathrm{S}$. ||

The above Lemma 1 is a generalization of an important result obtained by Derman [3].

The following lemma shows the structure of optimal expected total discounted cost function and it is used in the proof of theorems which present the structural properties of optimal replacement policy.

Lemma 2. Assume the conditions (1) and (2) hold. Then $V_\alpha(i)$ is a member of F.

Proof: The proof is carried out by mathematical induction. We first consider the n-period problem. Let $V_\alpha(i,n)$ be the minimum expected n period discounted cost when the system is in state i at the beginning. Then by letting $V_\alpha(i,0)=0$ for all $i \in I$, $V_\alpha(i,n)$ $(n \geq 1)$ satisfies a set of recursive equations:

$$V_\alpha(i_1,i_2,n)=\min\{A(i_1,i_2)+\alpha \sum_{j \in I} P_{(i_1,i_2)j} V_\alpha(j,n-1),$$

$$K+C_1(i_1)+A(0,i_2)+\alpha \sum_{j \in I} P_{(0,i_2)j} V_\alpha(j,n-1),$$

$$\text{(3.1)} \qquad K+C_2(i_2)+A(i_1,0)+\alpha \sum_{j \in I} P_{(i_1,0)j} V_\alpha(j,n-1),$$

$$K+C_1(i_1)+C_2(i_2)+A(0,0)+\alpha \sum_{j \in I} P_{(0,0)j} V_\alpha(j,n-1)\}.$$

For $n=1$ the result follows easily from condition (2). Suppose the result is true for n. Then under conditions (1) and (2), $V_\alpha(i,n+1)$ is a member of F by using the induction hypothesis, equation (3.1) and Lemma 1. This holds for all n, and as

$$\lim_{n \to \infty} V_\alpha(i,n)=V_\alpha(i),$$

$V_\alpha(i)$ is a member of F. ||

Next the structural properties of optimal replacement policy for units in a two-unit system are characterized. Now let $B^*(d)$ be the optimal region of the decision $d \in D$; that is, $B^*(d)=\{i \in I : \delta^*(i)=d\}$. The following theorems characterize the sets $B^*(d)$ under reasonable conditions.

Theorem 1. Assume the conditions (1), (2) and (3) hold. Then $B^*(0,0)$ is closed in the sense that $i\wedge j\in B^*(0,0)$ for all i and j in $B^*(0,0)$.

Proof: To show that $i\wedge j\in B^*(0,0)$ for all i and j in $B^*(0,0)$, we consider the four cases: (1) $i\wedge j=i$, (2) $i\wedge j=j$, (3) $i\wedge j=(i_1,j_2)$, and (4) $i\wedge j=(j_1,i_2)$. In the cases of (1) and (2), the result is obvious. In the case of (3) we have

$$\begin{aligned} V_\alpha(i\wedge j)=\min\{&A(i_1,i_2)+\alpha\sum_{k\in I}P_{(i_1,j_2)k}V_\alpha(k),\\ &K+C_1(i_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k),\\ &K+C_2(j_2)+A(i_1,0)+\alpha\sum_{k\in I}P_{(i_1,0)k}V_\alpha(k),\\ &K+C_1(i_1)+C_2(j_2)+A(0,0)+\alpha\sum_{k\in I}P_{(0,0)k}V_\alpha(k)\}.\end{aligned}$$

Then from $i_1\leqq j_1$ we obtain

$$\begin{aligned} &K+C_1(i_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)-\{A(i_1,j_2)+\alpha\sum_{k\in I}P_{(i_1,j_2)k}V_\alpha(k)\}\\ &\geqq K+C_1(j_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)-\{A(j_1,j_2)+\alpha\sum_{k\in I}P_{(j_1,j_2)k}V_\alpha(k)\}\\ &\geqq 0.\end{aligned}$$

The first inequality follows from the conditions (1),(2) and (3) and Lemmas 1 and 2. The last inequality is true since $j\in B^*(0,0)$. Similarly from $i_2\geqq j_2$ we obtain

$$K+C_2(j_2)+A(i_1,0)+\alpha\sum_{k\in I}P_{(i_1,0)k}V_\alpha(k)-\{A(i_1,j_2)+\alpha\sum_{k\in I}P_{(i_1,j_2)k}V_\alpha(k)\}\geqq 0,$$

and

$$\begin{aligned} &K+C_1(i_1)+C_2(j_2)+A(0,0)+\alpha\sum_{k\in I}P_{(0,0)k}V_\alpha(k)-\{A(i_1,j_2)+\alpha\sum_{k\in I}P_{(i_1,j_2)k}V_\alpha(k)\}\\ &\geqq 0.\end{aligned}$$

Thus in the case of (3) we have $i\wedge j\in B^*(0,0)$. Case (4) is proved similarly to the case (3). Then this completes the proof of this theorem. ||

Theorem 2. Assume the conditions (1), (2) and (3) hold. Then for each $d\in D_R=\{(1,0),(0,1),(1,1)\}$, $B^*(d)$ is closed in the sense that $i\vee j\in B^*(d)$ for all i and j in $B^*(d)$.

Proof: First in the case of $d=(1,0)$ we show $i\vee j\in B^*(d)$ for all i and j in $B^*(d)$. If for all i and j in $B^*(1,0)$, $i\vee j=i$ or $i\vee j=j$, then the result is obvious. If $i\vee j=(i_1,j_2)$, then from $i_1\geqq j_1$ using all the conditions of the theorem and Lemmas 1 and 2, we can easily show that

$$A(i_1,j_2)+\alpha\sum_{k\in I}P_{(i_1,j_2)k}V_\alpha(k)-\{K+C_1(i_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)\}$$

$$\geqq A(j_1,j_2)+\alpha\sum_{k\in I}P_{(j_1,j_2)k}V_\alpha(k)-\{K+C_1(j_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)\}$$

$$\geqq 0.$$

Similarly from $i_1\geqq j_1$ and $j\in B^*(1,0)$ we obtain

$$K+C_2(j_2)+A(i_1,0)+\alpha\sum_{k\in I}P_{(i_1,0)k}V_\alpha(k)$$
$$-\{K+C_1(i_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)\}$$
$$\geqq K+C_2(j_2)+A(j_1,0)+\alpha\sum_{k\in I}P_{(j_1,0)k}V_\alpha(k)$$
$$-\{K+C_1(j_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)\}\geqq 0,$$

and

$$K+C_1(i_1)+C_2(j_2)+A(0,0)+\alpha\sum_{k\in I}P_{(0,0)k}V_\alpha(k)$$
$$-\{K+C_1(i_1)+A(0,j_2)+\alpha\sum_{k\in I}P_{(0,j_2)k}V_\alpha(k)\}\geqq 0.$$

In the case of $d=(0,1)$ we may show the result by the same method. If $d=(1,1)$, then we can easily show the result. ||

Theorem 3. Assume the conditions (1), (2) and (3) hold.

1) If $i=(i_1,i_2)\in B^*(1,d_2)$ for $d_2\in D_2$, then $(i_1',i_2)\in B^*(1,d_2)$ for $i_1'\geqq i_1$.

2) If $i=(i_1,i_2)\in B^*(d_1,1)$ for $d_1\in D_1$, then $(i_1,i_2')\in B^*(d_1,1)$ for $i_2'\geqq i_2$.

Proof: 1) For $i=(i_1,i_2)\in I$, let

(3.2) $$\Lambda_K^1(i_1,i_2)=\min\{R(i_1,i_2;0,0),R(i_1,i_2;0,1)\},$$

(3.3) $$\Lambda_R^1(i_1,i_2)=\min\{R(i_1,i_2;1,0),R(i_1,i_2;1,1)\}.$$

Now we obtain easily that the difference $\Lambda_K^1(i_1,i_2)-\Lambda_R^1(i_1,i_2)$ is non-decreasing in i_1 for each i_2 by Lemmas 1 and 2. Therefore from $i\in B^*(1,d_2)$ for $d_2\in D_2$, we have

$$\Lambda_K^1(i_1',i_2)-\Lambda_R^1(i_1',i_2)\geqq\Lambda_K^1(i_1,i_2)-\Lambda_R^1(i_1,i_2)\geqq 0.$$

Since a difference $R(i_1,i_2;1,0)-R(i_1,i_2;1,1)$ is constant for any i_1, the above inequalities imply $i'=(i_1',i_2)\in B^*(1,d_2)$ for $d_2\in D_2$. 2) is proved similarly to 1). ||

Theorem 4. If all the conditions (1), (2) and (3) hold, then there exists a stationary control limit policy with respect to unit U_r ($r=1,2$) minimizing the expected total discounted cost of the replacement model for units in a two-unit system.

Proof: From Theorem 3 $\Lambda_K^1(i_1,i_2)-\Lambda_R^1(i_1,i_2)$ is non-decreasing in i_1 for each i_2. Hence there exists a set of critical numbers $n_1^*(i_2)$ for each i_2

such that the decision to replace the unit U_1 is optimal if and only if the state i_1 of unit U_1 is no less than $n_1^*(i_2)$, which is a control limit policy with respect to unit U_1. Similarly for unit U_2. ||

The above theorem states that an optimal replacement policy has the form of a 2-dimensional control limit policy introduced by Hatoyama [4], and when the failed units are immediately replaced the state space I is divided into at most four regions. The control limit with respect to unit U_1 is given by

$$(3.4)\qquad n_1^*(\,_2)=\min\{i_1:\Lambda_K^1(i_1,i_2)\geqq\Lambda_R^1(i_1,i_2)\}.$$

When the bracket of the right hand side of (3.4) is empty, we define $n_1^*(i_2)=L_1+1$. Similarly for unit U_2

$$(3.5)\qquad n_2^*(i_1)=\min\{i_2:\Lambda_K^2(i_1,i_2)\geqq\Lambda_R^2(i_1,i_2)\}.$$

We have the following corollary concerned with the properties of the control limit.

Corollary. Assume the conditions (1), (2) and (3) hold.

1) If $(L_1,i_2)\in B^*(1,d_2)$ for $d_2\in D_2$, then $n_2^*(0)\geqq n_2^*(L_1)$.
2) If $(i_1,L_2)\in B^*(d_1,1)$ for $d_1\in D_1$, then $n_1^*(0)\geqq n_1^*(L_2)$.

The above corollary is easily proved by comparing the appropriate terms in functional equation (2.1). In the case of $L_2=1$ this corollary states that an optimal replacement policy for unit U_1 is an (n,N) policy introduced by Radner and Jorgenson [6], where $n=n_1^*(L_2)$ and $N=n_1^*(0)$.

From the above theorems one realization of optimal regions $B^*(d)$ of the decision $d=(d_1,d_2)$ is illustrated in Fig.1. The boundary of optimal regions $B^*(1,1)$ and $B^*(0,1)$ is straight from Theorem 3. Similarly for optimal regions $B^*(1,1)$ and $B^*(1,0)$. The boundary curve of optimal regions $B^*(1,1)$ and $B^*(0,0)$ is non-increasing since the difference $R(i_1,i_2;0,0)-R(i_1,i_2;1,1)$ is a member of F.

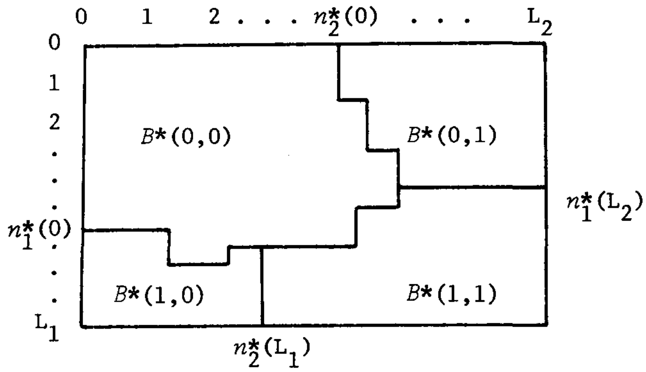

Fig. 1. A typical optimal replacement policy

Intuitively the optimal replacement policy $\delta^*(i)$ is a non-decreasing function with respect to the partial ordering $\geq$. However this conjecture is not verified. A counterexample that $\delta^*(i)$ is not always a non-decreasing function of i is illustrated later.

4. Examples

In this section we consider a two-unit system possessing stochastic independence and economic dependence. In this case since the transition probability of each unit is independent, the transition probability P_{ij} of a two-unit system is given by

$$P_{ij}=P^1_{i_1j_1}\cdot P^2_{i_2j_2}$$

where $P^r_{i_rj_r}$ is the transition probability of unit U_r (r=1,2). We assume that the condition introduced by Derman [3] holds, i.e., $\sum_{j=k}^{L_r} P_{i_rj}$ is non-decreasing in i_r for all $k\in I_r$. This condition is sufficient for the condition (1) from definition (1). Further we assume $A(i_1,i_2)=A_1(i_1)+A_2(i_2)$ and $K\neq 0$.

Table 1. Transition probability matrix P^r and L_r=7 (r=1,2)

	0	1	2	3	4	5	6	L_r
0	0.00	0.30	0.20	0.15	0.15	0.10	0.05	0.05
1	0.00	0.25	0.20	0.15	0.15	0.10	0.10	0.05
2	0.00	0.10	0.20	0.20	0.15	0.15	0.10	0.10
3	0.00	0.05	0.10	0.15	0.25	0.20	0.15	0.10
4	0.00	0.05	0.05	0.10	0.25	0.20	0.20	0.15
5	0.00	0.00	0.05	0.10	0.15	0.25	0.25	0.20
6	0.00	0.00	0.00	0.05	0.05	0.10	0.40	0.40
L_r	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 2. Operating cost $A_r(i_r)$ and Replacement cost $C_r(i_r)$

State (i_r)	0	1	2	3	4	5	6	L_r
Cost $A_r(i_r)$	0	5	10	15	20	25	30	50
Cost $C_r(i_r)$	30	30	30	30	30	30	30	50

To illustrate the optimal replacement policy of the preceding section, we consider numerical examples. The transition probability matrix P^r of unit U_r is given in Table 1. The operating cost $A(i)$ and the replacement cost $C_r(i_r)$ are given in Table 2. Then the conditions (1), (2) and (3) are satisfied. For reference, first consider the case where the units are stochastically and economically independent, i.e., suppose the set up cost is 2K when both units are replaced. Then the optimal replacement policy for units in a two-unit system is shown in Fig.2 in the case of K=10 and α=0.95. Notice that this optimal replacement policy for one unit doesn't depend on the state of the remaining unit (see [9]). On the other hand the optimal replacement policy for units in a two-unit system possessing economic dependence is shown in Fig.3 in the case of K=10 and α=0.95. This example shows that the optimal replacement policy has the form of a two-dimensional control limit policy with four regions. For example $\delta^*(5,1)=(1,0)$ and $\delta^*(5,2)=(0,0)$ are not non-decreasing. Therefore this example also shows that the monotonicity of optimal decision $\delta^*(i)$ does not always hold.

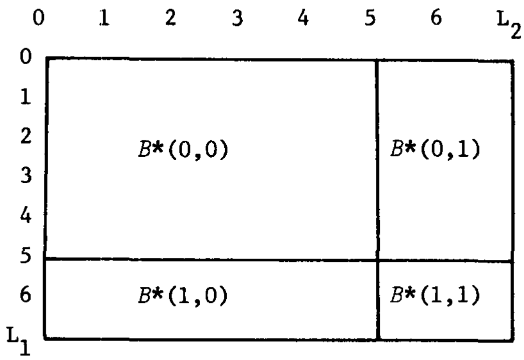

Fig. 2. Optimal replacement policy in the case where units are independent

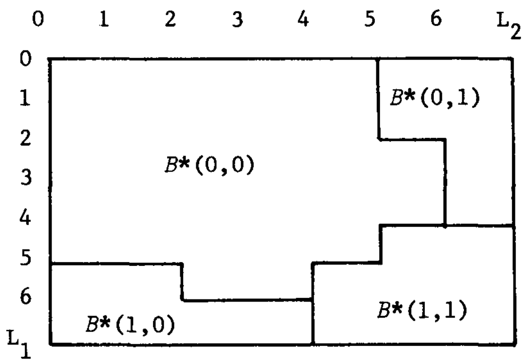

Fig. 3. Optimal replacement policy in the case where units aren't independent

5. Conclusion

We investigated the replacement policy for units in a two-unit system possessing stochastic and economic dependence. A decision is made at the beginning of each period to replace each unit or to keep it until the next decision epoch. Then we proved that the optimal replacement policy minimizing the expected total discounted cost is a two-dimensional control limit policy under reasonable conditions. Further we discussed the properties of the optimal region of the decision and showed that the state space is divided into at most four regions when the failed units are immediately replaced. It is future problem to find the structure of optimal replacement policy when units U_1 and U_2 have continuous distributions.

Acknowledgement

The authors are grateful to the referees for their valuable comments on this paper.

References

[1] Bansard, J. P., Descamps, J. L., Maarek, G., and Morihain, G.: Study of a Stochastic Renewal Process Applied to a Trigger-off Policy for Replacing Whole Sets of Components. in: J. Lawrence, ed., *Proc. 5th Intern. Conf. on Operational Research*, 1969. Tavistock, London, 1970.

[2] Berg, M.: Optimal Replacement Policies for Two-Unit Machines with Increasing Running Costs I. *Stochastic Processes and their Applications*, Vol. 4 (1976), 89-106.

[3] Derman, C.: *Finite State Markov Decision Processes*. Academic Press, New York, 1970.

[4] Hatoyama, Y.: Markov Maintenance Models with Control of Queue. *J. Operations Research Society of Japan*, Vol. 20 (1977), 164-181.

[5] Howard, R. A.: *Dynamic Programming and Markov Processes*. John Wiley, New York, 1960.

[6] Radner, R. and Jorgenson, D. W.: Optimal Replacement of a Single Part in the Presence of Several Monitored Parts. *Management Science*, Vol. 10 (1963), 70-84.

[7] Ross, S.: *Applied Probability Models with Optimization Applications*. Holden-Day, San Francisco, CA, 1970.

[8] Sethi, D.: Opportunistic Replacement Policies. in: C. P. Tsokos, ed., *The Theory and Applications of Reliability*, Vol. 1, 433-447.

[9] Vergin, R. C.: Optimal Renewal Policies for Complex Systems. *Naval Research Logistics Quarterly*, Vol. 15 (1968), 523-534.

[10] White, C. C.: Bounds on Optimal Cost for a Replacement Problem with Partial Observations. *Naval Research Logistics Quarterly*, Vol. 26 (1979), 415-422.

Mamoru OHASHI: Anan Technical College,
Minobayashi, Anan, Tokushima,
744, Japan.