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THE EFFICIENCY OF TWO-STAGE LINE

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Abstract This paper considers a two-stage parallel line in which one of stages has a station with an arbitrary operation time but the other stage has S stations with exponential operation times. First, the procedure to estimate the production rate is studied and the relation between the dual models is discussed. Next, the effect of the number of stations is evaluated by the imaginary buffer capacity. Consequently it is represented that this imaginary capacity has a linear asymptote and that this asymptote is obtained by the simple procedure and is very useful to evaluate the effect of the number of stations.

1. Introduction

Many papers have been published concerning the effects of the design factors of production lines on the production rate [1], [4], [5], [8], [9], [10], and the effects of the operation time distribution, the allocation of the operation time, the number of stages and the buffer capacity are represented. These results present the industrial engineers with the valuable advices to design the better production lines. Furthermore, the effect of the number of stations in the stage has been discussed by [6], and [12], and it is represented that paralleling is very effective to yield the high production rate when the buffer space is small. However, this effect has been discussed only for the special lines, i.e. the lines in which the operation time has the normal or exponential distribution and the number of stations in each stage is equal, so it is necessary to discuss the effect more elaborately.

In this paper, we consider the two-stage parallel lines in which one of stages has a station with an arbitrary operation time but another stage has S stations with exponential operation times in order to represent the basic effect of the stage consisting of the plural stations. First, the procedure to estimate the production rate is discussed by the imbedded Markov chain and the various characters for the dual models are represented. Next, the above parallel lines are compared with the corresponding series lines by the scale 'imaginary buffer capacity' introduced by [6], and it is shown that the asymptote for the imaginary buffer capacity is obtained by the simple procedure and that the asymptote is very useful to evaluate the effect of the number of stations within a small error.

2. Model

The production line discussed here consists of the buffer storage holding in-process works temporarily and the two stages, one of which has S stations to operate the work practically, as shown in Fig. 2.1. Each station in one stage has the same operation and the same operation time distribution. Then the line is defined by the buffer capacity M, the number S of stations and the operation time distribution $F_i(x)$ for stage i ($i=1, 2$). Especially in this paper we assume that, in Model 1, each station in stage 1 has an exponential distribution and stage 2 has an arbitrary distribution;

$$F_1(x) = 1 - \exp(-\lambda x)$$
$$F_2(x) = G(x) \quad ,$$

and, in Model 2, these distributions are exchanged;

$$F_1(x) = G(x)$$
$$F_2(x) = 1 - \exp(-\lambda x) \quad .$$

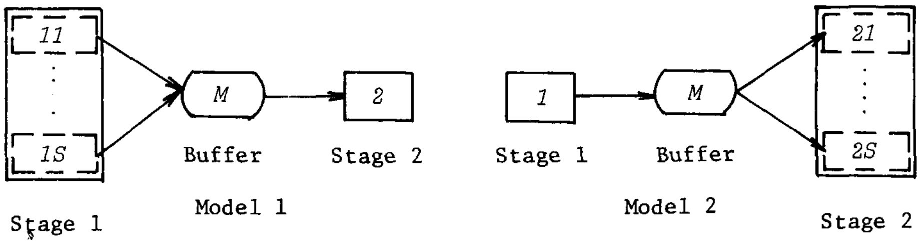

Fig. 2.1 Two-stage parallel line

Assume that there are infinite works ready to be operated in the first stage and there is an infinite buffer capacity just behind the second stage. This means that idling due to lack of input works is never occurred in the first stage and blocking is never occurred in the second stage because the completed work is immediately ejected from the stage. And assume that the operation times are mutually independent in each stage and station.

Moreover, the priority of the S stations must be interested, because it is required to decide in which station idling or blocking is occurred and released first. We adopt the random priority in these models, since the S stations have the same operation and the same operation time distribution and the priority has no effect on the production rate.

3. Formulation

3.1 Model 1

In this model we consider the states of the system at the time just before the completion of the n-th work in stage 2 ($n=1, 2, \ldots$). These states are defined by the number m of in-process works in the buffer and the number s of stations blocked in stage 1, and the state probabilities are represented as follows;

$P_n(m, 0)$: no station in stage 1 is in blocking and m in-process works are in the buffer ($0 \leq m \leq M$)

$P_n(M, s)$: s stations in stage 1 are in blocking and M in-process works are in the buffer ($1 \leq s \leq S$).

Then these states form the imbedded Markov chain and we can obtain the system equations by using the transition probabilities. We denote the state probabilities in a steady state condition by

$$(3.1) \qquad \begin{aligned} P_i &= \lim_{n\to\infty} P_n(i, 0) && (0 \leq i \leq M) \\ P_i &= \lim_{n\to\infty} P_n(M, i-M) && (M+1 \leq i \leq M+S) \end{aligned}$$

and the transition probability from the state that the summation of the number of in-process works in the buffer and the number of stations blocked in stage 1 is i to the state that the summation is j by $q_{i,j}$.

These transition probabilities can be represented by the probabilities $\alpha_{i,S}$ that i works are completed and ejected from the S stations in stage 1 to the buffer during the operation time in stage 2, $\beta_{i,j}$ that the $(S-j)$ stations in stage 1 are in blocking and the other i stations are blocked during the operation time in stage 2, and $\gamma_{i,j}$ that i works are completed and ejected from the S stations in stage 1 to the buffer to fill the buffer capacity and moreover the j stations are blocked during the operation time in stage 2, as follows;

$$
(3.2)\quad
\begin{aligned}
q_{i,j} &= 0 && (2\leqq i\leqq M+S,\ 0\leqq j\leqq i-2)\\
q_{0,j} &= q_{1,j} && (0\leqq j\leqq M+S)\\
q_{i,j} &= \alpha_{j+1-i,S} && (1\leqq i\leqq M,\ i-1\leqq j\leqq M-1)\\
q_{i,j} &= \beta_{j+1-i,M+S+1-i} && (M+1<i<M+S,\ i-1\leqq j\leqq M+S)\\
q_{i,j} &= \gamma_{M+1-i,j-M} && (1\leqq i\leqq M,\ M\leqq j\leqq M+S)
\end{aligned}
$$

where

$$
(3.3)\quad
\begin{aligned}
\alpha_{i,S} &= \int_0^\infty \frac{(S\lambda x)^i}{i!}\exp(-S\lambda x)\,dG(x)\\
\beta_{i,j} &= \int_0^\infty {}_jC_i(1-\exp(-\lambda x))^i\exp(-(j-i)\lambda x)\,dG(x)\\
\gamma_{i,j} &= \int_0^\infty\int_0^x {}_SC_j(1-\exp(-\lambda(x-y)))^j\exp(-(S-j)\lambda(x-y))\\
&\qquad \left\{\frac{(S\lambda)^i}{(i-1)!}y^{i-1}\exp(-S\lambda y)\right\}dy\,dG(x)\ .
\end{aligned}
$$

Consequently the system equations in a steady state condition are given by

$$
(3.4)\quad
\begin{aligned}
P_0 &= P_0\alpha_{0,S}+P_1\alpha_{0,S}\\
P_m &= P_0\alpha_{m,S}+P_1\alpha_{m,S}+P_2\alpha_{m-1,S}+\cdots+P_{m+1}\alpha_{0,S} \qquad (1\leqq m\leqq M-1)\\
P_M &= P_0\gamma_{M,0}+P_1\gamma_{M,0}+P_2\gamma_{M-1,0}+\cdots+P_M\gamma_{1,0}+P_{M+1}\beta_{0,S}\\
P_{M+s} &= P_0\gamma_{M,s}+P_1\gamma_{M,s}+P_2\gamma_{M-1,s}+\cdots+P_M\gamma_{1,s}+P_{M+1}\beta_{s,S}+P_{M+2}\beta_{s-1,S-1}+\\
&\qquad \cdots+P_{M+s+1}\beta_{0,S-s} \qquad (1\leqq s\leqq S-1)\\
P_{M+S} &= P_0\gamma_{M,S}+P_1\gamma_{M,S}+P_2\gamma_{M-1,S}+\cdots+P_M\gamma_{1,S}+P_{M+1}\beta_{S,S}+P_{M+2}\beta_{S-1,S-1}+\\
&\qquad \cdots+P_{M+S}\beta_{1,1}\ .
\end{aligned}
$$

From the first M equations in (3.4) we solve P_i $(1\leqq i\leqq M)$ and from the last $(S+1)$ equations we solve P_i $(M+1\leqq i\leqq M+S)$.

Now we introduce the generating function $G_1(Z)$, $G_2(Z)$ and $K(Z)$;

$$
(3.5)\quad G_1(Z) = \sum_{i=0}^{\infty} Z^i P_i
$$

$$
(3.6)\quad G_2(Z) = \sum_{i=0}^{S} Z^i P_{M+S-i}
$$

(3.7) $$K(Z) = \sum_{i=0}^{\infty} Z^i \alpha_{i,S} = U(S\lambda(1-Z)) \quad ,$$

where $G_1(\cdot)$ is defined for $M\to\infty$ and $U(\cdot)$ is the Laplace transform of $\frac{d}{dx}G(x)$. Then, for $0\leq M<\infty$, the first M equations are included in (3.8)

(3.8) $$G_1(Z) = K(Z)[P_0+\sum_{i=1}^{\infty} Z^{i-1}P_i] = K(Z)[P_0+\frac{1}{Z}(G_1(Z)-P_0)] \quad ,$$

and P_i $(1\leq i\leq M)$ is given by the coefficient of Z^i and is expressed in terms of P_0 if (3.9) can be expanded in a power series,

(3.9) $$G_1(Z) = P_0 \frac{1-Z}{1-Z/K(Z)} \quad .$$

On the other hand, the last $(S+1)$ equations are represented by

(3.10) $$\begin{aligned} G_2(Z) &= P_0\sum_{i=0}^{S} Z^i\gamma_{M,S-i}+\sum_{i=1}^{M} P_i\sum_{j=0}^{S} Z^j\gamma_{M+1-i,S-j}+\sum_{i=0}^{S-1} P_{M+S-i}\sum_{j=0}^{i+1} Z^j\beta_{i+1-j,i+1} \\ &= P_0 \int_0^{\infty}\int_0^{x} \frac{(S\lambda)^M}{(M-1)!} y^{M-1}(Z\exp(-\lambda x)+\exp(-\lambda y)-\exp(-\lambda x))^S dy dG(x) \\ &+ \sum_{i=1}^{M} P_i\int_0^{\infty}\int_0^{x}\frac{(S\lambda)^{M+1-i}}{(M-i)!} y^{M-i}(Z\exp(-\lambda x)+\exp(-\lambda y)-\exp(-\lambda x))^S dy dG(x) \\ &+ \int_0^{\infty}(Z\exp(-\lambda x)+1-\exp(-\lambda x))G_2(Z\exp(-\lambda x)+1-\exp(-\lambda x))dG(x) \\ &- P_M\int_0^{\infty}(Z\exp(-\lambda x)+1-\exp(-\lambda x))^{S+1}dG(x) \quad . \end{aligned}$$

Hence by using H_k

(3.11) $$H_k = \frac{1}{k!}\frac{d^k}{dZ^k}G_2(Z)\,|_{Z=1} \quad ,$$

we have

(3.12) $$\begin{aligned} H_k &= P_0\int_0^{\infty} {}_SC_k\exp(-k\lambda x)\{(\frac{S}{S-k})^M-\exp(-(S-k)\lambda x)\sum_{i=0}^{M-1}\frac{(S\lambda x)^{M-1-i}}{(M-1-i)!}(\frac{S}{S-k})^{i+1}\}dG(x) \\ &+ \sum_{i=1}^{M} P_i\int_0^{\infty} {}_SC_k\exp(-k\lambda x)\{(\frac{S}{S-k})^{M+1-i}-\exp(-(S-k)\lambda x)\cdot \\ &\quad \sum_{j=0}^{M-i}\frac{(s\lambda x)^{M-i-j}}{(M-i-j)!}(\frac{S}{S-k})^{j+1}\}dG(x) + (H_{k-1}+H_k)\int_0^{\infty}\exp(-k\lambda x)dG(x) \\ &- P_M\int_0^{\infty} {}_{S+1}C_k\exp(-k\lambda x)dG(x) \qquad (1\leq k\leq S-1) \end{aligned}$$

$$H_S = P_M$$

and finally we have

(3.13) $$H_{k-1} = \frac{1-\beta_{0,k}}{\beta_{0,k}} H_k - \frac{1}{\beta_{0,k}} \Psi_k ,$$

where

$$\Psi_k = P_0 \, {}_SC_K\{(\frac{S}{S-k})^M \beta_{0,k} - \sum_{i=0}^{M-1} (\frac{S}{S-k})^{i+1} \alpha_{M-1-i,S}\}$$

$$+ \sum_{i=1}^{M} P_i \, {}_SC_k\{(\frac{S}{S-k})^{M+1-i} \beta_{0,k} - \sum_{j=0}^{M-i} (\frac{S}{S-k})^{j+1} \alpha_{M-i-j,S}\} - P_M \, {}_{S+1}C_k \beta_{0,k}$$

$$(1 \leq k \leq S-1)$$

$$\Psi_S = P_0 \alpha_{M,S} + \sum_{i=1}^{M} P_i \alpha_{M+1-i,S} - P_M \, {}_{S+1}C_S \beta_{0,S} .$$

The above equation is the linear difference equation, so that H_k $(0 \leq k \leq S-1)$ can be solved and expressed in terms of P_i $(0 \leq i \leq M)$ and each P_i $(M+1 \leq i \leq M+S)$ is represented by

(3.14) $$P_i = \sum_{k=M+S-i}^{S} (-1)^{k+i-M-S} \, {}_kC_{M+S-i} \, H_k .$$

However, from (3.6) and (3.11) we have

$$H_0 = G_2(Z)|_{Z=1} = \sum_{i=M}^{M+S} P_i ,$$

so that normalizing, i.e.

(3.15) $$1 = \sum_{i=0}^{M+S} P_i = \sum_{i=0}^{M-1} P_i + H_0 ,$$

gives the actual probabilities for P_0, and hence P_i $(0 \leq i \leq M+S)$ is completely decided.

From these state probabilities we can obtain the idling time distribution $I(x)$ in stage 2, the mean idling time $\overline{I}$, the production rate R and the mean number $\overline{M}$ of in-process works in the buffer as follows;

(3.16) $$I(x) = 1 - P_0 \exp(-S\lambda x) \qquad (x \geq 0)$$

(3.17) $$\overline{I} = P_0 \, (1/S\lambda)$$

(3.18) $$R = 1/(\text{mean output interval in stage 2}) = S\lambda/(P_0+\rho)$$

(3.19) $$\overline{M} = \sum_{i=0}^{M-1} i \cdot P_i + M \sum_{i=M}^{M+S} P_i ,$$

where

$$\rho = \frac{\text{mean operation time in stage 2}}{1/S\lambda} .$$

3.2 Model 2

This is the dual model for Model 1 and, like Model 1, the states of the system form the imbedded Markov chain if we consider the states at the time just before the completion of the n-th work in stage 1.

These states are defined by the number m of in-process works in the buffer and the number s of stations operating in stage 2, so that the state probabilities are represented as follows;

$P_n^*(0,s)$: s stations in stage 2 are in operating and no in-process work is in the buffer $(0 \leq s \leq S)$

$P_n^*(m,S)$: S stations in stage 2 are in operating and m in-process works are in the buffer $(1 \leq m \leq M)$.

Then we can formulate the system equations in a steady state condition by the state probabilities P_i^* $(0 \leq i \leq M+S)$ and the transition probabilities $q_{i,j}^*$ from the state that the summation of the number of stations operating in stage 2 and the number of in-process works in the buffer is i to the state that the summation is j

$$(3.20) \qquad \begin{aligned} P_i^* &= \lim_{n\to\infty} P_n^*(0,\ i) && (0 \leq i \leq S) \\ P_i^* &= \lim_{n\to\infty} P_n^*(i-S,\ S) && (S+1 \leq i \leq S+M) . \end{aligned}$$

In this model the transition probabilities are represented by the probabilities $\alpha_{i,S}^*$ that idling is never occurred and i works are completed in the S stations in stage 2 during the operation time in stage 1, $\beta_{i,j}^*$ that $(S-j)$ stations in stage 2 are in idling and from the other j stations i works are completed during the operation time in stage 1, and $\gamma_{i,j}^*$ that i works are completed in the S stations in stage 2 to become the buffer empty and moreover j works are completed during the operation time in stage 1, as follows;

$$(3.21) \qquad \begin{aligned} q_{i,j}^* &= 0 && (0 \leq i \leq S+M-2,\ i+2 \leq j \leq S+M) \\ q_{S+M,j}^* &= q_{S+M-1,j}^* && (0 \leq j \leq S+M) \\ q_{i,j}^* &= \alpha_{i+1-j,S}^* && (S \leq i \leq S+M-1,\ S+1 \leq j \leq i+1) \\ q_{i,j}^* &= \beta_{i+1-j,i+1}^* && (0 \leq i \leq S-1,\ 0 \leq j \leq i+1) \\ q_{i,j}^* &= \gamma_{i+1-S,S-j}^* && (S \leq i \leq S+M-1,\ 0 \leq j \leq S) \end{aligned} ,$$

where

$$\alpha^*_{i,S} = \int_0^\infty \frac{(s\lambda x)^i}{i!} \exp(-S\lambda x)dG(x)$$

$$(3.22) \qquad \beta^*_{i,j} = \int_0^\infty {}_jC_i \ (1-\exp(-\lambda x))^i \exp(-(j-i)\lambda x)dG(x)$$

$$\gamma^*_{i,j} = \int_0^\infty\int_0^x {}_SC_j \ (1-\exp(-\lambda(x-y)))^j \exp(-(S-j)\lambda(x-y)) \cdot \left\{\frac{(S\lambda)^i}{(i-1)!} \ y^{i-1} \exp(-S\lambda y)\right\}dydG(x) \quad .$$

Therefore, by denoting $P^t_i = P^*_{S+M-i}$ and $q^t_{i,j} = q^*_{M+S-i,M+S-j}$, (3.21) is

$$(3.23) \qquad \begin{aligned} q^t_{i,j} &= 0 && (2\leqq i\leqq M+S,\ 0\leqq j\leqq i-2) \\ q^t_{0,j} &= q^t_{1,j} && (0\leqq j\leqq M+S) \\ q^t_{i,j} &= \alpha^*_{j+1-i,S} && (1\leqq i\leqq M,\ i-1\leqq j\leqq M-1) \\ q^t_{i,j} &= \beta^*_{j+1-i,M+S+1-i} && (M+1\leqq i\leqq M+S,\ i-1\leqq j\leqq M+S) \\ q^t_{i,j} &= \gamma^*_{M+1-i,j-M} && (1\leqq i\leqq M,\ M\leqq j\leqq M+S) \end{aligned}$$

and hence it is represented that the state probabilities P^t_i $(0\leqq i\leqq M+S)$ and the transition probabilities $q^t_{i,j}$ $(0\leqq i,j\leqq M+S)$ form the same system equations with (3.4). Furthermore, in the dual models we have

$$(3.24) \qquad \begin{aligned} \alpha_{i,S} &= \alpha^*_{i,S} \\ \beta_{i,j} &= \beta^*_{i,j} \\ \gamma_{i,j} &= \gamma^*_{i,j} \quad , \end{aligned}$$

so that the state probabilities can be obtained from the dual model, i.e. Model 1, and are represented as follows;

$$(3.25) \qquad P^t_i = P^*_{S+M-i} = P_i \qquad (0\leqq i\leqq M+S)$$

Consequently we can obtain the blocking time distribution $B^*(x)$ in stage 1, the mean blocking time $\overline{B^*}$, the production rate R^* and the mean number $\overline{M^*}$ of inprocess works in the buffer as follows;

$$(3.26) \qquad B^*(x) = 1 - P^*_{S+M}\exp(-S\lambda x) \qquad (x\geqq 0)$$

$$(3.27) \qquad \overline{B^*} = P^*_{S+M} \ (1/S\lambda)$$

(3.28) $R^* = 1/(\text{mean output interval in stage 1}) = S\lambda/(P^*_{S+M}+\rho^*)$

(3.29) $\bar{M}^* = \sum_{i=S+1}^{S+M} (i-S)\, P^*_i$

where

$$\rho^* = \frac{\text{mean operation time in stage 1}}{1/S\lambda} .$$

Furthermore from the comparison between the above equations and (3.16) ∿ (3.19) we have the following conclusion for the dual models;

(3.30)
$$B^*(x) = I(x)$$
$$\bar{B}^* = \bar{I}$$
$$R^* = R$$
$$\bar{M}^* + \bar{M} = M .$$

4. Imaginary Buffer Capacity

In this section we study the effect of the number S of stations by the numerical results and the imaginary buffer capacity, and show that the asymptote for the imaginary buffer capacity can be obtained by the simple procedure and is very useful to represent the effect of S. The imaginary capacity is the capacity which is required to yield the same production rate of the considered parallel line in the corresponding series line, i.e. $S=1$ and each stage has the same operation time distribution and the same mean time with the parallel line.

Tables 4.1 ∿ 3 show the production rates for the various parallel lines, where the operation times are given by q-Erlang or constant distribution with the mean $1/\mu$. From these results it is appeared that the production rate increases as S increases but the rate of increase decreases and that the effect of S is larger when the buffer capacity M is small and the variance of the operation time is large. Furthermore in the unbalanced lines it is appeared that the production rate and the effect of S are larger and the rate of convergence to the maximum production rate yielded as $M\to\infty$ is fast when $S\lambda>\mu$.

These situations can be simply explained by the imaginary buffer capacity M_I as shown in Tables 4.4 ∿ 6, where the capacity M_I is estimated by applying the production rates of the series line to Newtons interpolation formula. That is, the imaginary capacity increases as S increases but the rate of increase decreases, and the effect of S is large when q is small. Furthermore, in the unbalanced lines, the imaginary capacity and the effect of S are larger when $S\lambda>\mu$, and they are also larger than those for the balanced line.

Like this we can discuss the effect of the number of stations by the imaginary buffer capacity instead of the production rate, so that it is convenient to use this capacity if it is calculated by the easier procedure than that in Section 3. And, better still, it is expected that the imaginary capacity has the linear asymptote as M becomes large and that this asymptote gives the approximation value of M_I within a small error for small M as well as large M. Therefore the effect of S can be discussed by this asymptote.

Now we present the procedure to estimate the asymptote on the assumption that $G(x)$ is q-Erlang distribution with the mean $1/\mu$.

First we consider the zeros of the denominator of (3.9). They are obtained by

$$(4.1) \qquad Y(Z) = 1 - Z/K(Z) = 0 \quad ,$$

where

$$K(Z) = U(S\lambda(1-Z)) = (1+\frac{\rho}{q}(1-Z))^{-q}$$

$$\rho = S\lambda/\mu \quad .$$

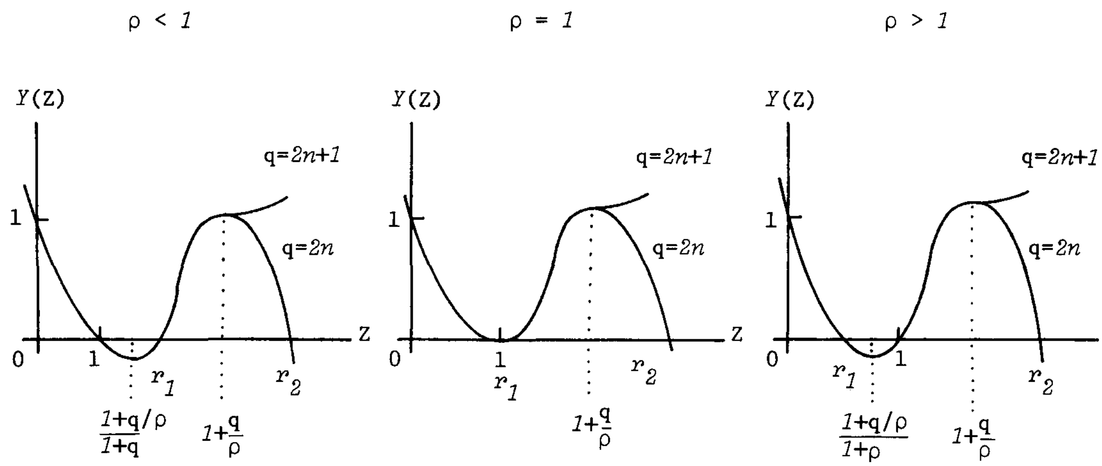

Fig. 4.1 The real roots

This equation has (q+1) roots but, as shown in Fig. 4.1, the number of the real roots is only two or three according to the value q. Therefore, if the roots are denoted by r_i $(i=0,1,2,\ldots,q)$ and especially the real roots are denoted by r_0, r_1, r_2 or r_0, r_1, the real roots exist in the range shown in Table 4.7 and we have (4.2) instead of (3.9),

$$(4.2) \qquad G_1(Z) = P_0\left[\frac{A_1}{1-\frac{1}{r_1}Z} + \frac{A_2}{1-\frac{1}{r_2}Z} + \ldots\ldots + \frac{A_q}{1-\frac{1}{r_q}Z}\right] ,$$

where A_i $(i=1,2,\ldots,q)$ is constant and $\sum_{i=1}^{q} A_i=1$. From this equation P_i $(0\leqq i\leqq M)$ is given by

$$(4.3) \qquad P_i = P_0 \sum_{\ell=1}^{q} A_\ell \left(\frac{1}{r_\ell}\right)^i .$$

Next, we consider the absolute value for the imaginary roots r_i $(i=2,3,\ldots,q$ or $i=3,4,\ldots,q)$. By using $\omega=1+(\rho/q)(1-Z)$, the equation (4.1) is re-written by

$$(4.4) \qquad 0 = 1 - \left(1+\frac{q}{\rho}(1-\omega)\right)\omega^q$$

$$= (1-\omega)\ \left(1+\omega+\omega^2+\ldots.+\omega^{q-1} - \frac{q}{\rho}\omega^q\right) ,$$

and hence

$$(4.5) \qquad Z = 1$$

$$(4.6) \qquad \frac{q}{\rho}\left(1+\frac{\rho}{q}(1-Z)\right)^q = 1+\left(1+\frac{\rho}{q}(1-Z)\right)+\ldots.+\left(1+\frac{\rho}{q}(1-Z)\right)^{q-1} .$$

Equation (4.6) contains the imaginary roots, so that by calculating the absolute value we have

$$(4.7) \qquad 1 = \frac{\rho}{q}\left| \frac{1}{1+\frac{\rho}{q}(1-Z)} + \ldots. + \frac{1}{\left(1+\frac{\rho}{q}(1-Z)\right)^q} \right|$$

$$\leqq \frac{\rho}{q}\left(\left|\frac{1}{1+\frac{\rho}{q}(1-Z)}\right| + \ldots. + \left|\frac{1}{1+\frac{\rho}{q}(1-Z)}\right|^q\right)$$

$$\leqq \frac{\rho}{q}\left(\frac{1}{\left|1+\frac{\rho}{q}(1-|Z|)\right|} + \ldots. + \frac{1}{\left|1+\frac{\rho}{q}(1-|Z|)\right|^q}\right) ,$$

where the equality is given by $Z = r_1$. Therefore, if Z is a complex number, we have

$$(4.8) \qquad \left|1 + \frac{\rho}{q}(1-|Z|)\right| < 1 + \frac{\rho}{q}(1-r_1)$$

and finally $|Z|$ is represented as follows;

$$(4.9) \qquad r_1 < |Z| < 2\left(1+\frac{q}{\rho}\right) - r_1 .$$

This means that the absolute value for the imaginary roots is larger than r_1 which is minimum in the real roots besides $r_0=1$.

On the other hand, by using (3.4), Ψ_k in (3.13) is

$$(4.10)\qquad \Psi_k = {}_SC_k\left(\frac{S}{S-k}\right)^M[\beta_{0,k}P_0+\beta_{0,k}\sum_{i=1}^{M}\left(\frac{S-k}{S}\right)^{i-1}P_i - \sum_{i=0}^{M-1}\left(\frac{S-k}{S}\right)^{i}P_i]$$

$$- \beta_{0,k}\ {}_{S+1}C_k\ P_M \qquad (1\leq k\leq S-1)$$

$$\Psi_S = P_M - \beta_{0,S}P_{M+1} - \beta_{0,S}\ {}_{S+1}C_S\ P_M \quad ,$$

and hence, by substituting (4.3), this equation becomes

$$(4.11)\qquad \Psi_k = {}_SC_k\ P_0\sum_{\ell=1}^{q} A_\ell \frac{\left(1-\frac{\beta_{0,k}}{r_\ell}\right)\left(\frac{1}{r_\ell}\right)^M}{1-\frac{S-k}{S}\frac{1}{r_\ell}} - \beta_{0,k}\ {}_{S+1}C_k\ P_0\sum_{\ell=1}^{q} A_\ell\left(\frac{1}{r_\ell}\right)^M$$

$$(1\leq k\leq S) \quad .$$

Furthermore from (3.13) H_0 is given by

$$(4.12)\qquad H_0 = \prod_{i=1}^{S}\phi_i P_M - \sum_{i=0}^{S-1}\psi_{i+1}\prod_{j=0}^{i}\phi_j$$

$$\psi_i = \Psi_i/\beta_{0,i}$$

$$\phi_i = \frac{1-\beta_{0,i}}{\beta_{0,i}} \qquad (i=1,2,\ldots,S)$$

$$\phi_0 = 1$$

and by substituting (4.11) we have

$$(4.13)\qquad H_0 = P_0\sum_{\ell=1}^{q} A_\ell\left(\frac{1}{r_\ell}\right)^M\left[\frac{1+\frac{1}{S}\frac{1}{r_\ell}}{1-\frac{S-1}{S}\frac{1}{r_\ell}} - \sum_{i=1}^{S-1} E_i\ {}_SC_{i+1}\ \cdot\right.$$

$$\left.\left\{\frac{i+1}{S^2}\ \frac{\left(\frac{1}{r_\ell}\right)^2}{\left(1-\frac{S-i}{S}\frac{1}{r_\ell}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_\ell}\right)}\right\}\right]$$

$$E_i = \prod_{j=1}^{i}\phi_j \quad .$$

Therefore normalizing condition (3.15) gives the actual probability for P_0 and we have

(4.14) $$\frac{1}{P_0} = \sum_{\ell=1}^{q} A_\ell \sum_{i=0}^{M+1} \left(\frac{1}{r_\ell}\right)^i + \sum_{\ell=1}^{q} A_\ell \left(\frac{1}{r_\ell}\right)^{M+2} \left[\frac{\frac{S-1}{S}}{1-\frac{S-1}{S}\frac{1}{r_\ell}} - \sum_{i=1}^{S-1} E_i \, {}_{S-1}C_i \frac{1}{S\left(1-\frac{S-i}{S}\frac{1}{r_\ell}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_\ell}\right)}\right] .$$

Especially in the series line, i.e. $S=1$, P_0 is represented by P_{01}

(4.15) $$\frac{1}{P_{01}} = \sum_{\ell=1}^{q} A_\ell \sum_{i=0}^{M+1} \left(\frac{1}{r_\ell}\right)^i .$$

Now we can solve the imaginary buffer capacity M_I. In the above equation P_0 and P_{01} are the function of M, so that this imaginary capacity is given by the following equality;

(4.16) $$R = \frac{S\lambda}{P_0(M) + \rho} = \frac{S\lambda}{P_{01}(M_I) + \rho}$$

and finally

(4.17) $$P_0(M) = P_{01}(M_I) .$$

To estimate the asymptote for M_I, let study the behavior of P_0 and P_{01} as M becomes large. From the discussion about the roots of (4.1), it is appeared that $1/r_1$ has the maximum value among $1/|r_i|\,(i=1,2,\ldots,q)$, so that (4.15) is rewritten as

(4.18) $$\frac{1}{P_{01}} = \sum_{\ell=1}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} - \left(\frac{1}{r_1}\right)^{M+2} \left\{\frac{A_1}{1-\frac{1}{r_1}} + \sum_{\ell=2}^{q} \frac{A_\ell\left(\frac{1/r_\ell}{1/r_1}\right)^{M+2}}{1-\frac{1}{r_\ell}}\right\} ,$$

and hence we have the asymptote for P_{01} as M becomes large as follows;

(4.19) $$\frac{1}{P_{01}} \to \sum_{\ell=2}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} + A_1 \frac{1-\left(\frac{1}{r_1}\right)^{M+2}}{1-\frac{1}{r_1}} \qquad \rho \neq 1 ,$$

$$\to \sum_{\ell=2}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} + A_1(M+2) \qquad \rho = 1 .$$

By the same way, (4.14) is rewritten as

$$(4.20)\quad \frac{1}{P_0} = \sum_{\ell=1}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} - \left(\frac{1}{r_1}\right)^{M+2} \left\{ \frac{A_1}{1-\frac{1}{r_1}} + \sum_{\ell=2}^{q} \frac{A_\ell\left(\frac{1/r_\ell}{1/r_1}\right)^{M+2}}{1-\frac{1}{r_\ell}} \right\}$$

$$+ \left(\frac{1}{r_1}\right)^{M+2} \Bigg[A_1 \left\{ \frac{\frac{S-1}{S}}{1-\frac{S-1}{S}\frac{1}{r_1}} - \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i \right.$$

$$\left. \frac{1}{S\left(1-\frac{S-i}{S}\frac{1}{r_1}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_1}\right)} \right\}$$

$$+ \sum_{\ell=2}^{q} A_\ell \left(\frac{1/r_\ell}{1/r_1}\right)^{M+2} \left\{ \frac{\frac{S-1}{S}}{1-\frac{S-1}{S}\frac{1}{r_\ell}} - \right.$$

$$\left. \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i \frac{1}{S\left(1-\frac{S-i}{S}\frac{1}{r_\ell}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_1}\right)} \right\} \Bigg]$$

and we have the asymptote for P_0 as follows;

$$(4.21)\quad \frac{1}{P_0} \to \sum_{\ell=2}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} + A_1 \frac{1-\left(\frac{1}{r_1}\right)^{M+2}}{1-\frac{1}{r_1}}$$

$$+ A_1 \left(\frac{1}{r_1}\right)^{M+2} \left\{ \frac{\frac{S-1}{S}}{1-\frac{S-1}{S}\frac{1}{r_1}} - \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i \right.$$

$$\left. \frac{1}{S\left(1-\frac{S-i}{S}\frac{1}{r_1}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_1}\right)} \right\} \qquad \rho \neq 1 ,$$

$$\to \sum_{\ell=2}^{q} \frac{A_\ell}{1-\frac{1}{r_\ell}} + A_1 (M+2)$$

$$+ A_1 \left\{ S-1- \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i \frac{1}{S\left(1-\frac{S-i}{S}\right)\left(1-\frac{S-1-i}{S}\right)} \right\}$$

$$\rho = 1 .$$

Therefore, from (4.17), the asymptote for M_I is given by the following simple equation;

$$(4.22)\qquad M_I = M + \frac{1}{\log(\frac{1}{r_1})}\log\left[1-\left(1-\frac{1}{r_1}\right)\left\{\frac{\frac{S-1}{S}}{1-\frac{S-1}{S}\frac{1}{r_1}} - \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i\, \frac{1}{S\left(1-\frac{S-i}{S}\frac{1}{r_1}\right)\left(1-\frac{S-1-i}{S}\frac{1}{r_1}\right)}\right\}\right] \qquad \rho \neq 1$$

$$= M + S - 1 - \sum_{i=1}^{S-1} E_i\, {}_{S-1}C_i\, \frac{S}{i(i+1)} \qquad \rho = 1$$

This equation appears that the imaginary buffer capacity has the linear asymptote with the slope 1 as expected in Tables 4.4 ∿ 6. Furthermore, it appears that the asymptote is decided only by the minimum real root r_1 and, especially for $\rho=1$, this asymptote is completely decided since $r_1=1$. In Table 4.8, the asymptotes for the various production lines are represented. From the comparison between Table 4.8 and Tables 4.4 ∿ 6, it is appeared that the asymptote given by (4.22) presents the approximation values on the accuracy within 0.01 for $M \geq 1$, so that this is very useful to evaluate the effect of S and also the other system parameters.

5. Conclusion

We have discussed the two-stage parallel line in which one of stages has a station but the other has S stations.

First, the procedure to estimate the production rate is represented and the relations between the dual models are discussed. This appears that the dual models have the same production rate and the various characters for the one model can be derived from the other dual model.

Next, we introduce the imaginary buffer capacity to evaluate the effect of the number of stations and show that the asymptote for the imaginary capacity can be obtained by the simple procedure and presents the good approximation value for the imaginary capacity. Therefore we can evaluate the effect of the number of stations by this asymptote.

Table 4.1 Production rate for the balanced line ($1/S\lambda$= $1/\mu$= 1.0)

$G(x)$	S	M=0	M=1	M=2	M=4	M=6	M=8	M=10
	1	0.6667	0.7500	0.8000	0.8571	0.8889	0.9091	0.9231
q=1	2	0.7143	0.7778	0.8182	0.8667	0.8947	0.9130	0.9259
Erlang	4	0.7630	0.8084	0.8392	0.8783	0.9021	0.9182	0.9297
	1	0.7133	0.8046	0.8525	0.9011	0.9256	0.9404	0.9503
q=5	2	0.7579	0.8266	0.8654	0.9071	0.9290	0.9426	0.9518
Erlang	4	0.8021	0.8507	0.8804	0.9145	0.9334	0.9455	0.9539
	1	0.7217	0.8138	0.8608	0.9076	0.9308	0.9447	0.9540
q-10	2	0.7653	0.8345	0.8727	0.9130	0.9339	0.9467	0.9554
Erlang	4	0.8085	0.8574	0.8867	0.9198	0.9379	0.9493	0.9572
	1	0.7311	0.8237	0.8696	0.9143	0.9362	0.9492	0.9578
Const.	2	0.7734	0.8431	0.8805	0.9192	0.9389	0.9509	0.9590
	4	0.8154	0.8646	0.8934	0.9253	0.9425	0.9532	0.9606

Table 4.2 Production rate for the unbalanced line ($1/S\lambda$=1.0, $1/\mu$=2.0)

$G(x)$	S	M=0	M=1	M=2	M=3	M=4	M=5	M=6
	1	0.4286	0.4667	0.4839	0.4921	0.4961	0.4980	0.4990
q=1	2	0.4546	0.4783	0.4894	0.4947	0.4974	0.4987	0.4994
Erlang	4	0.4773	0.4889	0.4945	0.4973	0.4986	0.4993	0.4997
	1	0.4547	0.4885	0.4968	0.4991	0.4997	0.4999	0.5000
q=5	2	0.4773	0.4937	0.4982	0.4995	0.4999	0.5000	0.5000
Erlang	4	0.4917	0.4977	0.4993	0.4998	0.4999	0.5000	0.5000
	1	0.4626	0.4912	0.4979	0.4995	0.4999	0.5000	0.5000
q=10	2	0.4807	0.4954	0.4989	0.4997	0.4999	0.5000	0.5000
Erlang	4	0.4938	0.4984	0.4996	0.4999	0.5000	0.5000	0.5000
	1	0.4683	0.4938	0.4988	0.4998	0.4999	0.5000	0.5000
Const.	2	0.4843	0.4969	0.4994	0.4999	0.5000	0.5000	0.5000
	4	0.4951	0.4990	0.4998	0.5000	0.5000	0.5000	0.5000

Table 4.3 Production rate for the unbalanced line ($1/S\lambda$=2.0, $1/\mu$=1.0)

$G(x)$	S	M=0	M=1	M=2	M=3	M=4	M=5	M=6
	1	0.4286	0.4667	0.4839	0.4921	0.4961	0.4980	0.4990
q=1	2	0.4483	0.4754	0.4880	0.4941	0.4971	0.4985	0.4993
Erlang	4	0.4653	0.4832	0.4918	0.4959	0.4980	0.4990	0.4995
	1	0.4461	0.4821	0.4940	0.4980	0.4993	0.4998	0.4999
q=5	2	0.4625	0.4874	0.4957	0.4986	0.4995	0.4998	0.4999
Erlang	4	0.4759	0.4918	0.4972	0.4991	0.4997	0.4999	0.5000
	1	0.4489	0.4842	0.4951	0.4984	0.4995	0.4999	0.5000
q=10	2	0.4647	0.4889	0.4966	0.4989	0.4997	0.4999	0.5000
Erlang	4	0.4775	0.4929	0.4978	0.4993	0.4998	0.4999	0.5000
	1	0.4519	0.4864	0.4962	0.4989	0.4997	0.4999	0.5000
Const.	2	0.4670	0.4905	0.4973	0.4992	0.4998	0.4999	0.5000
	4	0.4791	0.4939	0.4983	0.4995	0.4999	0.5000	0.5000

Table 4.4 Imaginary buffer capacity for the balanced line ($1/S\lambda$= $1/\mu$=1.0)

$G(x)$	S	M=0	M=1	M=2	M=4	M=6	M=8	M=10
	2	0.51	1.50	2.50	4.50	6.50	8.50	10.50
q=1	3	0.89	1.89	2.89	4.89	6.89	8.89	10.89
Erlang	5	1.51	2.51	3.51	5.51	7.51	9.51	11.51
	2	0.42	1.40	2.39	4.39	6.39	8.39	10.39
q=5	3	0.71	1.70	2.69	4.69	6.69	8.69	10.69
Erlang	5	1.18	2.17	3.18	5.17	7.18	9.17	11.18
	2	0.40	1.38	2.37	4.37	6.37	8.37	10.37
q=10	3	0.69	1.67	2.66	4.66	6.66	8.66	10.66
Erlang	5	1.13	2.12	3.12	5.12	7.12	9.12	11.12
	2	0.36	1.35	2.35	4.35	6.35	8.35	10.35
Const.	3	0.64	1.63	2.63	4.63	6.53	8.63	10.63
	5	1.08	2.07	3.07	5.06	7.06	9.06	11.07

Table 4.5 Imaginary buffer capacity for the unbalanced line ($1/S\lambda=1.0, 1/\mu=2.0$)

$G(x)$	S	$M=0$	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$
q=1 Erlang	2	0.59	1.58	2.59	3.58	4.59	5.58	6.59
	3	1.08	2.08	3.08	4.08	5.08	6.08	7.08
	5	1.94	2.94	3.94	4.94	5.94	6.94	7.94
q=5 Erlang	2	0.48	1.47	2.47	3.47	4.48		
	3	0.88	1.88	2.88	3.88	4.88		
	5	1.60	2.60	3.60	4.61			
q=10 Erlang	2	0.46	1.45	2.45	3.45	4.46		
	3	0.84	1.84	2.84	3.84	4.85		
	5	1.54	2.55	3.54	4.56			
Const.	2	0.44	1.42	2.43	3.42			
	3	0.80	1.79	2.80	3.79			
	5	1.47	2.48	3.46				

Table 4.6 Imaginary buffer capacity for the unbalanced line ($1/S\lambda=2.0, 1/\mu=1.0$)

$G(x)$	S	$M=0$	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$
q=1 Erlang	2	0.42	1.41	2.42	3.41	4.41	5.41	6.41
	3	0.71	1.71	2.71	3.71	4.71	5.71	6.72
	5	1.14	2.14	3.14	4.14	5.14	6.14	7.14
q=5 Erlang	2	0.33	1.32	2.32	3.31	4.31	5.32	
	3	0.56	1.54	2.54	3.54	4.54		
	5	0.88	1.85	2.85	3.84	4.85		
q=10 Erlang	2	0.32	1.30	2.30	3.30	4.31		
	3	0.54	1.51	2.51	3.51			
	5	0.84	1.81	2.80	3.80			
Const.	2	0.31	1.28	2.28	3.28			
	3	0.51	1.48	2.48	3.48			
	5	0.79	1.76	2.76	3.75			

Table 4.7 The range in which the real roots exist

q	$\rho < 1$	$\rho = 1$	$\rho > 1$
$q = 2n$	$r_0 = 1$ $\frac{1 + q/\rho}{1 + q} < r_1 < 1 + \frac{q}{\rho}$ $1 + \frac{q}{\rho} < r_2$	$r_0 = 1$ $r_1 = 1$ $1 + q < r_2$	$r_0 = 1$ $0 < r_1 < \frac{1 + q/\rho}{1 + q}$ $1 + \frac{q}{\rho} < r_2$
$q = 2n+1$	$r_0 = 1$ $\frac{1 + q/\rho}{1 + q} < r_1 < 1 + \frac{q}{\rho}$	$r_0 = 1$ $r_1 = 1$	$r_0 = 1$ $0 < r_1 < \frac{1 + q/\rho}{1 + q}$

Table 4.8 The asymptote $M_I = M + a$

$G(x)$	ρ	a				
		S=2	S=3	S=4	S=5	S=10
q=1 Erlang	2.0	0.585	1.078	1.524	1.939	3.792
	1.0	0.500	0.889	1.219	1.510	2.660
	0.5	0.415	0.711	0.945	1.141	1.825
q=5 Erlang	2.0	0.471	0.876	1.249	1.602	3.223
	1.0	0.390	0.692	0.948	1.175	2.067
	0.5	0.315	0.534	0.705	0.847	1.333
q=10 Erlang	2.0	0.449	0.839	1.200	1.543	3.126
	1.0	0.371	0.660	0.905	1.122	1.975
	0.5	0.299	0.507	0.669	0.803	1.261
Const.	2.0	0.424	0.797	1.144	1.476	3.017
	1.0	0.351	0.626	0.859	1.065	1.878
	0.5	0.282	0.478	0.631	0.757	1.185

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