

## THE INTERCHANGEABILITY OF THE SOJOURN AND DELAY TIMES IN A $GI/M/1 \rightarrow /M/1(0)$ QUEUE

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*Abstract* This paper is concerned with a system of two queues in tandem where each queue has a single exponential server. The rates of the servers may differ. The first queue has an infinite waiting room, whereas the second queue has no waiting room. Initially the system is empty. Customers enter the first queue according to a renewal input process, and then pass through the second queue.

We are concerned with the sojourn time of a customer in the system. We show that the queues are interchangeable, in the sense that the order of queues does not affect the sojourn and delay distributions.

### 1. Introduction

The output process of the following tandem queueing system has been recently studied by Weber [4]. The system of queues in tandem, denoted by  $\cdot/M_1/1 \rightarrow /M_2/1 \rightarrow \dots \rightarrow /M_K/1$ , consists of  $K$  queues of  $\cdot/M/1$  type arranged in a serial order. Each queue has an infinite waiting room and a single exponential server. Customers enter the first queue according to an arbitrary stochastic input process (need not to be a renewal process), and then pass through the queues in order. In [4] the output process has been proved to have the following interesting property:

- (a) The order of the queues does not affect the distribution of any statistics of the output process (such as the  $n$ -th customer departure time).

In other words, for the above system the order of the queues does not affect 'queueing behavior'.

This is not the case with blocking in general, i.e., the case including some queues with a finite waiting room, as shown in the following counterexample.

*Counterexample* (by courtesy of Dr. S. Fujii). Consider the system

$\cdot/M_1/1 \rightarrow M_2/1 \rightarrow M_3/1$ , and suppose that the first server, the second one and the third one have rate 1, 2 and 3, respectively, and that a queue always exists in front of the first server, but no queues are allowed in front of other servers (System 1). By calculating on the basis of [1], then, we have

$$[\text{the mean equilibrium interdeparture time}] = 179/150$$

and

$$[\text{the equilibrium interdeparture time variance}] = 26209/22500.$$

For System 2, which is obtained by interchanging the second server and the third one in the System 1, we similarly have

$$[\text{the mean equilibrium interdeparture time}] = 139/120$$

and

$$[\text{the equilibrium interdeparture time variance}] = 16279/14400.$$

Hence this system is not featured by (a).

In case  $K=2$ , however, even with blocking, it is considered that the system is featured by similar property (although weaker than (a)). For example, in the case where the first queue has an infinite waiting room, but the second has a finite, and where the input process to the system is Erlang type, the following property has been numerically obtained by Tumura and Ishikawa [3].

- (b) The order of queues does not affect the stationary distribution of the customers' number in the system.

This paper aims at obtaining a result similar to (b) for the delay and sojourn times in a typical system with blocking (meaning the system with no intermediate queue) under interarrival times identically and independently distributed. A similar result stronger than (b) is obtained subsequently for the distribution of the customers' number immediately after the departure of a customer in this system.

## 2. Description of the System

This paper is concerned with the following two-stage tandem queueing system. There are two service facilities (or servers for short) arranged in tandem. Each customer arriving at the system receives the service by the first server (server 1), then the second (server 2), before leaving the system. An infinite queue is allowed to form before the server 1, whereas no queue before the server 2 is allowed. If the server 2 is busy, therefore, when a service is completed to a customer by the server 1, this customer stays at the first stage and blocks further service until the server 2

becomes free. The service discipline is defined on FCFS basis. The  $n$ -th customer ( $C_n$ ) arrives at the instant  $T_n$  and has the service time  $S_{k,n}$  by the server  $k$  ( $k=1,2$ ), and we define  $A_n = T_n - T_{n-1}$ . It is assumed that  $A_1, A_2, \dots, S_{1,1}, S_{1,2}, \dots, S_{2,1}, S_{2,2}, \dots$  are mutually independent, the  $A_n$  are identically distributed random variables (r.v.'s) with a distribution function (d.f.)  $A(t)$ , and the  $S_{k,n}$  are also identically distributed r.v.'s with the d.f.  $1 - \exp(-\mu_k t)$  ( $k=1,2$ ). For such a system, a notation GI/M<sub>1</sub>/1 → M<sub>2</sub>/1(0) is employed.

For the system GI/M<sub>2</sub>/1 → M<sub>1</sub>/1(0), which is obtained by interchanging two service facilities in the system GI/M<sub>1</sub>/1 → M<sub>2</sub>/1(0), we denote by the  $S_{k,n}^*$  the service time of  $C_n$  at the  $k$ -th stage ( $k=1,2$ ).

### 3. Interchangeability

For the system GI/M<sub>1</sub>/1 → M<sub>2</sub>/1(0), let  $W_n$  make the departure time of  $C_n$  from the first stage minus the arrival time of  $C_n$  at the first stage. Using  $S_{1,n}, S_{2,n}$  and  $W_n$  we also define other r.v.'s as follows:

$$(3.1) \quad \begin{aligned} S_n &= W_n + S_{2,n} \quad (\text{the sojourn time of } C_n \text{ at the system}), \\ D_n &= W_n - S_{1,n} \quad (\text{the delay time of } C_n \text{ at the system}). \end{aligned}$$

We similarly define  $W_n^*, S_n^*$  and  $D_n^*$  for the system GI/M<sub>2</sub>/1 → M<sub>1</sub>/1(0).

The objective of this section, then, is to prove the following theorem.

**Theorem.** Let both systems start from scratch, so that  $W_1 = S_{1,1}$  and  $W_1^* = S_{1,1}^*$ . Then

$$(3.2) \quad S_n \stackrel{d}{=} S_n^*,$$

$$(3.3) \quad D_n \stackrel{d}{=} D_n^*,$$

where the sign  $\stackrel{d}{=}$  denotes the equality of distribution.

**Proof:** In order to prove (3.2) it is sufficient to show that for any real and positive number  $\theta$

$$(3.4) \quad \phi_n(\theta)\Psi_2(\theta) = \phi_n^*(\theta)\Psi_1(\theta),$$

where

$$(3.5) \quad \begin{aligned} \phi_n(\theta) &= E(\exp(-\theta W_n)), \quad \phi_n^*(\theta) = E(\exp(-\theta W_n^*)), \\ \Psi_1(\theta) &= E(\exp(-\theta S_{2,n}^*)) \quad (= \frac{\mu_1}{\theta + \mu_1}), \\ \Psi_2(\theta) &= E(\exp(-\theta S_{2,n})) \quad (= \frac{\mu_2}{\theta + \mu_2}). \end{aligned}$$

We have the following recurrence relation for  $W_n$  (cf. [2]).

$$(3.6) \quad W_n = S_{1,n} \vee (S_{1,n} \vee S_{2,n-1} - A_n + W_{n-1}),$$

where  $X \vee Y$  denotes the maximum of r.v.'s  $X$  and  $Y$ . From (3.6),

$$(3.7) \quad \begin{aligned} & \Pr(W_{n+1} \leq t | A_{n+1} = a) \\ &= \int_0^a \Pr(S_{1,n+1} \leq t) \Pr(S_{2,n} \leq t + a - y) d\Pr(W_n \leq y) + \\ & \quad \int_a^{t+a} \Pr(S_{1,n+1} \leq t + a - y) \Pr(S_{2,n} \leq t + a - y) d\Pr(W_n \leq y). \end{aligned}$$

By taking account of  $\Pr(S_{k,n} \leq t) = 1 - \exp(-\mu_k t)$  ( $k=1,2$ ), we can obtain

$$(3.8) \quad \begin{aligned} & d_t \Pr(W_{n+1} \leq t | A_{n+1} = a) \\ &= d_t \Pr(W_n \leq t + a) - d_t \{e^{-\mu_2(t+a)} \int_0^{t+a} e^{\mu_2 y} d\Pr(W_n \leq y)\} - \\ & \quad d_t \{\Pr(W_n \leq a) e^{-\mu_1 t}\} + d_t \{e^{-\mu_2 a} \int_0^a e^{\mu_2 y} d\Pr(W_n \leq y)\} e^{-(\mu_1 + \mu_2)t} - \\ & \quad d_t \{e^{-\mu_1(t+a)} \int_a^{t+a} e^{\mu_1 y} d\Pr(W_n \leq y)\} + d_t \{e^{-(\mu_1 + \mu_2)(t+a)} \\ & \quad \int_a^{t+a} e^{(\mu_1 + \mu_2)y} d\Pr(W_n \leq y)\}. \end{aligned}$$

Multiplying both sides of (3.8) by  $\exp(-\theta t)$ , integrating over  $[0, \infty)$  and unconditioning, we can obtain

$$(3.9) \quad \begin{aligned} \phi_{n+1}(\theta) &= -C_1 \frac{\theta \mu_1}{(\mu_1 + \mu_2 + \theta)(\mu_2 + \theta)} + C_2 \{\psi_1(\theta) + \psi_2(\theta) - \psi_{12}(\theta)\} + \\ & \quad C_3 \psi_1(\theta), \end{aligned}$$

where

$$(3.10) \quad \begin{aligned} C_1 &= \int_0^\infty dA(a) \int_0^a e^{-\mu_2(a-y)} d\Pr(W_n \leq y), \quad C_2 = \int_0^\infty dA(a) \int_a^\infty e^{-\theta(y-a)} d\Pr(W_n \leq y), \\ C_3 &= \int_0^\infty \Pr(W_n \leq a) dA(a). \end{aligned}$$

and

$$(3.11) \quad \psi_{12}(\theta) = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \theta}.$$

Similarly,

$$(3.12) \quad \begin{aligned} \phi_{n+1}^*(\theta) &= -C_1^* \frac{\mu_2 \theta}{(\mu_1 + \mu_2 + \theta)(\mu_1 + \theta)} + C_2^* \{\psi_1(\theta) + \psi_2(\theta) - \psi_{12}(\theta)\} + \\ & \quad C_3^* \psi_2(\theta), \end{aligned}$$

where

$$(3.13) \quad \begin{aligned} C_1^* &= \int_0^\infty dA(a) \int_0^a e^{-\mu_1(a-y)} d\Pr(W_n^* \leq y), \quad C_2^* = \int_0^\infty dA(a) \int_a^\infty e^{-\theta(y-a)} \\ & \quad d\Pr(W_n^* \leq y), \\ C_3^* &= \int_0^\infty \Pr(W_n^* \leq a) dA(a). \end{aligned}$$

Based on such preparations, we prove (3.4) by induction on  $n$ . For  $n=1$  (3.4) is true since  $\phi_1(\theta) = \psi_1(\theta)$  and  $\phi_1^*(\theta) = \psi_2(\theta)$ . Suppose that (3.4) is true for  $n$ , i.e.,

$$(3.14) \quad \phi_n^*(\theta) = \frac{\psi_2(\theta)}{\psi_1(\theta)} \phi_n(\theta) = \frac{\mu_2}{\mu_1} \phi_n(\theta) + \frac{\mu_2(\mu_1 - \mu_2)}{\mu_1(\mu_2 + \theta)} \phi_n(\theta) ;$$

we shall show that it is true for  $n+1$ . Inversion of both sides of (3.14) yields

$$(3.15) \quad d\Pr(W_n^* \leq t) = \frac{\mu_2}{\mu_1} d\Pr(W_n \leq t) + \frac{\mu_2(\mu_1 - \mu_2)}{\mu_1} \int_0^t e^{-\mu_2(t-y)} d\Pr(W_n < y) dt.$$

By appropriately combining  $C_i$  with  $C_j^*$  ( $i, j=1, 2, 3$ ) by means of (3.15) we have

$$(3.16) \quad C_1^* = \frac{\mu_2}{\mu_1} C_1, \quad C_2^* = \frac{\psi_2(\theta) \cdot (\mu_1 - \mu_2)}{\mu_1} C_1 + \frac{\psi_2(\theta)}{\psi_1(\theta)} C_2, \quad C_3^* = C_3 - \frac{\mu_1 - \mu_2}{\mu_1} C_1.$$

By substituting (3.16) into (3.12) and using (3.9) we can obtain

$$(3.17) \quad \phi_{n+1}^*(\theta) \psi_1(\theta) = \phi_{n+1}(\theta) \psi_2(\theta).$$

Hence the induction is complete and the first assertion has been proved.

To prove the second assertion, it is noted that a combination (3.1) with (3.6) yields

$$(3.18) \quad D_n = 0 \vee (R_n + W_{n-1} - A_n),$$

where  $R_n = S_{1,n} \vee S_{2,n-1} - S_{1,n}$ .

We similarly define  $R_n^*$  for the system GI/M<sub>2</sub>/1 → M<sub>1</sub>/1(0), so that

$$(3.19) \quad D_n^* = 0 \vee (R_n^* + W_{n-1} - A_n).$$

Then we have

$$(3.20) \quad \gamma_n(\theta) \psi_1(\theta) = \gamma_n^*(\theta) \psi_2(\theta),$$

where  $\gamma_n(\theta) = E(\exp(-\theta R_n))$  and  $\gamma_n^*(\theta) = E(\exp(-\theta R_n^*))$ . This is calculated in Appendix.

It follows from (3.4) and (3.20) that

$$(3.21) \quad \gamma_n(\theta) \phi_{n-1}(\theta) = \gamma_n^*(\theta) \phi_{n-1}^*(\theta),$$

so that

$$(3.22) \quad R_n + W_{n-1} \stackrel{d}{=} R_n^* + W_{n-1}^*.$$

From (3.18), (3.19) and (3.22) we can obtain (3.3). The proof is, therefore, complete.

For both systems the number of customers left behind by a departing customer is identical with the number of customers arriving during the stay of the customer. Thus, if we let  $L_{n+}$  ( $L_{n+}^*$ ) be the number of customers

immediately after the  $n$ -th departure in the system  $GI/M_1/1 \rightarrow M_2/1(0)$  ( $GI/M_2/1 \rightarrow M_1/1(0)$ ), we immediately have from the above theorem

Corollary. Let both systems start from scratch. Then

$$(3.23) \quad L_{n+} \stackrel{d}{=} L_{n+}^*.$$

## Appendix

From the definition of  $R_n$  we have

$$(A.1) \quad \Pr(R_n \leq x) = \begin{cases} 0 & \text{for } x < 0. \\ \int_0^\infty \Pr(S_{2,n-1} \leq x+y) d\Pr(S_{1,n} \leq y) & \text{for } x \geq 0. \end{cases}$$

Using  $\Pr(S_{k,n} \leq t) = 1 - \exp(-\mu_k t)$  for  $t \geq 0$  ( $k=1,2$ ), the integral of the right hand side of (A.1) becomes

$$1 - \frac{\mu_1}{\mu_1 + \mu_2} e^{-\mu_2 x}.$$

Thus,

$$(A.2) \quad \begin{aligned} \gamma_n(\theta) &= E(\exp(-\theta R_n)) = \int_{0-}^\infty e^{-\theta x} d\Pr(R_n \leq x) \\ &= \int_{0-}^{0+} e^{-\theta x} d\Pr(R_n \leq x) + \int_{0+}^\infty e^{-\theta x} d\Pr(R_n \leq x) \\ &= \frac{\mu_2}{\mu_1 + \mu_2} + \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)(\theta + \mu_2)}, \end{aligned}$$

so that

$$(A.3) \quad \gamma_n(\theta) \psi_1(\theta) = \frac{\mu_1 \mu_2 (\mu_1 + \mu_2 + \theta)}{(\theta + \mu_1)(\theta + \mu_2)(\mu_1 + \mu_2)}$$

Since the right hand side of (A.3) is symmetric with respect to  $\mu_1$  and  $\mu_2$ , we find that (3.29) holds.

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