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A SUMMARY OF PERIODIC REPLACEMENT WITH MINIMAL REPAIR AT FAILURE

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Abstract Several periodic replacement policies with minimal repair at failures are summarized: 1) A policy for a unit with random and wearout failures. 2) Two modified policies where if a failure occurs just before the replacement time, then (i) a unit remains failed, (ii) a unit is replaced by a new one. 3) Three imperfect preventive maintenance (pm) policies where (i) a unit after pm has the same failure rate as before pm with a certain probability, (ii) the age of a unit becomes x units of time younger at pm, (iii) the age of a unit after pm reduces to at at pm. Expected cost rates for each model are obtained and optimum policies are discussed. Some examples for the above models are presented.

1. Introduction

Barlow and Hunter [1] considered the following replacement policy: A unit is replaced periodically at scheduled times kT (k = 1, 2, ...). After each failure, only minimal repair is made so that the failure rate remains undisturbed by any repair of failures between successive replacements. This policy is commonly used with complex systems such as computers and airplanes. Holland and McLean [6] provided a practical procedure for applying the policy to large motors and small electrical parts. Morimura [8] has modified the policy in the way of the version that a unit is replaced at the k^{th} failure and the $(k - 1)^{th}$ previous failures are corrected with minimal repair. Further, Tilquin and Cléroux [17] introduced the adjustment costs which increase with the age of a unit. Tahara and Nishida [15] also introduced the breakdown cost suffered for a failed unit which is replaced at the first failure after some age.

In this paper, we summarize the known results of the policy, and consider extended and modified models which could be applicable to practical fields. For instance, we consider the policy for a used unit of age x and for a unit with random and wearout failures, and a discrete time policy where a unit operates at discrete times. Further, we consider two modifications of the policy in which any failed unit just before the scheduled replacement undergoes no repair. Finally, three imperfect preventive maintenance models with minimal repair at failures are presented. We discuss optimum policies which minimize the expected cost rates for each model. Some useful remarks for optimum policies are further made.

2. Known Results and Remarks

A unit is replaced at scheduled times kT ($k = 1, 2, \ldots$) and any unit is as good as new after replacement. Only minimal repair is made when the unit fails between periodic replacements. So that, the failure rate of the unit remains undisturbed by any repair of failures. Assume that the repair and replacement times are negligible.

Suppose that the failure times of each unit are independent, and have a density $f(t)$ and a distribution $F(t)$. Then, the following results were obtained by [2, p. 96]: The expected cost rate is

$$(2.1) \qquad C(T) = \frac{c_1 \int_0^T r(t)dt + c_2}{T},$$

where $r(t)$ = failure rate of the failure time distribution $F(t)$, i.e., $r(t) \equiv f(t)/\overline{F}(t)$ where $\overline{F} \equiv 1 - F$,

c_1 = cost of minimal repair,

c_2 = cost of scheduled replacement.

The purpose is to seek an optimum replacement time T^* which minimizes the expected cost rate $C(T)$. Differentiating $C(T)$ with respect to T and setting it equal to zero imply

$$(2.2) \qquad Tr(T) - \int_0^T r(t)dt = c_2/c_1.$$

Suppose that the failure rate $r(t)$ is monotonely increasing. Then, if a solution T^* to (2.2) exists, it is unique, and the resulting cost is

$$(2.3) \qquad C(T^*) = c_1 r(T^*).$$

Further, equation (2.2) can be rewritten as

$$(2.2') \qquad \int_0^T t dr(t) = c_2/c_1.$$

Thus, if $\int_0^\infty t dr(t) > c_2/c_1$ then there exists a solution to (2.2).

Remarks

(i) Suppose that $r(t)$ is monotonely increasing. Then, the optimum time T^* is not greater than that of the standard age replacement model [2, p. 85] in which the expected cost rate is

$$(2.4) \qquad C_A(T) = \frac{c_1 F(T) + c_2}{\int_0^T \overline{F}(t)dt},$$

and the optimum time is given by a solution of the equation

$$(2.5) \qquad r(T)\int_0^T \overline{F}(t)dt - F(T) = c_2/c_1.$$

For, we easily have the inequality

$$\int_0^T [r(T) - r(t)]F(t)dt \geq 0,$$

since $r(t)$ is increasing. Therefore

$$(2.6) \qquad Tr(T) - \int_0^T r(t)dt \geq r(T)\int_0^T \overline{F}(t)dt - F(T).$$

(ii) When we adopt the total expected cost as an appropriate objective function for an infinite time span, we should evaluate values of all future costs by using a discount rate. We apply the continuous discounting to the costs at the times when these costs occur actually. Let α be a positive discount rate and $C(T;\alpha)$ be the total expected cost for the policy. In this case, equations (2.1), (2.2), and (2.3) are rewritten, respectively, as follows:

$$(2.7) \qquad C(T;\alpha) = \frac{c_1\int_0^T e^{-\alpha t}\, r(t)dt + c_2 e^{-\alpha T}}{1 - e^{-\alpha T}},$$

$$(2.8) \qquad \frac{1 - e^{-\alpha T}}{\alpha} r(T) - \int_0^T e^{-\alpha t}\, r(t)dt = \frac{c_2}{c_1},$$

$$(2.9) \qquad C(T^*;\alpha) = (c_1/\alpha)r(T^*) - c_2.$$

Note that $\lim_{\alpha\to 0} \alpha C(T;\alpha) = C(T)$ which is the expected cost rate without discounting.

(iii) Consider a system consisting of n identical units which operate independently each other. Assume that all together are replaced at times kT (k = 1, 2, ...) and each failed unit between replacements undergoes minimal repair. Then, the expected cost rate is

$$(2.10) \qquad C(T;n) = \frac{nc_1\int_0^T r(t)dt + c_2}{T},$$

where c_1 = cost of minimal repair for one failed unit,

c_2 = cost of scheduled replacement for all units.

(iv) Consider the same policy for a used unit. A unit is replaced at times kT by the same used unit of age x, where x is previously specified. Then, the expected cost rate is, from [12],

$$(2.11)\qquad C(T;x) = \frac{c_1\int_x^{T+x} r(t)dt + c_2(x)}{T},$$

where $c_2(x)$ = acquisition cost of a used unit of age x. In this case, equations (2.2) and (2.3) are rewritten as

$$(2.12)\qquad Tr(T+x) - \int_x^{T+x} r(t)dt = c_2(x)/c_1,$$

$$(2.13)\qquad C(T^*;x) = c_1 r(T^*+x).$$

Next, consider the problem that it is the most economical to use a unit of what is the age. Suppose that x is a variable and inversely, T is constant, and $c_2(x)$ is differentiable. Then, differentiating $C(T;x)$ with respect to x and setting it equal to zero imply

$$(2.14)\qquad r(T+x) - r(x) = -c_2'(x)/c_1,$$

which is a necessary condition that a finite x minimizes $C(T;x)$ for a fixed T.

(v) Consider a unit which operates at discrete times n (n = 1, 2, ...). The unit is replaced at times kN (k = 1, 2, ...) and any failed unit between replacements undergoes minimal repair. Note that N corresponds to T in the continuous time model. Let $\{p_n\}_{n=1}^{\infty}$ denote the discrete failure distribution that the unit fails at time n. Then, the expected cost rate is

$$(2.15)\qquad C(N) = \frac{c_1\sum_{n=1}^{N} r(n) + c_2}{N} \qquad (N = 1, 2, \ldots),$$

where $r(n)$ = failure rate of the discrete failure distribution, i.e., $r(n) \equiv p_n / \sum_{j=n}^{\infty} p_j$ (n = 1, 2, ...).

We can convert the known results in continuous case to the discrete model as follows: From the two inequalities $C(N+1) \geq C(N)$ and $C(N) < C(N-1)$ (N = 1, 2, ...), we have, respectively,

$$(2.16)\qquad L(N) \geq c_2/c_1 \text{ and } L(N-1) < c_2/c_1,$$

where

$$L(N) = \begin{cases} Nr(N+1) - \sum_{n=1}^{N} r(n) & (N = 1, 2, \ldots), \\ 0 & (N = 0). \end{cases}$$

It is easily seen that if $r(n)$ is monotonely increasing then $L(N)$ is monotonely increasing. Thus, if a solution N^* to (2.16) exists, it is unique and minimizes the expected cost rate $C(N)$.

Example

Suppose that the failure time distribution is a discrete Weibull with a shape parameter 2, i.e., $p_n = q^{(n-1)^2} - q^{n^2}$ ($n = 1, 2, \ldots; 0 < q < 1$) (see [14]). Then, we have $r(n) = 1 - q^{2n-1}$ ($n = 1, 2, \ldots$), which is monotonely increasing from $1 - q$ to 1. From (2.16), an optimum replacement time N^* is given by a maximum of N which satisfies

$$\frac{q}{1 - q^2} \{1 - [1 + N(1 - q^2)]q^{2N}\} \geq \frac{c_2}{c_1} .$$

For example, if $q = 0.95$ then we have $N^* = 2, 4, 5, 8, 11, 14, 17$ for each $c_2/c_1 = 0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0$, respectively.

3. Replacement Policy with Random and Wearout Failures

Mine and Kawai [7] considered a modified replacement policy for a unit with random and wearout failures, where an operating unit enters a wearout failure period at a fixed time T_0, after it has operated continuously in a random failure period. We assume that the unit is replaced at scheduled time $T + T_0$, where T_0 is constant and previously given, and it undergoes only minimal repair at failures between replacements.

Suppose that the unit has a constant failure rate λ in a random failure period and $\lambda + r(t)$ in a wearout failure period. Then, the expected cost rate is given by

$$C_1(T;T_0) = c_1\lambda + \frac{c_1\int_0^T r(t)dt + c_2}{T + T_0} . \tag{3.1}$$

Thus, if $r(t)$ is monotonely increasing and there exists a solution T^* which satisfies

$$(T + T_0)r(T) - \int_0^T r(t)dt = c_2/c_1, \tag{3.2}$$

then it is unique and the resulting cost is

(3.3) $$C_1(T^*;T_0) = c_1[\lambda + r(T^*)].$$

Further, it is easy to see that T^* is a decreasing function of T_0 since the left-hand side of (3.2) is increasing in T_0 for a fixed T. Thus, an optimum time T^* is less than the optimum time given by (2.2).

4. Modified Replacement Policies

Suppose that the unit fails just before one of the scheduled replacement times. Then, it may be wasteful to repair the failed unit and may be wise to replace it at the next scheduled replacement. That is, if a failure occurs in an interval $(kT - T_d, kT)$ $(0 \leq T_d \leq T)$, the unit is not repaired in this interval and is replaced at scheduled time kT. The unit will be down for the time interval from its failure to the replacement. Cox [4] considered a similar model of block replacement where the replacement of a failed unit just before the scheduled time is postponed untill the next scheduled replacement.

The mean time between failure and its replacement when a failure occurs in an interval $(T - T_d, T)$ is

$$\frac{\int_{T-T_d}^{T} (T - t)dF(t)}{\overline{F}(T-T_d)} = \frac{\int_{T-T_d}^{T} [F(t) - F(T-T_d)]dt}{\overline{F}(T-T_d)} .$$

Thus, the expected cost rate is

(4.1) $$C_2(T_d;T) = \frac{c_1\int_0^{T-T_d} r(t)dt + c_2 + c_3\int_{T-T_d}^{T} [F(t) - F(T-T_d)]dt/\overline{F}(T-T_d)}{T} ,$$

where c_3 = cost of the time elapsed between failure and its replacement per unit of time.

Suppose that a constant T minimizes the expected cost rate $C(T)$ in (2.1), i.e., T is a solution of equation (2.2). Then, differentiating $C_2(T_d;T)$ with respect to T_d and setting it equal to zero for a fixed $T > 0$, we have

(4.2) $$\frac{\int_{T-T_d}^{T} \overline{F}(t)dt}{\overline{F}(T-T_d)} = \frac{c_1}{c_3} .$$

Thus, if $r(t)$ is monotonely increasing and $\int_0^T \overline{F}(t)dt > c_1/c_3$, then there exists a unique T_d^* which satisfies (4.2), and T_d^* is an increasing function of T. Conversely, if $\int_0^T \overline{F}(t)dt \leq c_1/c_3$, then $T_d^* = T$, i.e., a failed unit is not repaired and is replaced only at scheduled time, and

$$C_2(T;T) = \frac{c_2 + c_3\int_0^T F(t)dt}{T} . \tag{4.3}$$

In the above policy, it may be wise to replace a failed unit at scheduled time without repairing, but we can not sometimes leave a failed unit as it is until the scheduled replacement time. To overcome this, we consider the following model: If the unit fails in an interval $(T - T_d;T)$ then it is replaced by a new one before a scheduled replacement time. Tahara and Nishida [14] called the policy as the (t, T)-policy.

The expected cost rate is, from [15]

$$C_3(T_d;T) = \frac{c_1\int_0^{T-T_d} r(t)dt + c_2 + c_4[F(T) - F(T-T_d)]/\overline{F}(T-T_d)}{T - T_d + \int_{T-T_d}^{T} \overline{F}(t)dt/\overline{F}(T-T_d)} , \tag{4.4}$$

where c_4 = additional replacement cost caused by failure. Note that $C_3(T;T)$ agrees with [2, p. 87], which is the expected cost of the standard age replacement model.

Suppose that $c_2 + c_4 > c_1 > c_4$ and $r(t)$ is monotonely increasing. Then, by the method similar to the one of [15], the following results are obtained: There exists a unique T_d^* $(0 < T_d^* < T)$ which satisfies

$$c_1\left[\frac{(T - T_d)\overline{F}(T-T_d)}{\int_{T-T_d}^{T} \overline{F}(t)dt} - \int_0^{T-T_d} r(t)dt\right] - c_4 \frac{(T - T_d)\overline{F}(T)}{\int_{T-T_d}^{T} \overline{F}(t)dt} = c_2 + c_4 - c_1, \tag{4.5}$$

and the resulting minimum cost is

$$C_3(T_d^*;T) = \frac{c_1\overline{F}(T-T_d^*) - c_4\overline{F}(T)}{\int_{T-T_d^*}^{T} \overline{F}(t)dt} . \tag{4.6}$$

Further, if $c_1 \le c_4$ then $T_d^* = 0$, viz., the unit undergoes only minimal repair until the scheduled replacement, and we have $C_3(0;T) = C(T)$. If $c_1 > c_2 + c_4$ then $T_d^* = T$, viz., the unit is replaced at failure or at time T, whichever occurs first, after its installation, and

$$C_3(T;T) = \frac{c_2 + c_4F(T)}{\int_0^T \overline{F}(t)dt} . \tag{4.7}$$

Example

Suppose that the failure time distribution is a gamma distribution with a shape parameter 2, i.e., $F(t) = 1 - (1 + t)e^{-t}$. Then, the failure rate is $t/(1 + t)$ and is monotonely increasing from 0 to 1. Table 1 shows the optimum replacement time T^* which minimizes $C(T)$ in (2.1) for c_1 = 2, 4, 6, 8, 10, 15, 20 when we assume that $c_2 = 5$, $c_3 = 15$, and $c_4 = 4$. Further, when we put $T = T^*$, $T^* - T_d^*$, $C_2(T_d^*;\ T^*)$, and $T^* - T_d^*$, $T_3(T_d^*;\ T^*)$ are computed. It has been shown that both $T^* - T_d^*$ are decreasing in the minimal repair cost c_1, but the expected cost rates are increasing in c_1.

Table 1. Dependence of the minimal repair cost c_1 in T^*, $C(T^*)$, $T^* - T_d^*$, $C_2(T_d^*;T^*)$, and $T^* - T_d^*$, $C_3(T_d^*;T^*)$ when $c_2 = 5$, $c_3 = 15$, and $c_4 = 4$

minimal repair cost c_1	T^*	$C(T^*)$	$T^* - T_d^*$	$C_2(T_d^*;T^*)$	$T^* - T_d^*$	$C_3(T_d^*;T^*)$
2	31.1	1.94	31.0	1.93	31.1	1.94
4	7.4	3.52	7.1	3.46	7.4	3.52
6	4.2	4.84	3.7	4.58	1.4	4.37
8	2.9	5.97	2.3	5.35	0.3	4.69
10	2.3	6.99	1.5	5.84	0	4.88
15	1.6	9.16	0.3	6.15	0	5.46
20	1.2	11.03	0	6.19	0	6.05

5. Imperfect Preventive Maintenance Policies

Barlow and Hunter [1] considered the preventive maintenance (pm) policy in which a failed unit between periodic pm's undergoes minimal repair. Earlier results of optimum pm policies have been summarized in [9]. However, almost all models have assumed that a unit is as good as new after any pm. In practice, this assumption is often not true: A unit after pm usually might be younger at pm, and occasionally might be worce than before pm because of faulty procedures.

In this section, we consider the following three imperfect pm policies for a unit with minimal repair at failures:

(i) A unit after pm has the same failure rate as before pm or is as good as new with certain probabilities.

(ii) The age of a unit becomes x units of time younger at each pm.

(iii) The age of a unit after pm reduces to at when it was t before pm.

Assume that the unit is maintained preventively at scheduled times kT

(k = 1, 2, ...), and undergoes only minimal repair at failures between pm's. Further, assume that the repair and pm times are negligible.

(i) Model A

Suppose that the unit after pm has the same failure rate as it has been before pm with probability $p(0 \leq p < 1)$ and is as good as new with probability $\bar{p}$ ($\equiv 1 - p$). The pm action does not make any improvement in the condition of the unit with probability p, because of wrong adjustments, bad parts, damage done during pm, and so on. Helvic [5] applied such an imperfect pm to the periodic maintenance of fault tolerant computing systems. The expected cost rate is, from [10],

$$(5.1)\qquad C_4(T;p) = \frac{c_1(\bar{p})^2 \sum_{j=1}^{\infty} p^{j-1} \int_0^{jT} r(t)dt + c_2}{T},$$

where c_1 = cost of minimal repair,

c_2 = cost of scheduled pm.

Suppose that $r(t)$ is monotonely increasing. Then, if $\int_0^{\infty} t dr(t) > c_2/[c_1(\bar{p})^2]$ then there exists a finite and unique T^* which satisfies

$$(5.2)\qquad \sum_{j=1}^{\infty} p^{j-1} \int_0^{jT} t dr(t) = c_2/[c_1(\bar{p})^2],$$

and the resulting cost is

$$(5.3)\qquad C_4(T^*;p) = c_1(\bar{p})^2 \sum_{j=1}^{\infty} p^{j-1} jr(jT).$$

(ii) Model B

Suppose that the age of the unit becomes x units of time younger at each pm. where x ($0 \leq x \leq T$) is constant and previously specified. Further, suppose that the unit is replaced if it operates for the time interval NT where N is a positive integer. Then, the expected cost rate is easily given by

$$(5.4)\qquad C_4(T,N;x) = \frac{c_1 \sum_{j=0}^{N-1} \int_{j(T-x)}^{T+j(T-x)} r(t)dt + (N-1)c_2 + c_3}{NT},$$

where c_3 = cost of scheduled replacement at time NT, where $c_3 > c_2$.

Suppose that N is constant and T is a variable on $(0, \infty)$. A necessary condition that a finite T^* minimizes $C_4(T,N;x)$ is that it satisfies

$$(5.5)\qquad \sum_{j=0}^{N-1} \int_{j(T-x)}^{T+j(T-x)} t dr(t) = \frac{(N-1)c_2 + c_3}{c_1}.$$

Next, suppose that T is constant. Further, $C_4(T,0;x) = \infty$ formally for simplicity of analysis. Then, a necessary condition that there exists a finite and unique N^* which minimizes $C_4(T,N;x)$ is that N^* satisfies $C_4(T,N+1;x) \geq C_4(T,N;x)$ and $C_4(T,N;x) < C_4(T,N-1;x)$ $(N = 1, 2, \ldots)$. Thus, from these inequalities, we have, respectively,

$$(5.6) \qquad L(N;x) \geq (c_3 - c_2)/c_1 \text{ and } L(N-1,x) < (c_3 - c_2)/c_1 \quad (N = 1, 2, \ldots),$$

where

$$L(N;x) \equiv \begin{cases} N\int_{N(T-x)}^{T+N(T-x)} r(t)dt - \sum_{k=0}^{N-1} \int_{k(T-x)}^{T+k(T-x)} r(t)dt & (N = 1, 2, \ldots), \\ 0 & (N = 0). \end{cases}$$

Further, we have

$$(5.7) \qquad L(N+1;x) - L(N;x) = (N+1)\left[\int_{(N+1)(T-x)}^{T+(N+1)(T-x)} r(t)dt - \int_{N(T-x)}^{T+N(T-x)} r(t)dt\right].$$

Thus, if $r(t)$ is monotonely increasing, then $L(N;x)$ is also an increasing function of N from (5.7). Therefore, if $L(\infty;x) \geq (c_3 - c_2)/c_1$ then an optimum number N^* of pm cycles is given by a minimum value of N which satisfies $L(N;x) \geq (c_3 - c_2)/c_1$, and otherwise, we make no replacement.

(iii) Model C

Suppose that the age of the unit after pm reduces to at $(0 \leq a \leq 1)$ when it was t before pm, i.e., the age becomes $t(1 - a)$ units of time younger at each pm. Then, the expected cost rate is

$$(5.8) \qquad C_4(T,N;a) = \frac{c_1 \sum_{k=0}^{N-1} \int_{A_kT}^{(A_k+1)T} r(t)dt + (N-1)c_2 + c_3}{NT},$$

where $A_k \equiv a + a^2 + \ldots + a^k$ $(k = 1, 2, \ldots)$, and $A_0 \equiv 0$.

We can have similar results to ones of Model B: Equations (5.5) and $L(N;a)$ are rewritten as, respectively,

$$(5.9) \qquad \sum_{k=0}^{N-1} \int_{A_kT}^{(A_k+1)T} t dr(t) = \frac{(N - 1)c_2 + c_3}{c_1},$$

$$L(N;a) = N\int_{A_NT}^{(A_N+1)T} r(t)dt - \sum_{k=0}^{N-1} \int_{A_kT}^{(A_k+1)T} r(t)dt \qquad (N = 1, 2, \ldots).$$

It is noted that all models are identical and agree with Section 2 when $p = 0$ in Model A, $N = 1$ and $x = T$ in Model B, and $N = 1$ and $a = 0$ in Model C.

Example

Suppose that the failure time has a Weibull distribution with a shape parameter β, i.e., $F(t) = 1 - \exp(-\lambda t^{\beta})$ $(\lambda > 0,\ \beta > 1)$. Then, the failure rate is $r(t) = \lambda\beta t^{\beta-1}$, which is monotonely increasing, taking the values from 0 to ∞. Thus, we have the following results for each model.

(i) Model A

The expected cost rate is, from (5.1),

$$C_4(T;p) = \frac{c_1\bar{p}\lambda T^{\beta}g(\beta) + c_2}{T},$$

where $g(\beta) \equiv \bar{p}\sum_{j=1}^{\infty} p^{j-1}j^{\beta}$ which represents the β^{th} moment of the geometric distribution with parameter p. The optimum pm time is, from (5.2),

$$T^* = \left[\frac{c_2}{c_1\bar{p}\lambda(\beta - 1)g(\beta)}\right]^{1/\beta}.$$

(ii) Model B

The expected cost rate is, from (5.4),

$$C_4(T,N;x) = \frac{c_1\lambda \sum_{k=0}^{N-1} \{[T + k(T-x)]^{\beta} - [k(T-x)]^{\beta}\} + (N-1)c_2 + c_3}{NT}.$$

From (5.5),

$$\sum_{j=0}^{N-1} \{[T + j(T-x)]^{\beta} - [j(T-x)]^{\beta}\} = \frac{(N-1)c_2 + c_3}{c_1\lambda(\beta-1)},$$

where the left-hand side is monotonely increasing in T, taking the values from 0 to ∞. Thus, the optimum pm time T^* exists uniquely, which satisfies (5.5). Further, the left-hand side is also decreasing in x for a fixed T, and hence, the optimum pm time T^* is an increasing function of x. Thus, putting $x = 0$ and $x = T$ in (5.5), we have the lower and upper limits:

$$\frac{1}{N}\left[\frac{(N-1)c_2 + c_3}{c_1\lambda(\beta-1)}\right]^{1/\beta} \le T^* \le \left[\frac{1}{N}\frac{(N-1)c_2 + c_3}{c_1\lambda(\beta-1)}\right]^{1/\beta}.$$

(iii) Model C

The expected cost rate is, from (5.8),

$$C_4(T,N;a) = \frac{c_1\lambda \sum_{k=0}^{N-1} T^{\beta}[(A_k+1)^{\beta} - A_k^{\beta}] + (N-1)c_2 + c_3}{NT},$$

and the optimum pm time T^* exists uniquely, which satisfies

$$\sum_{k=}^{N-1} T^{\beta}[(A_k+1)^{\beta} - A_k^{\beta}] = \frac{(N-1)c_2 + c_3}{c_1\lambda(\beta-1)} .$$

Until now, we have assumed in Models B & C that x and a are constant. Actually, these would depend on the cost or the time spent for pm. To take one example, it is supposed in Model C that the age of the unit after pm decreases in proportion to cost or time for pm. Then, some simple functions of a are: (1) $a = 1 - (c_2/c_3)$ for $0 < c_2 \leq c_3$, (2) $a = \exp(-\theta c_2)$ for $\theta > 0$, $c_2 > 0$, (3) $a = \exp(-\theta y)$ for $\theta > 0$, where y is the time taken for pm. Other functions could be formed by the resources consumed in pm. When $a = \exp(-\theta y)$, A_k in (5.8) is given by

$$A_k = \frac{\exp(-\theta y) - \exp[-\theta(k+1)y]}{1 - \exp(-\theta y)} .$$

In particular, if we take sufficient time for pm, i.e., $y \to \infty$, then $C_4(T,N;a) = C(T)$ in (2.1). Inversely, if we take no time for pm, i.e., $y \to 0$ then

$$\text{(5.10)} \qquad C_4(T,N;a) = \frac{c_1\int_0^{NT} r(t)dt + c_3}{NT} .$$

Similar discussions are made for Model B.

6. Concluding Remarks

We have summarized the periodic replacement models with minimal repair at failures. In particular, three imperfect pm models are theoretically new and could be applied to more practical fields. Throughout this paper, we have assumed that the failure rate remains undisturbed by any repair of failures between replacements. Actually, this assumption is often not true. It is usually said that the unit after minimal repair might be worce than before failure.

Suppose that the age of the unit after minimal repair becomes at ($a \geq 0$) when it was t before failure. If $a < 1$ then the unit is younger at minimal repair and if $a > 1$ then it is worce than before failure. Then, the expected number of failures during the interval (0, T] is easily given by

$$\text{(6.1)} \qquad M(T;a) = \sum_{k=1}^{\infty} \int \cdots \int_{t_1<t_2<\cdots<t_k\leq T} \frac{f(at_1+t_2-t_1)}{\overline{F}(at_1)} \frac{f[a^2t_1 + a(t_2-t_1)+t_3-t_2]}{\overline{F}[a^2t_1+a(t_2-t_1)]}$$

$$\cdots \frac{f[a^{k-1}t_1+a^{k-2}(t_2-t_1)+\ \ldots\ +a(t_{k-1}-t_{k-2})+t_k-t_{k-1}]}{\overline{F}[a^{k-1}t_1+a^{k-2}(t_2-t_1)+\ldots+a(t_{k-1}-t_{k-2})]}\, dt_1\, dt_2 \ldots dt_k.$$

It is evident that

$$M(T;0) = M(T),$$

$$M(T;1) = \int_0^T r(t)dt,$$

where $M(t)$ = renewal function of the failure time distribution $F(t)$, i.e., $M(t) \equiv \sum_{k=1}^{\infty} F^{(k)}(t)$ where $F^{(k)}$ is the k-fold convolution of $F(t)$ with itself.

Thus, the expected cost rate is

$$(6.2) \qquad C(T;a) = \frac{c_1 M(T;a) + c_2}{T}.$$

When $a = 0$, the unit becomes always new at each minimal repair and the model corresponds to block replacement. When $a = 1$, the failure rate is not disturbed by each minimal repair and this corresponds to the model in this paper. However, in general, it is very difficult to make discussions about optimum policies for the model.

We have not treated block replacement appeared in [3, 4, 16]. The policies in this paper could be applied to other replacement models. For instance, we can combine a block replacement policy and this policy. That is, a failed unit is replaced by a new one during $(0, T-T_d]$ and undergoes minimal repair during $(T-T_d, T)$ for $0 \le T_d \le T$. Then, the expected cost rate is, from [11],

$$(6.3) \qquad C_5(T_d;T) = \frac{c_1\{\overline{F}(T-T_d)\int_{T-T_d}^{T} r(t)dt + \int_0^{T-T_d}[\int_{T-T_d}^{T} r(t-u)dt]\overline{F}(T-T_d-u)dM(u)\} + c_2 + c_5 M(T-T_d)}{T}$$

where c_5 = cost of replacement for a failed unit.

References

[1] Barlow, R.E. and Hunter, L.C.: Optimum Preventive Maintenance Policies. *Operations Research*, Vol.8, No.1 (1960), 90-100.

[2] Barlow, R.E. and Proschan, F.: *Mathematical Theory of Reliability*. John Wiley & Sons, New York, 1965.

[3] Berg, M. and Epstein, B.: A Modified Block Replacement Policy. *Naval Research Logistics Quarterly*, Vol.23, No.1 (1976), 15-24.

[4] Cox, D.R.: *Renewal Theory*. Methuen, London, 1963.

[5] Helvic, B.E.: Periodic Maintenance on the Effect of Imperfectness. *Proceedings of 1980 International Symposium on Fault-Tolerant Computing*. 1980, 204-206.

[6] Holland, C.W. and McLean, R.A.: Applications of Replacement Theory. *AIIE Trans.*, Vol.7, No.1 (1975), 42-47.

[7] Mine, H. and Kawai, H.: Preventive Replacement of a 1-unit System with a Wearout State. *IEEE Trans. on Reliability*, Vol.R-23, No.1 (1974), 24-28.

[8] Morimura, H.: On Some Preventive Maintenance Policies for IFR. *J. Operations Research Soc. of Japan*, Vol.12, No.3 (1970), 94-124.

[9] Nakagawa, T.: Optimum Preventive Maintenance Policies for Repairable Systems. *IEEE Trans. on Reliability*, Vol.R-26, No.3 (1977), 168-173.

[10] Nakagawa, T.: Optimum Policies When Preventive Maintenance is Imperfect. *IEEE Trans. on Reliability*, Vol.R-28, No.4 (1979), 331-332.

[11] Nakagawa, T.: A Summary of Block Replacement Policies. *R.A.I.R.O. Recherche Opérationnelle*, Vol.13, No.4 (1979), 351-361.

[12] Nakagawa, T.: Optimum Replacement Policies for a Used Unit. *J. Operations Research Soc. of Japan*, Vol.22, No.4 (1979), 338-346.

[13] Nakagawa, T.: A Summary of Imperfect Preventive Maintenance Policies with Minimal Repair. *R.A.I.R.O. Recherche Opérationnelle*, Vol.14, No.3 (1980), 249-255.

[14] Nakagawa, T. and Osaki, S.: The Discrete Weibull Distribution. *IEEE Trans. on Reliability*, Vol.R-24, No.5 (1975), 300-301.

[15] Tahara, A. and Nishida, T.: Optimal Replacement Policy for Minimal Repair Model. *J. Operations Research Soc. of Japan*, Vol.18, No.3 & 4 (1975), 113-124.

[16] Tango, T.: A Modified Block Replacement Policy Using Less Reliable Items. *IEEE Trans. on Reliability*, Vol.R-28, No.5 (1979), 400-401.

[17] Tilquin, C. and Cléroux, R.: Periodic Replacement with Minimal Repair at Failure and Adjustment Costs. *Naval Research Logistics Quarterly*, Vol.22, No.2 (1975), 243-254.

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