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A SEQUENTIAL ALLOCATION PROBLEM WITH PERISHABLE GOODS

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Abstract A decision maker will allocate his goods to appearing customers during the given periods. There are finite types of customers. Each type of customer appears with a given probability at the beginning of each period. When the customer appears, the decision maker sells some of his goods to acquire the expected reward which depends on the number of goods sold and the type of appearing customer. Unsold goods at the end of each period perish at the beginning of next period with some probability. The objective is to find a sequence of optimal number of goods to be sold which maximizes the total expected reward. Some properties of an optimal policy are investigated and some simple examples are presented.

1. Introduction

A decision maker will allocate his goods to appearing customers during the given periods. The customer appears one by one at the beginning of each period with some fixed probability. The decision maker obtains the reward which depends on the number of goods sold and the appearing customer. How does he allocate his goods to maximize the total expected reward?

Sequential allocation problems described above were studied by many authors. In [1,2,3,4] it was assumed that the unsold goods at the end of each period did not perish. The present paper considers the sequential allocation problem where the unsold goods at the end of each period stochastically perish at the beginning of next period.

An outline of the paper is as follows: In Section 2 our problem is formulated into a dynamic programming one. In Section 3 some properties of an optimal policy are discussed. In Section 4 some examples are presented to illustrate our model.

2. Model and Formulation

The decision maker will allocate M units of his goods to appearing customers during given periods N. There are I types of customers. The customer appears one by one at the beginning of each period, where the type i customer appears with probability $r_i(n)$ ($i=1,\dots,I$, $n=1,\dots,N$)(n denotes the number of remaining periods), and no customer with probability $r_0(n)$ ($r_0(n)+\cdots+r_I(n)=1$). When he sells j units of goods ($j=0,\dots,M$) to the type i customer, he obtains the immediate reward $R_i(n,j)$, where $R_i(n,j)$ is assumed to be a nondecreasing concave function of j and $R_i(n,0)=0$. Each of unsold goods at the end of each period perishes at the beginning of next period with a given probability. The objective is to find an optimal sequential allocation procedure which maximizes the total expected reward by allocating M units of goods to the appearing customers during N periods.

Consider, for example, a company which trades in perishable goods such as fresh vegetable and raw fish or up-to-date electronic devices, and customers who place an order with the company for the perishable goods. The length of each period may be appropriately determined so that at most one order is placed at each period. Unsold goods at the end of each period become valueless with a given ratio at the beginning of next period because of their perishing nature. The ratio may be regarded as a perishing probability. Yet the model discussed in the last paragraph is not applicable to this example directly since the reward function $R_i(n,\cdot)$ in the model is independent of the quantity of order of customer. If the quantities of order of customer can be classified into finite classes, then by determining reward functions and probabilities of appearance appropriately we can interpret the i-th customer with j-th quantity of order as an (i,j) type of customer. Then the model can be applicable to this example. Also, in the case of trading in electronic devices in stead of perishable goods, the model may be applicable, because some ratio of unsold electronic devices become valueless by the rapid development of electronic technique.

Concentrating on our model we define the following notations:

$\mathrm{p}^n=(p_n,p_{n-1},\dots,p_1)$: the perishing probability vector (see Fig.1),

$V_n(m;i,\mathrm{p}^{n-1})$=the maximum total expected reward when there are n periods remaining, m units of goods are on hand, the type i customer is appearing, the perishing probability vector is given by p^{n-1}, and an optimal policy follows.

$\bar{V}_n(m, p^n)$=the maximum total expected reward when m units of goods are on hand and the perishing probability vector is given by p^n at the end of the $(n+1)$-st period from the final period, and an optimal policy follows.

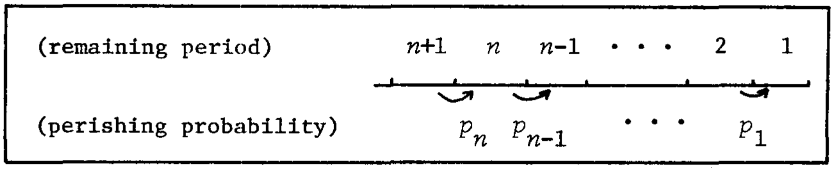

Fig. 1.

Using these notations, the following recursive relations are derived by the principle of optimality:

$$(1)\qquad V_n(m;i,p^{n-1})=\max_{j=0,\dots,m}\{R_i(n,j)+\bar{V}_{n-1}(m-j,p^{n-1})\},$$

$$(2)\qquad \bar{V}_n(m,p^n)=\sum_{k=0}^{m}\binom{m}{k}p_n^{m-k}(1-p_n)^k(r_0(n)\bar{V}_{n-1}(k,p^{n-1})+\sum_{i=1}^{I}r_i(n)V_n(k;i,p^{n-1})),$$

$$(i=1,\dots,I,\quad n=1,\dots,N,\quad m=0,\dots,M),$$

where $\bar{V}_0(\cdot,\cdot)=\bar{V}.(0,\cdot)=0$.

If there are n periods remaining, m units of goods are on hand, the type i customer is appearing, the perishing probability vector is given by p^{n-1}, and j units of goods (j=0,...,m) are sold to the customer, then the immediate expected reward to the decision maker is given by $R_i(n,j)$ and the maximum total expected reward from the resultant state is $\bar{V}_{n-1}(m-j,p^{n-1})$. Hence equation (1) follows. Equation (2) is obtained since the number of utilizable goods at the beginning of the n-th period from the final period has a binomial distribution with parameters m and $1-p_n$.

It should be noted that an optimal policy is presented if the recursive relations (1) and (2) are solved.

3. Structure of Optimal Policy

We can find an optimal policy if we solve the equations (1) and (2) recursively. Unfortunately we cannot solve them explicitly in general. So we develop some properties of an optimal policy.

As was shown in the previous paper [3], the value of j that maximizes the braces of the right hand side of equation (1) is the optimal number to be sold to the type i customer. To determine one of the optimal policies we define $k(n,m;i,\mathrm{p}^{n-1})$ as the smallest value of j that maximizes the braces of the right hand side of equation (1), that is,

$$k(n,m;i,\mathrm{p}^{n-1})=\min[t;\ \max_{j=0,\ldots,m}\{R_i(n,j)+\bar{V}_{n-1}(m-j,\mathrm{p}^{n-1})\}$$
$$=R_i(n,t)+\bar{V}_{n-1}(m-t,\mathrm{p}^{n-1})].$$

Using this notation it is the optimal policy that allocates $k(n,m;i,\mathrm{p}^{n-1})$ units of goods to the type i customer when there are n periods remaining, m units of goods are on hand, the type i customer is appearing, and the perishing probability vector is given by p^{n-1}.

If $\mathrm{p}^n=(0,\ldots,0)$, then the model is just the same as discussed in [3]. We introduce the following lemma which shows the concavity of $\bar{V}_n(\cdot,\mathrm{p}^n)$ and $V_n(\cdot;i,\mathrm{p}^{n-1})$.

Lemma 1. $\bar{V}_n(m,\mathrm{p}^n)$ and $V_n(m;i,\mathrm{p}^{n-1})$ are concave functions of m.

Proof: An easy calculation yields

$$(\bar{V}_n(m+1,\mathrm{p}^n)-\bar{V}_n(m,\mathrm{p}^n))-(\bar{V}_n(m+2,\mathrm{p}^n)-\bar{V}_n(m+1,\mathrm{p}^n))$$
$$=(1-p_n)^2\sum_{k=0}^{m}\binom{m}{k}p_n^{m-k}(1-p_n)^k[r_0(n)((\bar{V}_{n-1}(k+1,\mathrm{p}^{n-1})-\bar{V}_{n-1}(k,\mathrm{p}^{n-1}))$$
$$-(\bar{V}_{n-1}(k+2,\mathrm{p}^{n-1})-\bar{V}_{n-1}(k+1,\mathrm{p}^{n-1})))$$
$$+\sum_{i=1}^{I}r_i(n)((V_n(k+1;i,\mathrm{p}^{n-1})-V_n(k;i,\mathrm{p}^{n-1}))$$
$$-(V_n(k+2;i,\mathrm{p}^{n-1})-V_n(k+1;i,\mathrm{p}^{n-1})))].$$

Note that $V_n(m;i,\mathrm{p}^{n-1})=\max_{j=0,\ldots,m}\{R_i(n,j)+\bar{V}_{n-1}(m-j,\mathrm{p}^{n-1})\}$ is concave with respect to m if $\bar{V}_{n-1}(\cdot,\mathrm{p}^{n-1})$ is concave since $R_i(n,\cdot)$ is a nondecreasing concave function. These facts and the concavity of $V_1(m;i)=R_i(1,m)$ prove the result of this lemma by induction. Q.E.D.

The following theorem presents the monotonicity of the optimal number of goods to be sold to the customer with respect to the number of goods on hand. The result seems to meet with our intuition that the more units of goods are on hand the more units of goods should be sold.

Theorem 1. $k(n,m;i,\mathrm{p}^{n-1}) \leq k(n,m+1;i,\mathrm{p}^{n-1}) \leq k(n,m;i,\mathrm{p}^{n-1})+1$

for $n=1,\ldots,N$, $m=0,\ldots,M-1$, $i=1,\ldots,I$, and any p^{n-1}.

This theorem is proved similarly to Proposition 2 and Theorem 3 in [1], since $R_i(n,\cdot)$ and $\bar{V}_{n-1}(\cdot,\mathrm{p}^{n-1})$ are concave.

One may conjecture that the optimal number to be sold increases as perishing probabilities increase, that is, $k(n,m;i,\mathrm{p}^{n-1}) \geq k(n,m;i,\mathrm{q}^{n-1})$ for $\mathrm{p}^{n-1}=(p_{n-1},\ldots,p_1) \geq \mathrm{q}^{n-1}=(q_{n-1},\ldots,q_1)$. But the conjecture is not always true as seen in the example of Section 4-2. We present a sufficient condition under which the conjecture is satisfied. First we need the following lemma.

Lemma 2. If $\bar{V}_{n-1}(m,\mathrm{q}^{n-1})-\bar{V}_{n-1}(m-1,\mathrm{q}^{n-1}) \geq ((1-p_n)/(1-q_n))A_n\bar{V}_{n-1}(1,\mathrm{p}^{n-1})$ and $R_i(n,m)-R_i(n,m-1) \geq ((1-p_n)/(1-q_n))A_nR_i(n,1)$ for $i=1,\ldots,I$ and some $A_n \geq 0$, then

$$\bar{V}_n(m,\mathrm{q}^n)-\bar{V}_n(m-1,\mathrm{q}^n) \geq A_n\bar{V}_n(1,\mathrm{p}^n),\ \text{where}\ \mathrm{q}^n=(q_n,\ldots,q_1),\ \mathrm{p}^n=(p_n,\ldots,p_1),$$

$\mathrm{q}^{n-1}=(q_{n-1},\ldots,q_1)$, and $\mathrm{p}^{n-1}=(p_{n-1},\ldots,p_1)$.

Proof: It should be noted that $\bar{V}_{n-1}(k+1,\mathrm{q}^{n-1})-\bar{V}_{n-1}(k,\mathrm{q}^{n-1}) \geq ((1-p_n)/(1-q_n))A_n\bar{V}_{n-1}(1,\mathrm{p}^{n-1})$ for $k=0,\ldots,m-1$ by the concavity of $\bar{V}_{n-1}(\cdot,\mathrm{q}^{n-1})$ and the assumption of this lemma. Also that

$$\begin{aligned} V_n(k+1;i,\mathrm{q}^{n-1})-V_n(k;i,\mathrm{q}^{n-1}) &\geq V_n(m;i,\mathrm{q}^{n-1})-V_n(m-1;i,\mathrm{q}^{n-1}) \\ &= \max\{R_i(n,j_0+1)-R_i(n,j_0), \\ &\qquad \bar{V}_{n-1}(m-j_0,\mathrm{q}^{n-1})-\bar{V}_{n-1}(m-1-j_0,\mathrm{q}^{n-1})\} \\ &\geq ((1-p_n)/(1-q_n))A_n\max\{R_i(n,1),\bar{V}_{n-1}(1,\mathrm{p}^{n-1})\} \end{aligned}$$

for $k=0,\ldots,m-1$ where $j_0=k(n,m-1;i,\mathrm{q}^{n-1})$ since the equality of the above expression follows from Theorem 1. Therefore we have

$$\begin{aligned} \bar{V}_n(m,\mathrm{q}^n)-\bar{V}_n(m-1,\mathrm{q}^n) &= (1-q_n)\textstyle\sum_{k=0}^{m-1}\binom{m-1}{k}q_n^{m-1-k}(1-q_n)^k\cdot \\ &\qquad [r_0(n)(\bar{V}_{n-1}(k+1,\mathrm{q}^{n-1})-\bar{V}_{n-1}(k,\mathrm{q}^{n-1})) \\ &\qquad +\textstyle\sum_{i=1}^{I}r_i(n)(V_n(k+1;i,\mathrm{q}^{n-1})-V_n(k;i,\mathrm{q}^{n-1}))] \\ &\geq (1-p_n)A_n[r_0(n)\bar{V}_{n-1}(1,\mathrm{p}^{n-1}) \\ &\qquad +\textstyle\sum_{i=1}^{I}r_i(n)\max\{R_i(n,1),\bar{V}_{n-1}(1,\mathrm{p}^{n-1})\}] \\ &= A_n\bar{V}_n(1,\mathrm{p}^n). \end{aligned}$$

Q.E.D.

Using Lemma 2, the following theorem is proved.

Theorem 2. If $(R_i(s,m)-R_i(s,m-1))/R_i(s,1) \geq \prod_{t=s}^{n-1}((1-p_t)/(1-q_t))$ for $i=1,\ldots$

.,I and $s=1,\ldots,n-1$, then

$$k(n,j;i,\mathrm{p}^{n-1})\geqq k(n,j;i,\mathrm{q}^{n-1}) \text{ for } j=1,\ldots,m \text{ and } i=1,\ldots,\mathrm{I}, \text{ where}$$

$\mathrm{p}^{n-1}=(p_{n-1},\ldots,p_1)$ and $\mathrm{q}^{n-1}=(q_{n-1},\ldots,q_1)$.

Proof: It should be noted that if

$$(3)\qquad \bar{V}_{n-1}(k+1,\mathrm{q}^{n-1})-\bar{V}_{n-1}(k,\mathrm{q}^{n-1})\geqq\bar{V}_{n-1}(k+1,\mathrm{p}^{n-1})-\bar{V}_{n-1}(k,\mathrm{p}^{n-1}) \text{ for } k=0,\ldots$$

$,j-1$, then $k(n,j;i,\mathrm{p}^{n-1})\geqq k(n,j;i,\mathrm{q}^{n-1})$ since inequality (3) implies that

$$(R_i(n,j_0)+\bar{V}_{n-1}(j-j_0,\mathrm{p}^{n-1}))-(R_i(n,k)+\bar{V}_{n-1}(j-k,\mathrm{p}^{n-1}))$$
$$\geqq(R_i(n,j_0)+\bar{V}_{n-1}(j-j_0,\mathrm{q}^{n-1}))-(R_i(n,k)+\bar{V}_{n-1}(j-k,\mathrm{q}^{n-1}))>0$$

for $k=0,\ldots,j_0-1$, where $j_0=k(n,j;i,\mathrm{q}^{n-1})$. By the concavity of $\bar{V}_{n-1}(\cdot,\mathrm{q}^{n-1})$,

$$(4)\qquad \bar{V}_{n-1}(j,\mathrm{q}^{n-1})-\bar{V}_{n-1}(j-1,\mathrm{q}^{n-1})\geqq\bar{V}_{n-1}(1,\mathrm{p}^{n-1})$$

implies inequality (3). Therefore the theorem is proved if we show inequality (4) for $j=m$. Noting the assumption that

$$R_i(s,m)-R_i(s,m-1)\geqq(\Pi_{t=s}^{n-1}((1-p_t)/(1-q_t)))R_i(s,1) \text{ for } i=1,\ldots,\mathrm{I}$$

and $s=1,\ldots,n-1$, the inequality (4) for $j=m$ follows from Lemma 2. Q.E.D.

It is remarked that $(R_i(n,m)-R_i(n,m-1))/R_i(n,1)$ is decreasing with respect to m and is less than or equal to 1 since $R_i(n,\cdot)$ is concave.

Remark. If $R_i(s,\cdot)$ is linear for $i=1,\ldots,\mathrm{I}$, $s=1,\ldots,n$, and $\mathrm{p}^{n-1}\geqq\mathrm{q}^{n-1}$, then the assumption of Theorem 2 is satisfied.

Corollary. If there exist $R_i(m)$ and $f(n)>0$ such that $R_i(n,m)=R_i(m)f(n)$ for $n=1,\ldots,\mathrm{N}$, $m=0,\ldots,\mathrm{M}$, and $i=1,\ldots,\mathrm{I}$, and if $(R_i(m)-R_i(m-1))/R_i(1)$ $\geqq(1-p_{n-1})/(1-q_{n-1})$, then

$$k(n,j;i,\mathrm{p}^{n-1})\geqq k(n,j;i,\mathrm{q}^{n-1}) \text{ for } j=1,\ldots,m \text{ and } i=1,\ldots,\mathrm{I}, \text{ where}$$

$\mathrm{p}^{n-1}=(p_{n-1},\ldots,p_1)$ and $\mathrm{q}^{n-1}=(q_{n-1},\ldots,q_1)$ with $\mathrm{p}^{n-1}\geqq\mathrm{q}^{n-1}$.

If, in the example described in Section 2, $R_i(n,m)=\alpha_n R_i(n+1,m)$ $(0<\alpha_n\leqq 1)$ for $n=1,\ldots,\mathrm{N}-1$, $m=0,\ldots,\mathrm{M}$, and $i=1,\ldots,\mathrm{I}$, then the first assumption of Corollary is satisfied.

4. Examples

This section presents three simple examples. One of them deals with the case of linear $R_i(n,\cdot)$ and the others are represented by numerical values.

4.1. The case of linear $R_i(n,\cdot)$

Suppose that $R_i(n,j)=a(i,n)j$ for $i=1,\ldots,I$, $n=1,\ldots,N$ and $0<a(1,n)\leqq\cdots\leqq a(I,n)$ for $n=1,\ldots,N$. In this case the optimal policy can be derived explicitly and has a simple form.

Let $g^{(n)}(z)=r_0(n)z+\sum_{i=1}^{I}r_i(n)\max\{z,a(i,n)\}$ for $n=1,\ldots,N-1$, $g_0=0$, and $g_n(\mathrm{p}^n)=(1-p_n)g^{(n)}(g_{n-1}(\mathrm{p}^{n-1}))$ for $n=1,\ldots,N-1$ where $\mathrm{p}^n=(p_n,\ldots,p_1)$. Also let $i^*(n,\mathrm{p}^{n-1})=\min\{i;\ a(i,n)>g_{n-1}(\mathrm{p}^{n-1})\}$. Then one can easily show that

$$\bar{V}_n(m,\mathrm{p}^n)=g_n(\mathrm{p}^n)m \text{ and}$$

$$k(n,m;i,\mathrm{p}^{n-1})=\begin{cases} m & \text{for } i\geqq i^*(n,\mathrm{p}^{n-1}),\\ 0 & \text{for } i<i^*(n,\mathrm{p}^{n-1}).\end{cases}$$

Now we examine some properties of the optimal policy. From Remark of Theorem 2, $k(n,m;i,\mathrm{p}^{n-1})\geqq k(n,m;i,\mathrm{q}^{n-1})$ if $\mathrm{p}^{n-1}\geqq\mathrm{q}^{n-1}$, where $\mathrm{p}^{n-1}=(p_{n-1},\ldots,p_1)$ and $\mathrm{q}^{n-1}=(q_{n-1},\ldots,q_1)$. In addition, if $a(i,n)=a(i)f(n)$ $(0<f(1)\leqq\cdots\leqq f(\mathrm{N}))$, $(1-p_1)/f(2)\leqq(1-p_2)/f(3)\leqq\cdots\leqq(1-p_{\mathrm{N}-1})/f(\mathrm{N})$, and $\sum_{i=k}^{I}r_i(1)\leqq\sum_{i=k}^{I}r_i(2)\leqq\cdots\leqq\sum_{i=k}^{I}r_i(\mathrm{N}-1)$ for $k=1,\ldots,I$, then $i^*(1)\leqq i^*(2,\mathrm{p}^1)\leqq\cdots\leqq i^*(\mathrm{N},\mathrm{p}^{\mathrm{N}-1})$ (see Appendix). The value of goods is not uniform in time and the quantity $(1-p_{n-1})/f(n)$ may represent the rate of utilizable goods measured in value instead of number at the beginning of next period. So the above assumptions state as follows: As the length of remaining periods is getting shorter, the rate of utilizable goods measured in value instead of number at the beginning of next period is getting smaller and it becomes stochastically more difficult for the decision maker to find the desirable customers. Under these assumptions the monotonicity of optimal number of goods to be sold to the customer with respect to the number of remaining periods is satisfied. But the monotonicity is not always satisfied as will be shown in 4.3.

4.2. Numerical example I

Suppose that I=3 and $(r_0,r_1,r_2,r_3)=(0.01,0.49,0.3,0.2)$. Also $R_i(n,j)$ (assumed to be independent of n) is defined in Table 1. It is shown that $k(2,4;2,(0.2))=2\nleqq k(2,4;2,(0.35))=1$ and $k(2,4;3,(0.2))=2\nleqq k(2,4;3,(0.35))=1$. This concludes that the optimal number to be sold to the customer is not always a monotone function of the perishing probability.

4.3. Numerical example II

If I=3, $(r_0,r_1,r_2,r_3)=(0.2,0.3,0.3,0.2)$, $R_1(n,j)=j$, $R_2(n,j)=2j$, and $R_3(n,j)=4j$, then we have $i^*(1)=1$, $i^*(2,(0.2))=2$, $i^*(3,(0.2,0.2))=2$, $i^*(4,(0.99,0.2,0.2))=1$, and $i^*(5,(0.2,0.99,0.2,0.2))=2$, that is, $k(5,m;1,(0.2,0.99,0.2,0.2))=0\leqq k(4,m;1,(0.99,0.2,0.2))=m\nleqq k(3,m;1,(0.2,0.2))=0\leqq k(2,m;1,(0.2))=0\leqq k(1,m;1)=m$, which shows that the optimal number to be sold to the customer is not always a monotone function of the number of remaining periods.

Table 1. $R_i(\cdot,j)$

i \ j	0	1	2	3	4
1	0	100	100.4	100.7	100.9
2	0	120	128	128.4	128.6
3	0	150	160	165	165.3

5. Conclusion

We considered a discrete-time finite horizon sequential allocation problem with perishable goods. In Section 3, it was proved that the optimal number to be sold to the customer is a nondecreasing function of the number of goods on hand. We also derived a sufficient condition under which the optimal number to be sold to the customer increases as some of perishing probabilities increase. In Section 4, the optimal policy was obtained explicitly for the case of linear expected reward. Some numerical examples showed that the optimal number to be sold to the customer was not always a monotone function either of the number of remaining periods or of the perishing probability.

It is a future problem to find an appropriate condition which guarantees the monotonicity of the optimal number to be sold to the customer with respect to the number of remaining periods.

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Appendix.

If $a(i,n)=a(i)f(n)$ $(0<f(1)\leqq\cdots\leqq f(N))$, $(1-p_1)/f(2)\leqq(1-p_2)/f(3)\leqq\cdots\leqq$

$(1-p_{N-1})/f(N)$, and $\sum_{i=k}^{I} r_i(1) \leqq \sum_{i=k}^{I} r_i(2) \leqq \cdots \leqq \sum_{i=k}^{I} r_i(N-1)$ for $k=1,\ldots,I$, then $i^*(1) \leqq i^*(2,p^1) \leqq \cdots \leqq i^*(N,p^{N-1})$.

Proof: Note that the appendix is obvious if we show

$$0=g_0/f(1) \leqq g_1(p^1)/f(2) \leqq \cdots \leqq g_{N-1}(p^{N-1})/f(N). \tag{5}$$

To show (5), we need the following Lemma:

Lemma 3. If $0<a(1) \leqq \cdots \leqq a(I)$, $0<f(1) \leqq \cdots \leqq f(N)$, and $\sum_{i=k}^{I} r_i(1) \leqq \sum_{i=k}^{I} r_i(2) \leqq \cdots \leqq \sum_{i=k}^{I} r_i(N-1)$ for $k=1,\ldots,I$, then $g^{(1)}(z) \leqq g^{(2)}(z) \leqq \cdots \leqq g^{(N-1)}(z)$ for any z.

Proof of Lemma 3: Let $b(1,n)=\max\{0,a(1)f(n)-z\} \geqq 0$, and $b(j,n)=\max\{0,a(j)f(n)-z\}-\max\{0,a(j-1)f(n)-z\} \geqq 0$ for $j=2,\ldots,I$. Thus

$$\begin{aligned}
g^{(n)}(z) &= r_0(n)z+\sum_{i=1}^{I} r_i(n)\max\{z,a(i)f(n)\} \\
&= z+\sum_{i=1}^{I} r_i(n)\ \max\{0,a(i)f(n)-z\} \\
&\geqq z+\sum_{i=1}^{I} r_i(n)\ \max\{0,a(i)f(n-1)-z\} \\
&= z+\sum_{i=1}^{I} r_i(n)\sum_{j=1}^{i} b(j,n-1) \\
&= z+\sum_{j=1}^{I} b(j,n-1)\sum_{i=j}^{I} r_i(n) \\
&\geqq z+\sum_{j=1}^{I} b(j,n-1)\sum_{i=j}^{I} r_i(n-1) \\
&= z+\sum_{i=1}^{I} r_i(n-1)\max\{0,a(i)f(n-1)-z\}=g^{(n-1)}(z).
\end{aligned}$$

We employ an induction to prove (5). It is obvious that

$$0=g_0/f(1) \leqq g_1(p^1)/f(2).$$

Assume that

$$g_{n-1}(p^{n-1})/f(n) \leqq g_n(p^n)/f(n+1). \tag{6}$$

Then we have

$$\begin{aligned}
g_{n+1}(p^{n+1})/f(n+2) &= (1-p_{n+1})g^{(n+1)}(g_n(p^n))/f(n+2) \\
&\geqq (1-p_n)g^{(n+1)}(g_n(p^n))/f(n+1) \\
&\geqq (1-p_n)g^{(n+1)}(g_{n-1}(p^{n-1}))/f(n+1) \\
&\geqq (1-p_n)g^{(n)}(g_{n-1}(p^{n-1}))/f(n+1)=g_n(p^n)/f(n+1),
\end{aligned}$$

where the second inequality follows from the assumption (6) and the monotonicity of $g^{(n+1)}(\cdot)$, and the third from Lemma 3. Q.E.D.

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