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CHANCE CONSTRAINED SPANNING TREE PROBLEM

Hiroaki Ishii
Osaka University

Shōgo Shiode
Osaka University

Toshio Nishida
Osaka University

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Abstract We consider a stochastic version of minimal spanning tree problem in which edge costs are random variables. The problem is to find an optimal spanning tree and optimal probability level of a certain chance constraint. The problem is first transformed into a deterministic equivalent problem. Then its subproblem with positive parameter and further an auxiliary problem of subproblem are introduced. Finally, fully utilizing relations among these problems, we propose an algorithm which finds an optimal solution of the original problem in a polynomial order of its problem size.

1. Introduction

Until today the minimal spanning tree problem has been well studied and many efficient algorithms such as [3,6,7] are known. This paper generalizes it to a stochastic version of minimal spanning tree problem where edge costs are not constant, but random variables. The problem is to find an optimal spanning tree and optimal satisficing probability level of a certain chance constraint. In other words, the problem may be considered as a discrete version of [4].

Section 2 formulates the problem P_0 and gives its deterministic equivalent problem P. Section 3 introduces subproblem P^q and clarifies its relation to the original problem P. Further in order to solve P^q, its auxiliary problem P_R^q is introduced. The relation between P^q and P_R^q is also clarified. Fully utilizing these results in Section 3, Section 4 proposes a parametric type algorithm. Section 4 also shows that the algorithm finds an optimal spanning tree and optimal satisficing level in at most $O(m^2n^2)$ computational time where m is the number of edges and n is the number of vertices in a given graph G. Finally, Section 5 discusses more improvement of the algorithm.

2. Problem Formulation

Let $G=(N,E)$ denote undirected graph consisting of vertex set $N=\{v_1,v_2,\cdots,v_n\}$ and edge set $E=\{e_1,e_2,\cdots,e_m\}\subset N\times N$. Moreover cost c_j is attached to edge e_j. Spanning tree $T=T(N,S)$ of G is a partial graph satisfying the following conditions. (See [2] for details.)

(a) T has a same vertex set as G.

(b) $S\subseteqq T$. $|S|$=n-1 where $|S|$ denotes the cardinality of set S.

(c) T is connected.

T can be denoted with 0-1 variables $x_1,x_2,\cdots,x_m$ as follows.

$$T:\quad x_i=1 \quad e_i\in S$$
$$\quad\quad x_i=0 \quad e_i\notin S.$$

Conversely, if $\{e_i|x_i=1\}$ becomes a spanning tree of G with vertex set N, $X=(x_1,x_2,\cdots,x_m)$ is also called spanning tree hereafter in this paper.

Ordinary minimal spanning tree problem is to seek a spanning tree X minimizing $\sum_{j=1}^{m} c_j x_j$. In many real situations, however, c_j's may not be constant, rather random variables. So we consider the following stochastic version P_0 of minimal spanning tree problem.

$$P_0:\quad \text{Minimize } f-\lambda\alpha$$
$$\text{subject to (2.1) } \Pr\{\sum_{j=1}^{m} c_j x_j \leqq f\}\geqq\alpha$$
$$x_j=0 \text{ or } 1,\ j=1,\ldots,m,\ X\text{: spanning tree},$$
$$1\geqq\alpha>\frac{1}{2},$$

where each c_j is assumed to be distributed according to the normal distribution $N(\mu_j,\sigma_j^2)$ with mean μ_j and variance σ_j^2, and they are mutually independent. The probability level α is also decision variable representing a satisficing level of chance constraint (2.1) and λ is a positive constant. As is well known in the theory of stochastic programming ([5,8,9]), P_0 is equivalent to the following deterministic problem P. (For details, see Appendix.)

$$P:\quad \text{Minimize } g(X,q)\triangleq\sum_{j=1}^{m}\mu_j x_j+q(\sum_{j=1}^{m}\sigma_j^2 x_j)^{\frac{1}{2}}-\lambda F(q)$$
$$\text{subject to } x_j=\text{o or } 1,\ j=1,2,\ldots,m,\ X\text{: spanning tree},\ q>0,$$

where $F(\cdot)$ is the distribution fuction of the standard normal distribution

N(0,1) and $q=F^{-1}(\alpha)$.

3. Subproblem P^q and Its Auxiliary Problem P_R^q

First this section introduces the following subproblem P^q in order to solve P.

P^q: Minimize $\sum_{j=1}^{m}\mu_j x_j + q(\sum_{j=1}^{m}\sigma_j^2 x_j)^{\frac{1}{2}}$

subject to $x_j=0$ or 1, X: spanning tree.

Let X^q denote an optimal solution of P^q, X(q) set of all X^q and (X^*,q^*) an optimal solution of P. Further we define

$$D(X)=(\sum_{j=1}^{m}\sigma_j^2 x_j)^{\frac{1}{2}},\ E(X)=\sum_{j=1}^{m}\mu_j x_j \text{ and } D(q)=\{D(X^q)\,|\,X^q\in X(q)\}.$$

Though $q\to D(q)$ is a point to set mapping, the following discussions hold, however, even if we choose $D(X^q)$ corresponding to any X^q as a representative of D(q). Therefore, we denote above $D(X^q)$ as D^q simply as if D^q were unique.

Property 1. D^q is a nonincreasing function of q.

Proof: From the optimality of X^{q_1} and X^{q_2} for q_1 and $q_2>q_1$ respectively the following relations

$$(3.1)\quad \sum_{j=1}^{m}\mu_j x_j^{q_1} + q_1(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_1})^{\frac{1}{2}} \leqq \sum_{j=1}^{m}\mu_j x_j^{q_2} + q_1(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_2})^{\frac{1}{2}}$$

and

$$(3.2)\quad \sum_{j=1}^{m}\mu_j x_j^{q_2} + q_2(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_2})^{\frac{1}{2}} \leqq \sum_{j=1}^{m}\mu_j x_j^{q_1} + q_2(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_1})^{\frac{1}{2}}$$

hold. Subtracting the right hand side of (3.2) from the left hand of (3.1) and the left hand side of (3.2) from the right hand side of (3.1) respectively we have

$$(q_1-q_2)(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_1})^{\frac{1}{2}} \leqq (q_1-q_2)(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_2})^{\frac{1}{2}}$$

or

$$(3.3)\quad (q_1-q_2)\{(\sum_{j=1}^{m}\sigma_j^2 x_j^{q_1})^{\frac{1}{2}} - (\sum_{j=1}^{m}\sigma_j^2 x_j^{q_2})^{\frac{1}{2}}\} \leqq 0$$

Since $q_1-q_2<0$, (3.3) implies

$$(3.4)\quad D^{q_1} = (\sum_{j=1}^{m}\sigma_j^2 x_j^{q_1})^{\frac{1}{2}} \geqq (\sum_{j=1}^{m}\sigma_j^2 x_j^{q_2})^{\frac{1}{2}} = D^{q_2}$$ □

In order to solve P^q, now we define an auxiliary problem P_R^q with positive parameter R as follows.

$$P_R^q: \quad \text{Minimize} \quad R\sum_{j=1}^{m} \mu_j x_j + q\sum_{j=1}^{m} \sigma_j^2 x_j$$

subject to $x_j=0$ or 1, X: spanning tree.

Let $X^q(R)$ denote an optimal solution of P_R^q. Note that P_R^q is an ordinary minimal spanning tree problem with each edge cost $R\mu_j + q\sigma_j^2$. Thus $X^q(R)$ can be found by Prim [7], Kruskal [6] etc in $O(n^2)$.

Property 2. $D(X^q(R))$ is a nondecreasing function of R.

Proof: From the optimality of $X^q(R_1)$ and $X^q(R_2)$ for R_1 and $R_2 > R_1 > 0$ respectively, we can obtain

$$(3.5) \quad R_1\sum_{j=1}^{m} \mu_j x_j^q(R_1) + q\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1) \leqq R_1\sum_{j=1}^{m} \mu_j x_j^q(R_2) + q\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2)$$

and

$$(3.6) \quad R_2\sum_{j=1}^{m} \mu_j x_j^q(R_2) + q\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2) \leqq R_2\sum_{j=1}^{m} \mu_j x_j^q(R_1) + q\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1) \quad .$$

Dividing (3.5) by R_1 and (3.6) by R_2 respectively,

$$(3.7) \quad \sum_{j=1}^{m} \mu_j x_j^q(R_1) + \frac{q}{R_1}\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1) \leqq \sum_{j=1}^{m} \mu_j x_j^q(R_2) + \frac{q}{R_1}\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2)$$

and

$$(3.8) \quad \sum_{j=1}^{m} \mu_j x_j^q(R_2) + \frac{q}{R_2}\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2) \leqq \sum_{j=1}^{m} \mu_j x_j^q(R_1) + \frac{q}{R_2}\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1)$$

result. Subtracting the right hand side of (3.8) from the left hand side of (3.7) and the left hand side of (3.8) from the right hand side of (3.7) respectively,

$$q\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1) \leqq q\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2)$$

or

$$D(X^q(R_1))^2 = \sum_{j=1}^{m} \sigma_j^2 x_j^q(R_1) \leqq \sum_{j=1}^{m} \sigma_j^2 x_j^q(R_2) = D(X^q(R_2))^2$$

results since $q>0$ and $\left(\frac{1}{R_1} - \frac{1}{R_2}\right)>0$. That is

$$D(X^q(R_1)) \leqq D(X^q(R_2))$$

holds since $D(X^q(R_1))$, $D(X^q(R_2))>0$. □

Next the relation between P^q and P_R^q can be clarified.

Lemma 1. For $R \leqq 2D^q$ and any spanning tree $\bar{X}$ such that $D(\bar{X}) > D^q$,

$$R \sum_{j=1}^{m} \mu_j \bar{x}_j + q \sum_{j=1}^{m} \sigma_j^2 \bar{x}_j > R \sum_{j=1}^{m} \mu_j x_j^q + q \sum_{j=1}^{m} \sigma_j^2 x_j^q$$

holds.

Proof: From the optimality of X^q for P^q,

$$(3.9) \qquad \sum_{j=1}^{m} \mu_j x_j^q + q\left(\sum_{j=1}^{m} \sigma_j^2 x_j^q\right)^{\frac{1}{2}} \leqq \sum_{j=1}^{m} \mu_j \bar{x}_j + q\left(\sum_{j=1}^{m} \sigma_j^2 \bar{x}_j\right)^{\frac{1}{2}}$$

holds. Multiplying both hands of (3.9) by R such that $2D^q \geqq R > 0$ and rearranging (3.9) appropriately,

$$R \sum_{j=1}^{m} \mu_j x_j^q + q \sum_{j=1}^{m} \sigma_j^2 x_j^q \leqq R \sum_{j=1}^{m} \mu_j \bar{x}_j + q \sum_{j=1}^{m} \sigma_j^2 \bar{x}_j + q\varepsilon$$

results where $\varepsilon \triangleq \left(\sum_{j=1}^{m} \sigma_j^2 x_j^q\right) - \left(\sum_{j=1}^{m} \sigma_j^2 \bar{x}_j\right) + R\left\{\left(\sum_{j=1}^{m} \sigma_j^2 \bar{x}_j\right)^{\frac{1}{2}} - \left(\sum_{j=1}^{m} \sigma_j^2 x_j^q\right)^{\frac{1}{2}}\right\}$.

Then it is sufficient to prove $\varepsilon < 0$. Using D^q and $D(\bar{X})$, ε is rewritten as follows.

$$\varepsilon = (D^q)^2 - D(\bar{X})^2 + RD(\bar{X}) - RD^q = (D^q - D(\bar{X}))(D^q + D(\bar{X}) - R).$$

Since $D^q < D(\bar{X})$ from the assumption of this lemma and $D^q + D(\bar{X}) - R \leqq 0$,

$$\varepsilon < 0$$

is derived. □

Lemma 2. For $R \geqq 2D^q$ and any spanning tree such that $D(\hat{X}) < D^q$,

$$R \sum_{j=1}^{m} \mu_j x_j^q + q \sum_{j=1}^{m} \sigma_j^2 x_j^q < R \sum_{j=1}^{m} \mu_j \hat{x}_j + q \sum_{j=1}^{m} \sigma_j^2 \hat{x}_j$$

holds.

Proof: Assume contrary, i.e.,

$$(3.10) \qquad R \sum_{j=1}^{m} \mu_j x_j^q + q \sum_{j=1}^{m} \sigma_j^2 x_j^q \geqq R \sum_{j=1}^{m} \mu_j \hat{x}_j + q \sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \quad .$$

From the optimality of X^q,

$$(3.11) \qquad \sum_{j=1}^{m} \mu_j x_j^q + q\left(\sum_{j=1}^{m} \sigma_j^2 x_j^q\right)^{\frac{1}{2}} \leqq \sum_{j=1}^{m} \mu_j \hat{x}_j + q\left(\sum_{j=1}^{m} \sigma_j^2 \hat{x}_j\right)^{\frac{1}{2}}$$

holds. Then the assumption $D(\hat{X}) < D^q$ and (3.11) together implies

$$(3.12) \qquad \sum_{j=1}^{m} \mu_j x_j^q < \sum_{j=1}^{m} \mu_j \hat{x}_j .$$

Therefore from (3.10) and (3.12),

$$(3.13)\qquad R \leqq \frac{q\left(\sum_{j=1}^{m}\sigma_j^2 x_j^q - \sum_{j=1}^{m}\sigma_j^2\hat{x}_j\right)}{\sum_{j=1}^{m}\mu_j\hat{x}_j - \sum_{j=1}^{m}\mu_j x_j^q}$$

holds. Since

$$(3.14)\qquad \sum_{j=1}^{m}\mu_j\hat{x}_j - \sum_{j=1}^{m}\mu_j x_j^q \geqq q\left\{\left(\sum_{j=1}^{m}\sigma_j^2 x_j^q\right)^{\frac{1}{2}} - \left(\sum_{j=1}^{m}\sigma_j^2\hat{x}_j\right)^{\frac{1}{2}}\right\}$$

holds from (3.11), the relations (3.13) and (3.14) together imply

$$R \leqq \frac{q\left(\sum_{j=1}^{m}\sigma_j^2 x_j^q - \sum_{j=1}^{m}\sigma_j^2\hat{x}_j\right)}{\sum_{j=1}^{m}\mu_j\hat{x}_j - \sum_{j=1}^{m}\mu_j x_j^q} \leqq \frac{q\left\{\left(\sum_{j=1}^{m}\sigma_j^2 x_j^q\right) - \left(\sum_{j=1}^{m}\sigma_j^2\hat{x}_j\right)\right\}}{q\left\{\left(\sum_{j=1}^{m}\sigma_j^2 x_j^q\right)^{\frac{1}{2}} - \left(\sum_{j=1}^{m}\sigma_j^2\hat{x}_j\right)^{\frac{1}{2}}\right\}} = \frac{(D^q)^2 - (D(\hat{X}))^2}{D^q - D(\hat{X})} = D^q + D(\hat{X}) < 2D^q .$$

But this contradicts the assumption $R \geqq 2D^q$. Thus this lemma holds. []

Remark 1. All optimal solutions of $P^q_{2D^q}$ have the same value with respect to $D(\cdot)$ and $E(\cdot)$. Thus they have the same value with respect to $g(\cdot, q)$.

Theorem 1. $X^q(2D^q)$, an optimal solution of $P^q_{2D^q}$, is also optimal for P^q.

Proof: From Lemma 1 and Lemma 2, X^q is better than any $\hat{X}$ $(\bar{X})$ in Lemma 1 (Lemma 2) respectively for $P(2D^q)$. Further Remark 1 proves the optimality of X^q among spanning trees $\tilde{X}$'s such that $D(\tilde{X})=D^q$. []

Now let define $R_{ij}=(\sigma_j^2-\sigma_i^2)/(\mu_i-\mu_j)$ $(i,j=1,2,\ldots,m,\ i<j)$. Note that at the point $R=qR_{ij}$, the order of cost $R\mu_j+q\sigma_j^2$ changes. Rearranging qR_{ij} such that $0<qR_{ij}<\infty$ in increasing order, let

$$(3.15)\qquad R_1^q < R_2^q < \cdots < R_\ell^q$$

and $R_0^q \triangleq 0$ where ℓ is the number of different qR_{ij}'s belonging to the interval $(0, \infty)$. Note that the order of R_i^q, $i=0,1,2,\ldots,\ell$, and ℓ are independent of q.

Theorem 2. $X^{\bar{q}}(\bar{R})$ for $\bar{R}\in[R_i^{\bar{q}}, R_{i+1}^{\bar{q}}]$ is also an optimal solution of all P_R^q for $R\in[R_i^q, R_{i+1}^q]$ so long as the latter interval includes $\bar{R}$.

Proof: Let $T_R^{\bar{q}}$ be a corresponding spanning tree of $X^{\bar{q}}(\bar{R})$ i.e., $T_R^{\bar{q}}$ consists of N and edge set $E_R^{\bar{q}} \triangleq \{e_i \mid x^{\bar{q}}(\bar{R})_i=1\}$. Then from the optimality of $X^{\bar{q}}(\bar{R})$,

$$(3.16)\qquad \bar{R}\mu_r + \bar{q}\sigma_r^2 \geqq \bar{R}\mu_t + \bar{q}\,\sigma_t^2$$

must hold for any $e_t\in E_R^{\bar{q}}$ and $e_r\in \mathcal{E}(e_t, T_R^{\bar{q}})$ where $\mathcal{E}(e_t, T_R^{\bar{q}})=\{e_r \mid \text{loop in } \{e_r\}\cup T_R^{\bar{q}} \text{ contains edge } e_t\}$. By the definition of R_k^q, $k=1,2,\ldots,\ell$, order of edge cost does not change among the interval $[R_i^q, R_{i+1}^q]$. Thus once (3.16) holds for a certain $\bar{R}$ such that $\bar{R} \in [\bar{R}_i^q, \bar{R}_{i+1}^q]$, for any R on $[R_i^q, R_{i+1}^q]$ including $\bar{R}$, (3.16) also holds, i.e., $X_{\bar{R}}^q$ is optimal for P_R^q. □

4. Algorithm for P

let f(.) denote the probability density function of standard normal distribution. Then

(4.1) $$\frac{\partial g(X,q)}{\partial q} = \Big(\sum_{j=1}^{m} \sigma_j^2 x_j\Big)^{\frac{1}{2}} - \lambda f(q) \quad \text{and} \quad \frac{\partial^2 g(X,q)}{\partial q^2} = \frac{q}{\sqrt{2\pi}}\, e^{-\frac{1}{2}q^2} .$$

Theorem 3. g(X,q) is a convex function with respect to q > 0.

Proof: Since q > 0, (4.1) implies

(4.2) $$\frac{\partial^2 g(X,q)}{\partial q^2} > 0$$

for q > 0. This inequality (4.2) shows the convexity of g(X,q) with respect to q > 0. □

By Theorem 3, the optimal q=q(X) for each spanning tree X becomes as follows.

$$q(X) = \begin{cases} \sqrt{\log\left(\dfrac{\lambda^2}{2\pi\sum_{j=1}^{m}\sigma_j^2 x_j}\right)} & (\lambda \geqq \sqrt{2\pi}D(X)) \\ 0 & (\lambda < \sqrt{2\pi}D(X)) \end{cases} .$$

Based on q(X), transformation T(q) with respect to q > 0 is defined as follows.

$$T(q) = \begin{cases} \sqrt{\log\left(\dfrac{\lambda^2}{2\pi(D^q)^2}\right)} & (\lambda \geqq \sqrt{2\pi}\ D^q) \\ 0 & (\lambda < \sqrt{2\pi}\ D^q) \end{cases} .$$

Note that T(q) is also not necessarily unique. Again as D^q, the followings hold even if we use any D^q for T(q).

Property 3. T(q) is a nondecreasing function of q.

Proof: By Property 1,

$$\sqrt{\log\left(\frac{\lambda^2}{2\pi(D^q)^2}\right)}$$

is nondecreasing function with respect to q and this proves Property 4. □

Theorem 4. (X^*,q^*), an optimal solution of P, satisfies $q^*=T(q^*)$, $X^{q^*}=X^*$. That is, q^* is a fixed point with respect to $T(q)$.

Proof: $q^* \neq T(q^*)$ means $q^* \neq q(X^{q^*})$ and it impies

$$g(X^{q^*}, q^*) > g(X^{q^*}, q(X^{q^*})).$$

This contradicts optimality of q^*. □

Theorem 5. For q_1 and $q_2 = T(q_1)$,

$$q_1 > q_2 \longrightarrow q^* \notin (q_2, q_1]$$

and

$$q_1 < q_2 \longrightarrow q^* \notin [q_1, q_2)$$

hold.

Proof: If $q_1 > q_2$, for any $\hat{q} \in (q_2, q_1]$,

$$T(\hat{q}) - \hat{q} < T(\hat{q}) - q_2 \leqq T(q_1) - q_2 = 0$$

holds since $T(q)$ is a nondecreasing function of q. Therefore $\hat{q}$ does not satisfy the necessary condition of q^*. In case of $q_1 < q_2$, the proof can be similarly done. □

Now we are ready to construct our algorithm. In the algorithm, we use the followimg notations.

X^L: a minimal spanning tree for the case of each edge cost μ_j.

X^U: a maximal spanning tree for the case of each edge cost σ_j^2.

$q^L \triangleq q(X^L)$. $q^U \triangleq q(X^U)$.

[Algorithm]

Step 0: Set $q \leftarrow 1$ and calculate $R_0^q, \cdots, R_\ell^q$. Then set $C \leftarrow g(X^L, q^L)$, $\bar{X} \leftarrow X^L$, $\bar{q} \leftarrow q^L$ and $i \leftarrow 0$. Go to Step 1.

Step 1: Set $R \leftarrow \frac{1}{2}(R_i^q + R_{i+1}^q)$, find X_R^q and calculate $g(X_R^q, q(X_R^q))$. If $C > g(X_R^q, q(X_R^q))$, set $C \leftarrow g(X_R^q, q(X_R^q))$, $\bar{X} \leftarrow X_R^q$ and $\bar{q} \leftarrow q(X_R^q)$, and go to Step 2. Otherwise, go to Step 2 directly.

Step 2: Set $i \leftarrow i+1$. If $i=\ell$, go to Step 3. Otherwise return to Step 1.

Step 3: If $g(X^U, q^U) < C$, terminate after setting $X^* \leftarrow X^U$ and $q^* \leftarrow q^U$. otherwise terminate after setting $X^* \leftarrow \bar{X}$ and $q^* \leftarrow \bar{q}$.

Theorem 6. Above algorithm finds an optimal solution (X^*,q^*) in at most $O(m^2n^2)$ iterations.

Proof: (Validity) By Theorem 4, $X^* \in S^{q^*}$ holds where S^{q^*} is the set of all optimal solutions of P^{q^*}. Moreover by Theorem 1, $S^{q^*} \subset S^{q^*}_{2D}q^*$ holds where $S^{q^*}_{2D}q^*$ is the set of all optimal solutions of $P^{q^*}_{2D}q^*$. Above discussion and Theorem 2 together show X^* is included among X^q_R's for (q,R) such that $R \in [R^q_i, R^q_{i+1}]$, $i=1,\ldots,\ell$, $R < R^q_1$ and $R > R^q_\ell$, because of Remark 1. Further the order of R^q_i and ℓ are fixed independent of q. The algorithm tests all these candidates and find a minimal solution of them.

(Complexity) First note that the calculation of $R^q_1,\cdots,R^q_\ell$ can be done in at most $O(m^2 \log m)$. For each (q,R), X^q_R can be found in at most (n^2) if using Prim's algorithm [7] or Kruskal's one [6]. Clearly, the number of X^q_R checked by the algorithm is at most $\ell+2 < \frac{m(m-1)}{2} + 2$ in order to find (X^*,q^*). Thus in at most $O(m^2n^2)$ computational time, the algorithm finds (X^*,q^*). ▯

5. Conclusion

This paper considered a stochastic version of minimal spanning tree problem P_0. First P_0 was transformed into the deterministic equivalent problem P. Then subproblem P^q with a positive parameter q was introduced and its relation to P was clarified. Further, auxiliary problem P^q_R was introdued and its relation to P^q was also clarified. Based on these relations, parametric type algorithm which finds an optimal solution of P in at most $O(m^2n^2)$ computational time has been proposed.

Another type algorithm which alternatively changes q and X may be possible. Perhaps, this type algorithm may be more efficient than the algorithm and it is one of future research problems.

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Appendix. Transformation of P_0 into P ([1,5,8,9])

The chance constraint

$$\Pr\{\sum_{j=1}^{m} c_j x_j \leqq f\} \geqq \alpha$$

is transformed as follows.

(A1) $$\Pr\{\sum_{j=1}^{m} c_j x_j \leqq f\} \geqq \alpha \Longleftrightarrow \Pr\left\{\frac{\sum_{j=1}^{m}(c_j x_j - \mu_j x_j)}{(\sum_{j=1}^{m}\sigma_j^2 x_j^2)^{\frac{1}{2}}} \leqq \frac{f - \sum_{j=1}^{m}\mu_j x_j}{(\sum_{j=1}^{m}\sigma_j^2 x_j^2)^{\frac{1}{2}}}\right\} \geqq \alpha$$

Since $\sum_{j=1}^{m}(c_j x_j - \mu_j x_j)/(\sum_{j=1}^{m}\sigma_j^2 x_j^2)^{\frac{1}{2}}$ is a random variable according to the standard normal distribution, (A1) is further transformed into (A2).

(A2) $$(f - \sum_{j=1}^{m}\mu_j x_j)/(\sum_{j=1}^{m}\sigma_j^2 x_j^2)^{\frac{1}{2}} \geqq F^{-1}(\alpha)$$

where $F(\cdot)$ is the distribution function of standard normal distribution and $F^{-1}(\alpha)$ is its 100α percentile point. Therefore, setting $q \triangleq F^{-1}(\alpha) > 0$ (since

$1 \geqq \alpha > \frac{1}{2}$) and noting $x_j^2 = x_j$ ($0^2=0$, $1^2=1$) and minimum of f equals to

$$\sum_{j=1}^{m} \mu_j x_j + q\left(\sum_{j=1}^{m} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}$$

for each spanning tree X, P_0 is equivalent to the following deterministic problem P.

$$P: \quad \text{Minimize} \quad \sum_{j=1}^{m} \mu_j x_j + q\left(\sum_{j=1}^{m} \sigma_j x_j\right)^{\frac{1}{2}} - \lambda F(q)$$

subject to $x_j=0$ or 1, X: spanning tree, $q > 0$.

Hiroaki Ishii: Department of Applied Physics, Faculty of Engineering, Osaka University, Yamadaoka, Suita, Osaka, 565, Japan.