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ON EQUIVALENT-JOB FOR JOB-BLOCK IN $2\times n$ SEQUENCING PROBLEM WITH TRANSPORTATION-TIMES

Parkash Lal Maggu
M. S. College

Ghanshiam Das
M. S. College

Ravinder Kumar
M. S. College

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Abstract This paper has the objective to obtain optimal schedule principle for $2 \times n$ sequencing problem wherein transportation times of jobs and equivalent-jobs for job-blocks are assumed to occur. The optimal schedule rule is based upon a theorem as established in the paper.

1. Introduction

Bellman [1] and Johnson [2] considered a problem involving the scheduling of n jobs on two machines. Mitten [6, 7] and Johnson [3] treated a scheduling problem with arbitrary time lags. Maggu and Das [4] established "equivalent-job for job-blocks theorem" for $2\times n$ sequencing problem. In the '2-machine, n-job' makespan problem the concept of equivalent jobs for blocks in job sequencing was introduced by Maggu and Das as follows: Consider the job-sequence $S = (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n)$ of n jobs with the condition that jobs α_k and α_{k+1} must occur in the sequences as a block, i.e., if α_k is the i-th job then α_{k+1} must be the $(i+1)$-th job. Now it is possible to define a job β (say) with processing times $t_{\beta A}$ and $t_{\beta B}$ on two machine A and B respectively which can replace the jobs α_k and α_{k+1} for the purpose of finding the minimum schedule time. When β replaces α_k and α_{k+1} to produce a new sequence S', the completion times on both machines is changed by a value which is independent of the particular sequence S. Hence the substitution does not change the relative merit of different sequences.

Further, Maggu and Das [5] considered $2\times n$ flowshop problem wherein transportation times of jobs from one machine to another are assumed to occur. In all the preceding papers, except [5], the transportation times of jobs from one machine to another are neglected. Owing to theoretical and as well practical interest this paper has the object of providing a decomposition algorithm to obtain optimal schedule of jobs for the case of 2-machine n-job flow-shop problem wherein a block of ordered jobs is assumed to be an equivalent to a single job and, furthermore, the transportation times of jobs from one machine to another subsequent machine are given.

Now to meet the object of this paper to develop an algorithm producing an optimal schedule, for the two-machine n-job flow-shop problem, minimizing the total elapsed time we prepare the basis for the same in the form of a theorem as follows.

2. Description of Problem

Theorem: Consider a flow-shop problem consisting of two machines A and B, and a set of n-jobs to be processed on these machines. We are given processing time t_{iX} for each job i, on machine $X = A, B$. Each machine can handle at most one job at a time and the processing of each job must be finished on machine A before it can be processed on machine B. It has been assumed that the order of treatments in the process A and B are the same. Let t_i denote the transportation time of job i from machine A to B. In the transportation process, several jobs can be handled simultaneously. Let β be an equivalent job for a given ordered set of jobs (α_k, α_{k+1}). Then processing times $t_{\beta A}$ and $t_{\beta B}$ on machines A and B are given by

$$t_{\beta A} = (t_{\alpha_k A} + t_{\alpha_k}) + (t_{\alpha_{k+1} A} + t_{\alpha_{k+1}}) - \min(t_{\alpha_{k+1} A} + t_{\alpha_{k+1}}, t_{\alpha_k B} + t_{\alpha_k})$$

$$t_{\beta B} = (t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_{k+1} B} + t_{\alpha_{k+1}}) - \min(t_{\alpha_k B} + t_{\alpha_k}, t_{\alpha_{k+1} A} + t_{\alpha_{k+1}})$$

and the transportation time of β from machine A to B is given by $t_\beta=0$.

Proof: Consider the two sequences S and S' of jobs as:

$$S = (\alpha_1, \alpha_2, \alpha_3, \ldots\alpha_{k-1}, \alpha_k, \alpha_{k+1}, \alpha_{k+2}, \ldots\alpha_n)$$

$$S' = (\alpha_1, \alpha_2, \alpha_3, \ldots\alpha_{k-1}, \beta, \alpha_{k+2}, \ldots\ldots \alpha_n).$$

Let T_{iX} denote completion time of job i on machine X, when jobs are processed according to S. Let T'_{iX} denote completion time of job i on machine

X, when jobs are done according to S'.

Now it is clear to see that

$$T_{\alpha_k B} = \max(T_{\alpha_k A} + t_{\alpha_k}, T_{\alpha_{k-1}B}) + t_{\alpha_k B}$$

$$= \max(T_{\alpha_k A} + t_{\alpha_k} + t_{\alpha_k B}, T_{\alpha_{k-1}B} + t_{\alpha_k B})$$

$$T_{\alpha_{k+1}B} = \max(T_{\alpha_{k+1}A} + t_{\alpha_{k+1}}, T_{\alpha_k B}) + t_{\alpha_{k+1}B}$$

$$= \max(T_{\alpha_{k+1}A} + t_{\alpha_{k+1}}, T_{\alpha_k A} + t_{\alpha_k} + t_{\alpha_k B},$$

$$T_{\alpha_{k-1}B} + t_{\alpha_k B}) + t_{\alpha_{k+1}B}$$

$$= \max(T_{\alpha_{k+1}A} + t_{\alpha_{k+1}} + t_{\alpha_{k+1}B}, T_{\alpha_k A} + t_{\alpha_k} + t_{\alpha_k B} +$$

$$t_{\alpha_{k+1}B}, T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B}).$$

Again, we have

$$T_{\alpha_{k+2}B} = \max(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}}, T_{\alpha_{k+1}B}) + t_{\alpha_{k+2}B}$$

$$= \max(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}}, T_{\alpha_{k+1}A} + t_{\alpha_{k+1}} + t_{\alpha_{k+1}B},$$

$$T_{\alpha_k A} + t_{\alpha_k} + t_{\alpha_k B} + t_{\alpha_{k+1}B},$$

$$T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B}) + t_{\alpha_{k+2}B}$$

$$= \max(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}}, T_{\alpha_k A} + t_{\alpha_{k+1}A} + t_{\alpha_{k+1}} + t_{\alpha_{k+1}B},$$

$$T_{\alpha_k A} + t_{\alpha_k B} + t_{\alpha_k} + t_{\alpha_{k+1}B},$$

$$T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B}) + t_{\alpha_{k+2}B}.$$

Now,

$$\max(T_{\alpha_k A} + t_{\alpha_{k+1}A} + t_{\alpha_{k+1}} + t_{\alpha_{k+1}B},$$

$$T_{\alpha_k A} + t_{\alpha_k B} + t_{\alpha_k} + t_{\alpha_{k+1}B})$$

$$= T_{\alpha_k A} + \max(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}, t_{\alpha_k B} + t_{\alpha_k}) + t_{\alpha_{k+1}B}.$$

Hence

$$T_{\alpha_{k+2}B} = \max\ (T_{\alpha_{k+2}A} + t_{\alpha_{k+2}},$$

$$T_{\alpha_k A} + \max\ (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}},\ t_{\alpha_k B} + t_{\alpha_k}) +$$

$$t_{\alpha_{k+1}B},\ T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B}) + t_{\alpha_{k+2}B} \qquad (1)$$

Similarly for sequence S' we have

$$T'_{\beta B} = \max\ (T'_{\beta A} + t_{\beta},\ T'_{\alpha_{k-1}B}) + t_{\beta B}$$

$$= \max\ (T'_{\beta A} + t_{\beta} + t_{\beta B}\ ,\ T'_{\alpha_{k-1}B} + t_{\beta B})$$

$$T'_{\alpha_{k+2}B} = \max\ (T'_{\alpha_{k+2}A} + t_{\alpha_{k+2}},\ T'_{\beta B}) + t_{\alpha_{k+2}B}$$

$$= \max\ (T'_{\alpha_{k+2}A} + t_{\alpha_{k+2}},\ T'_{\beta A} + t_{\beta} + t_{\beta B}\ ,$$

$$T'_{\alpha_{k-1}B} + t_{\beta B}) + t_{\alpha_{k+2}B}\ . \qquad (2)$$

Now without loss of generality one can assume easily:

$$t_{\beta A} = (t_{\alpha_k A} + t_{\alpha_k}) + (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - C\ ,$$

$$t_{\beta} = 0\ ,$$

$$t_{\beta B} = (t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_{k+1}B} + t_{\alpha_{k+1}}) - C\ ,$$

with $C = \min\ (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}},\ t_{\alpha_k B} + t_{\alpha_k})$.

Now $T'_{\alpha_{k-1}A} = T_{\alpha_{k-1}A}\ ,\ T_{\alpha_{k-1}B} = T_{\alpha_{k-1}B}$

$$T'_{\alpha_{k+2}A} = T'_{\alpha_{k-1}A} + t_{\beta A} + t_{\alpha_{k+2}A}$$

$$= T_{\alpha_{k-1}A} + (t_{\alpha_k A} + t_{\alpha_k}) + (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - C$$

$$+ t_{\alpha_{k+2}A}$$

$$= T_{\alpha_{k-1}A} + t_{\alpha_k A} + t_{\alpha_{k+1}A} + t_{\alpha_{k+2}A} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)$$

$$T'_{\alpha_{k+2}A} = T_{\alpha_{k+2}A} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ . \qquad (3)$$

Hence

$$T'_{\alpha_{k+2}A} + t_{\alpha_{k+2}} = T_{\alpha_{k+2}A} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C) + t_{\alpha_{k+2}}$$

$$= T_{\alpha_{k+2}A} + t_{\alpha_{k+2}} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)$$

$$T'_{\beta A} + t_{\beta} + t_{\beta B} = T'_{\alpha_{k-1}A} + t_{\beta A} + t_{\beta} + t_{\beta B}$$

$$= T_{\alpha_{k-1}A} + (t_{\alpha_k A} + t_{\alpha_k}) + (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - C$$

$$+ 0 + (t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_{k+1}B} + t_{\alpha_{k+1}}) - C$$

$$= T_{\alpha_{k-1}A} + t_{\alpha_k A} + (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - C$$

$$+ (t_{\alpha_k B} + t_{\alpha_k}) + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)$$

Now $(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - C + (t_{\alpha_k B} + t_{\alpha_k})$

$$= (t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}) - \min\,(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,\ t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_k B} + t_{\alpha_k})\ .$$

$$= \max\,\{t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,\ t_{\alpha_k B} + t_{\alpha_k}\}$$

Hence

$$T'_{\beta A} + t_{\beta} + t_{\beta B} = T_{\alpha_{k-1}A} + t_{\alpha_k A} + \max\,(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,\ t_{\alpha_k B} + t_{\alpha_k}) + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ .$$

$$T'_{\alpha_{k-1}B} + t_{\beta B} = T'_{\alpha_{k-1}B} + (t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_{k+1}B} + t_{\alpha_{k+1}}) - C$$

$$= T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ .$$

Substituting in (2), we have

$$T'_{\alpha_{k+2}B} = \max\,(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ ,$$

$$T_{\alpha_{k-1}A} + t_{\alpha_k A} + \max\,(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,$$

$$t_{\alpha_k B} + t_{\alpha_k}) + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ ,$$

$$T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)$$
$$+ t_{\alpha_{k+2}B}$$
$$= \max\,(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ ,$$
$$T_{\alpha_k A} + \max\,(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,\ t_{\alpha_k B} + t_{\alpha_k})$$
$$+ t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)\ ,$$
$$T_{\alpha_{k-1}B} + t_{\alpha_k B} + t_{\alpha_{k+1}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C)$$
$$+ t_{\alpha_{k+2}B}$$
$$= \max\,(T_{\alpha_{k+2}A} + t_{\alpha_{k+2}}\ ,\ T_{\alpha_k A} + \max\,(t_{\alpha_{k+1}A} + t_{\alpha_{k+1}}\ ,$$
$$t_{\alpha_k B} + t_{\alpha_k}) + t_{\alpha_{k+1}B}\ ,\ T_{\alpha_{k-1}B} + t_{\alpha_k B} +$$
$$t_{\alpha_{k+1}B}) + t_{\alpha_{k+2}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C) \qquad (4)$$

From (1) and (4), we have

$$T'_{\alpha_{k+2}B} = T_{\alpha_{k+2}B} + (t_{\alpha_k} + t_{\alpha_{k+1}} - C) \qquad (5)$$

Let $D = t_{\alpha_k} + t_{\alpha_{k+1}} - C$

Then from (3) and (5), we have

$$T'_{\alpha_{k+2}B} = T_{\alpha_{k+2}B} + D \qquad (6)$$

$$T'_{\alpha_{k+2}A} = T_{\alpha_{k+2}A} + D \qquad (7)$$

From (6) and (7), it is clear that replacement of job-block (α_k, α_{k+1}) in S by job β increases the completion times of later job α_{k+2} by a constant D in S' as compared for that job (i.e. α_{k+2}) in S. Let T and T' be the completion times of sequences S and S' respectively. Then from the above discussion, it is easily observed that $T' = T + D$. Hence we can replace a single job β as equivalent to the job-block (α_k, α_{k+1}) of the given ordered job-pair α_k, α_{k+1} in S.

Note 1: Extension of the equivalent - job concept of a block consisting of an ordered set of $k \geq 2$ jobs is obvious.

Note 2: Following as in [4] it can be indicated that equivalency is associative but not commutative i.e.

$(\alpha_1, \alpha_2, \alpha_3) = ((\alpha_1, \alpha_2)\ \alpha_3) = (\alpha_1, (\alpha_2, \alpha_3))$ but $(\alpha_1, \alpha_2) \neq (\alpha_2, \alpha_1)$.

3. Decomposition Algorithm

The above theorem gives a numerical method to obtain an optimal sequence for the $2 \times n$ sequencing problem wherein concepts of Equivalent-job for job-block and transportation times of jobs are involved.

Let the given problem in the tableau form be described as follows:

job (i)	Machine A (A_i) $\xrightarrow{t_i}$ Machine B (B_i)
1	$A_1 \xrightarrow{t_1} B_1$
2	$A_2 \xrightarrow{t_2} B_2$
3	$A_3 \xrightarrow{t_3} B_3$
.	. .
.	. .
.	. .
n	$A_n \xrightarrow{t_n} B_n$

where t_i denotes the transportation time for job i from machine A to B, and A_i, B_i (as usual) denote the processing times of job i on machines A and B respectively. Let $\beta = (r, s, \ldots, u)$ be an equivalent-job of an ordered block of jobs $r, s, \ldots, u$. Then the algorithm to determine optimal schedule minimizing total elapsed time is decomposed into following steps:

Step (1): Obtain processing-times on machines A and B for each equivalent-job of a job-block by using above theorem. Also find the transportation time of each equivalent job.

Step (2): Now form a reduced problem replacing the given job-block in the original problem by their equivalent-jobs.

Step (3): Let G and H be fictitious machines. Then form a new reduced problem from step 2, where G_i and H_i are the processing times of job

i on G and H defined by $G_i = A_i + t_i$

$$H_i = B_i + t_i$$

Step (4): Find optimal sequence for the reduced problem in step 3.

Step (5): In the optimal sequence of step (4) replace back each equivalent-job by their ordered job-block. Now this sequence gives us the optimality for the original problem.

Justification of the Algorithm: Step (1) is justified from theorem. From step (2) we obtain a reduced two machine problem with intermediate transportation process equivalent to our original problem. When intermediate transportation exists under the assumption that several jobs can be handled simultaneously in the transportation process, the problem can be reduced to a two process problem with fictitious machines (which justifies the step (3) in the algorithm), a fact clearly stated in the references [1] (p.12), [2] (p.151) and [5] (p.4). Using [2] we can obtain optimal sequence of reduced problem as per step 3. Again step (5) is justified from the theorem.

Numerical Example: Consider the following tableau form of a 2×5 flow-shop problem where symbols t_i, A_i, B_i have the usual meanings as defined above.

job (i)	Machine A (A_i)	t_i	Machine B (B_i)
1	2	3 →	6
2	5	1 →	3
3	5	6 →	7
4	3	4 →	6
5	9	2 →	8

Let $\beta = (1, 3, 5)$. Then optimal sequence is obtained by following steps 1 through 5 of the Decomposition Alogrithm.

Let $\delta = (1, 3)$

Then $\beta = (\delta, 5)$.

Now as per step (1):

$$t_{\delta A} = (t_{\alpha_k A} + t_{\alpha_k}) + (t_{\alpha_{k+1} A} + t_{\alpha_{k+1}}) - \min\,(t_{\alpha_{k+1} A} + t_{\alpha_{k+1}}\,,\ t_{\alpha_k B} + t_{\alpha_k})$$

$= (2 + 3) + (5 + 6) - \min (5 + 6,\ 6 + 3)$

$= 5 + 11 - 9$

$= 7.$

$$t_{\delta B} = (t_{\alpha_k B} + t_{\alpha_k}) + (t_{\alpha_{k+1} B} + t_{\alpha_{k+1}}) - \min (t_{\alpha_{k+1} A} + t_{\alpha_{k+1}},\ t_{\alpha_k B} + t_{\alpha_k})$$

$= (6 + 3) + (7 + 6) - \min (5 + 6,\ 6 + 3)$

$= 9 + 13 - 9$

$= 13.$

$t_\delta = 0.$

Again

$$t_{\beta A} = (t_{\delta A} + t_\delta) + (t_{5A} + t_5) - \min (t_{\delta B} + t_\delta,\ t_{5A} + t_5)$$

$= (7 + 0) + (9 + 2) - \min (13 + 0,\ 9 + 2)$

$= 7 + 11 - 11$

$= 7.$

$$t_{\beta B} = (t_{\delta B} + t_\delta) + (t_{5B} + t_5) - \min (t_{\delta B} + t_\delta,\ t_{5A} + t_5)$$

$= (13 + 0) + (8 + 2) - \min (13 + 0,\ 9 + 2)$

$= 13 + 10 - 11$

$= 12.$

$t_\beta = 0.$

By step (2) replacing job-block (1, 3, 5) by equivalent job β, the reduced problem is:

job (i)	Machine A (A_i)	t_i →	Machine (B) (B_i)
β	7	0	12
2	5	1	3
4	3	4	6

By step (3), let G and H be fictitious machines then the new reduced problem is

job (i)	Machine G (G_i)	Machine H (H_i)
β	7	12
2	6	4
4	7	10

where $G_i = A_i + t_i$, $H_i = B_i + t_i$.

By step (4), the optimal sequence of newly reduced problem due to Johnson's procedure is (β, 4, 2) or (4, β, 2).
By step (5), replacing back β = (1, 3. 5), we have optimal sequence

(1, 3, 5, 4, 2) or (4, 1, 3, 5, 2).

The total elapsed time T for each optimal sequence is calculated in the following tableau forms:

job (i)	Machine A in - out	t_i →	Machine B in - out	
1	0 - 2	3 →	5 - 11	
3	2 - 7	6 →	13 - 20	
5	7 - 16	2 →	20 - 28	
4	16 - 19	4 →	28 - 34	
2	19 - 24	1 →	34 - 37	(= T) .
4	0 - 3	4 →	7 - 13	
1	3 - 5	3 →	13 - 19	
3	5 - 10	6 →	19 - 26	
5	10 - 19	2 →	26 - 34	
2	19 - 24	1 →	34 - 37	(= T) .

4. Observation

It may be observed that the problem dealt in this paper is thus basically the two process problem under the transportation process as studied in theorem 1 (c.f. [3]) wherein the concept of Block-job has been equipped with. It may be noted that here the job-block is formed equivalent to a given set of preordered jobs.

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Parkash Lal MAGGU:
Post Box 65, Saharanpur
247001, India