Journal of the Operations Research
Society of Japan
Vol. 24, No. 2, June 1981

SOME PROPERTIES OF PERISHABLE INVENTORY CONTROL SUBJECT TO STOCHASTIC LEADTIME

Toyokazu Nose
Osaka University

Hiroaki Ishii
Osaka University

Toshio Nishida
Osaka University

(Received January 19, 1980; Final January 6, 1981)

Abstract This paper discusses the problem of determining the optimal ordering policies and their properties with respect to a fixed-life perishable commodity subject to stochastic procurement leadtime. In addition, set-up cost for placing an order is introduced. Again, several properties of optimal ordering policies are derived. Finally, some further research problems are discussed.

1. Introduction

There exist so many commodities whose value does not remain constant over time during transportation, holding in stock, etc.. Bulinskaya [1], Van Zyl [8], etc., treated the cases where lifetimes are one and two periods of time, respectively. But they did not refer to the perishing cost when the optimal ordering policy is determined. Nahmias and Pierskalla [7] introduced perishing cost into the determination of optimal ordering policy for the two periods lifetime dynamic inventory model. Later, Fries [2] and Nahmias [5] generalized their model to the m periods lifetime model.

But most of those models assume that the lifetime of the product is fixed when the reordered perishable commodities are received. However this assumption is not realistic in many actual circumstances. For example, the average age of units of blood may vary from one day to the next when it is received from the central blood bank. For another example, the lifetime of certain types of foods such as fresh or processed product also tends to vary depending on various external factors, e.g., delay of transportation due to the traffic jam or some traffic accidents, change of weather etc..

Therefore it is reasonable to assume that if the procurement leadtime varies, the value of commodities may also be changeable. Taking the above view point into consideration, this paper discusses perishable inventory control under a certain maximum lifetime and the stochastic leadtime. Particularly the case of 0 or 1 period leadtime is considered and its optimal ordering policy is derived. Further the relationships between optimal ordering policy and some important factors, i.e., the probability of leadtime 0 or 1, the rate of excess perishability under the occurrence of leadtime 1 and the status of inventory on hand (newer or older), are analyzed. Next, considering the fixed ordering cost, some other characteristics of the ordering policy are obtained. Note that the model in this paper is a generalization of the one period horizon model of Nahmias [4].

Sections 2 states the assumptions and the formulations of the model, and the optimal ordering policy is obtained. Section 3 discusses the influences of leadtime, rate of excess perishability and on-hand stock upon the obtained optimal ordering policy. Section 4 gives a numerical example in order to illustrate the results of Sections 2 and 3. Finally, Section 5 concludes this paper and discusses further research problems.

2. Perishable inventory model

A periodic review inventory model is considered for one planning period horizon and single item. That is, ordering takes place at the start of a period and costs are incurred during a period, rather than continuously. The period length is arbitrary but fixed. And the followings are assumed throughout this paper.

(1) Maximum lifetime of the perishable commodity discussed in this paper is m finite periods. If the commodity has not been depleted by demand until the period it reaches age m, then it perishes and must be discarded at a specified per-unit cost.

(2) Demands D_j in succesive periods $j=1,2,\ldots$ are independent nonnegative random variables with known distribution $F_j(\cdot)$ and density $f_j(\cdot)$.

(3) Inventory is depleted by demand at the start of each period according to a FIFO policy and after commodities are placed into stock, deterioration proceeds monotonically one stage at each period.

(4) When procurement leadtime l is 0, the stock arrives new, that is, maximum lifetime m, and when $l=1$, the stock with lifetime $m-1$ or $m-2$ arrives. Leadtime 0 occurs with probability l_0 and 1 with l_1, where $l_0+l_1=1$, $l_0>0$, $l_1\geqq 0$, and when $l=1$, the stock with lifetime $m-1$ arrives at a constant rate α ($0\leqq\alpha\leqq 1$) and $m-2$

at $1-\alpha$. Here, $1-\alpha$ corresponds to the rate of excess perishability under the occurrence of leadtime 1.

In stocking perishable commodities, it is necessary to keep track of the amount of inventory on hand at each lifetime level. Concentrating on our model, we define the following notations;

$x_{(i)}$; the amount of commodity on hand with i periods of usable lifetime left,

X_p ; inventory level in stock, i.e., $X_p \triangleq (x_{(p)}, x_{(p-1)}, \ldots, x_{(1)})$, $1 \leq p \leq m-1$,

B_j ; after depleting all of the commodities $x_{(j)}, x_{(j-1)}, \ldots, x_{(1)}$, the total unsatisfied demand until period j which should be satisfied by the inventory commodities whose remaining lifetimes are greater than j, i.e.,

(2.1) $$B_j = [D_j + B_{j-1} - x_{(j)}]^+ \quad , \; 1 \leq j \leq m-1$$

where $B_0 = 0$ and $[b]^+ = max\,(b, 0)$.

Figure 1 shows this model.

$Q_n(u: X_{n-1})$; the probability that the sum of D_n and B_{n-1} is less than a real number u, i.e.,

(2.2) $$Q_n(u: X_{n-1}) = Pr\{D_n + B_{n-1} \leq u\}, \; 1 \leq n \leq m$$

where, if $u \leq 0$, then $Q_n(u: X_{n-1}) \triangleq 0$,

$A(X_{m-1}, y)$; the total expected cost when y is ordered under the current stock level X_{m-1}, i.e., including the fixed ordering, purchasing, holding, shortage and outdating costs,

$L(X_{m-1}, y)$; the total expected cost excluding the fixed ordering cost from $A(X_{m-1}, y)$,

K ; fixed ordering cost/order,

c ; purchasing cost/unit, $\quad$ h ; holding cost/unit,

p ; shortage cost/unit, $\quad$ r ; outdating cost/unit,

x ; total amount of inventory on hand, i.e., $x \triangleq \sum_{i=1}^{m-1} x_{(i)}$,

e_i ; i-th vector, i.e., $e_i \triangleq (0, \ldots, \overset{i}{1}, 0, \ldots, 0)$, $i = 1, 2, \ldots, m-1$.

From the definitions (1) and (2), the distribution function of the amount of fresh order y that perishes is

(2.3) $$Pr\{[y - (D_m + B_{m-1})]^+ \leq u\} = 1 - Q_m(y - u: X_{m-1}).$$

Note that y is the decision variable which represents the amount of ordered fresh commodity. From the definitions (2) and (3), the expected amount of units of y scheduled to outdate after m periods is derived as follows [4] :

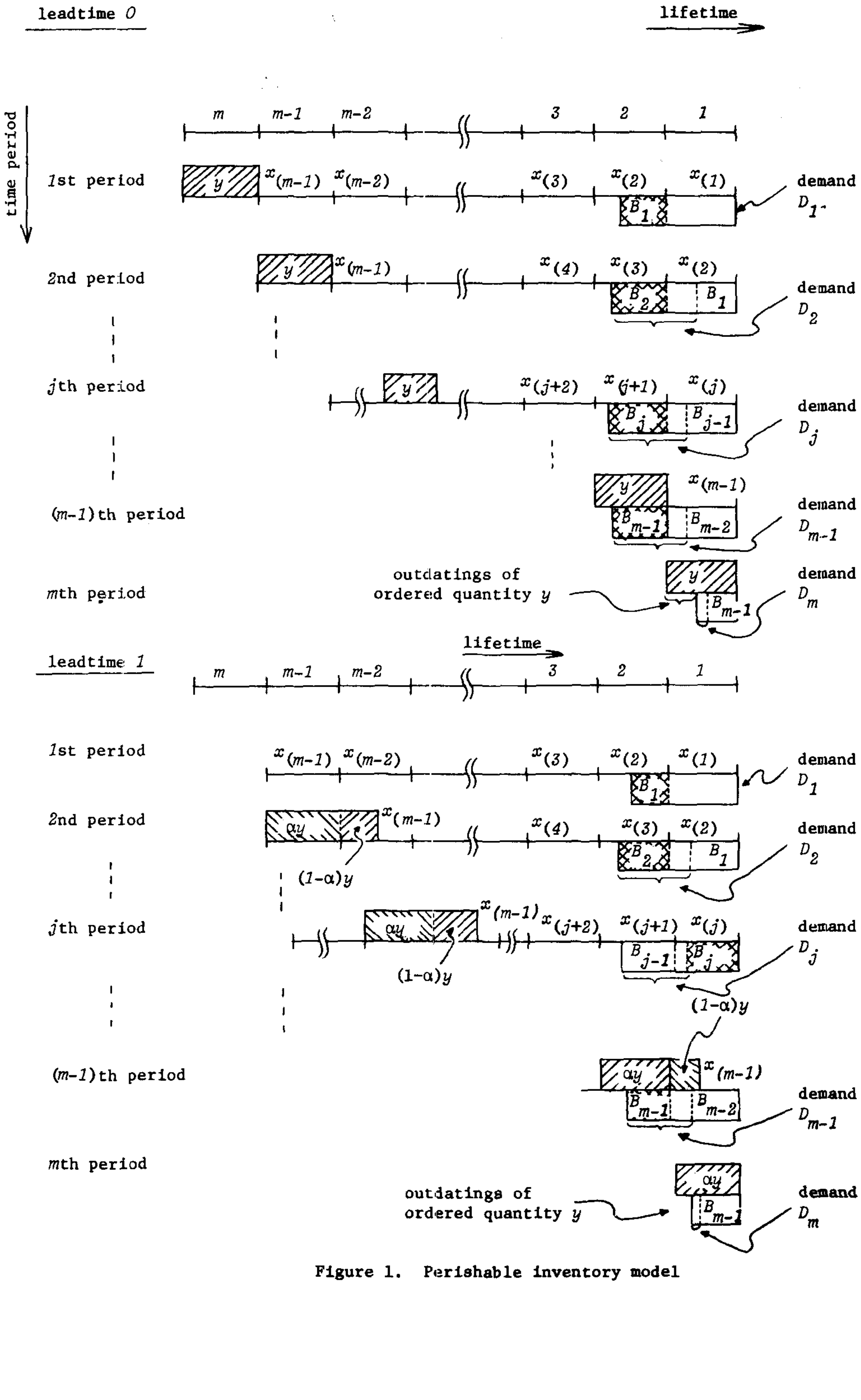

Figure 1. Perishable inventory model

(2.4) $$\int_0^\infty u d\{1-Q_m(y-u:X_{m-1})\}=\int_0^y Q_m(u:X_{m-1})du.$$

Then the total expected cost, $A(X_{m-1}, y)$, is established as the equations (2.5) and (2.6):

(2.5) $$A(X_{m-1}, y)= K\cdot\delta(y)+L(X_{m-1}, y),$$

where

$$\delta(y)=\begin{cases} 1 ; & y>0. \\ 0 ; & y=0. \end{cases}$$

Moreover,

(2.6) $$\begin{aligned} L(X_{m-1}, y)=c\cdot y+l_0[h\int_0^{x+y}&(x+y-u)\cdot f_1(u)du \\ &+p\int_{x+y}^\infty (u-x-y)\cdot f_1(u)du+r\cdot\int_0^y Q_m(u:X_{m-1})du] \\ &+l_1[h\int_0^x (x-u)\cdot f_1(u)du+p\int_x^\infty (u-x)\cdot f_1(u)du \\ &+r\int_{x_{(m-1)}}^{(1-\alpha)y+x_{(m-1)}} Q_{m-1}(u:X_{m-2})du \\ &+r\int_0^{\alpha y} Q_m(u:X_{m-1}+(1-\alpha)y.e_{m-1})du\ ]. \end{aligned}$$

In order to show the convexity of the cost function $L(X_{m-1},y)$, the following lemmas by Nahmias [4] are fully utilized. In the literature [4], $Q_n(u:X_{n-1})$ is formulated as the convolution with $f_i(\cdot)$;

(2.7) $$Q_n(u:X_{n-1}) = \int_0^u Q_{n-1}(v+x_{(n-1)}:X_{n-2})\ f_n(u-v)dv,\ 1 \leqq n \leqq m$$

where $Q_0(u) = 1$.

Especially, the eighth term of the equation (2.6) is expressed as the following lemma.

Lemma 1.

(2.8) $$Q_m(u:X_{m-1}+(1-\alpha)\ y\cdot e_{m-1}) = \int_0^u Q_{m-1}(v+x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(u-v)dv$$

where $Q_0(u) = 1$.

Lemma 2. ([4]) Assume that each demand distribution F_k possesses density f_k that is continuous everywhere. Then the functions $\partial Q_n(X_n)/\partial x_{(i)}$ are continuous over n dimensional real space R^n. If f_n has a jump at 0, then $\partial Q_n(X_n)/\partial x_{(i)}$ are all continuous over R^n for $i \leqq n-1$ and $\partial Q_n(X_n)/\partial x_{(n)}$ is continuous in all its arguments except a jump at $x_{(n)}=0$.

Next equation (2.9) guarantees the existence of an optimal ordering quantity $y(X_{m-1})$ under inventory on-hand stock level X_{m-1}. Here, $L(X_{m-1},y)$ is redefined in the area of $(-\infty,\infty)$. In the literature [3], authors showed that $L(X_{m-1},y)$ is a convex function of y and $y(X_{m-1})$ minimizing $L(X_{m-1},y)$ exists in $(-\infty,\infty)$, that is,

$$L(X_{m-1}, y(X_{m-1})) = \min_y [L(X_{m-1},y)]. \tag{2.9}$$

From the above equation, the value $y(X_{m-1})$ of decision variable y, which satisfies the following equation (2.10), is optimal.

$$\begin{aligned}\frac{\partial L(X_{m-1},y)}{\partial y} = c &+ l_0[(h+p)F_1(x+y)-p+rQ_m(y:X_{m-1})] \\ &+ l_1[(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y:X_{m-2})\{1-F_m(\alpha y)\} \\ &+ r\int_0^{\alpha y} Q_{m-1}(y+x_{(m-1)}-u:X_{m-2})f_m(u)du] = 0 .\end{aligned} \tag{2.10}$$

Further, the following optimal ordering policy which is shown in the equations (2.11) to (2.15) is also obtained by authors [3].

When $x_{(m-1)}=0$, the following policy is optimal; the critical order point, $\bar{x}$, is obtained uniquely as follows:

$$\bar{x} = \begin{cases} \hat{x} ; & \text{if } F_1(\hat{x}) = \dfrac{Pl_0-c}{(h+p)l_0} \text{ and } c<pl_0 . \\ 0 ; & \text{otherwise .}\end{cases} \tag{2.11}$$

Based on $\bar{x}$,

$$\begin{array}{ll} \text{order } y(X_{m-1}) ; & x < \bar{x} . \\ \text{not order ;} & \text{otherwise.}\end{array} \tag{2.12}$$

When $x_{(m-1)}>0$, the critical order point is derived to be $\bar{x}_{(m-1)}(X_{m-2}) \geqq 0$ as follows:

$$\bar{x}_{(m-1)}(X_{m-2}) = \begin{cases} \hat{x}_{(m-1)}(X_{m-2}) ; & \text{if } \hat{x}_{(m-1)}(X_{m-2}) \text{ satisfying (2.14) exists.} \\ 0 ; & \text{otherwise.}\end{cases} \tag{2.13}$$

where $\hat{x}_{(m-1)}(X_{m-2})$ is the solution of the following equation (2.14).

$$(2.14)\qquad c+l_0[hF_1(\Sigma_{i=1}^{m-2} x_{(i)}+\hat{x}_{(m-1)}(X_{m-2}))-p\{1-F_1(\Sigma_{i=1}^{m-2} x_{(i)}+\hat{x}_{(m-1)}(X_{m-2}))\}] +(1-\alpha)rl_1Q_{m-1}(\hat{x}_{(m-1)}(X_{m-2}):X_{m-2}) = 0 .$$

Then,

$$(2.15)\qquad \begin{cases} \text{order } y(X_{m-1}) \;;\; x_{(m-1)} < \bar{x}_{(m-1)}(X_{m-2}) \\ \text{not order} \;;\; \text{otherwise} \end{cases}$$

is optimal.

3. Properties on the optimal ordering policy

First, $Q_m^{(i)}(y:X_{m-1})$ is defined as follows :

$$Q_m^{(i)}(y:X_{m-1}) = \begin{cases} \dfrac{\partial Q_m(y:X_{m-1})}{\partial y} \;;\; i = 1, \\ \dfrac{\partial Q_m(y:X_{m-1})}{\partial x_{(m-i+1)}} \;;\; i = 2, \ldots, m. \end{cases}$$

Next inequality (3.1) which is presented by Nahmias [4] will be used in order to prove the followings :

$$(3.1)\qquad Q_m^{(i)}(y:X_{m-1}) \geqq Q_m^{(i+1)}(y:X_{m-1})$$

This inequality (3.1) and the preceding Lemma 1 together show the following relations.

Proposition 1. If $\alpha=0$ or 1, then

$$(3.2)\qquad -1 \leqq y^{(i)}(X_{m-1}) \leqq y^{(i+1)}(X_{m-1}) \leqq 0 \;;\; i = 2, \ldots, m-2,$$

holds. If $0 < \alpha \leqq \frac{1}{2}$,

$$(3.3)\qquad -2 \leqq y^{(i)}(X_{m-1}) \leqq y^{(i+1)}(X_{m-1}) \leqq 0 \;;\; i = 1, \ldots, m-2,$$

is obtained, where $y^{(i)}(X_{m-1})$ is defined as follows:

$$y^{(i)}(X_{m-1}) = \frac{\partial y(X_{m-1})}{\partial x_{(m-i)}} , \; i = 1, 2, \ldots , m-1.$$

Proof: See Appendix. □

This proposition implies that the optimal ordering quantity is more sensitive to the increase of newer on-hand inventory than that of the older on-hand inventory, and also implies that the increase of on-hand inventory by one unit derives the decrease of order quantity by less than one unit as shown in inequality (3.2), or by less than two units as shown in inequality (3.3).

Proposition 2.

$$(3.4)\qquad \frac{\partial}{\partial l_1}\left(\frac{L(X_{m-1},\ y)}{\partial y}\right)\Bigg|_{y=y\ (X_{m-1})} > 0$$

Proof: Let $y\ (X_{m-1})$ satisfies

$$\frac{\partial L(X_{m-1},\ y)}{\partial y}\Bigg|_{y=y\ (X_{m-1})} = 0,$$

then the following is obtained by the use of equation (2.10).

$$\frac{\partial}{\partial l_1}\left(\frac{\partial L(X_{m-1},\ y)}{\partial y}\right)\Bigg|_{y=y\ (X_{m-1})} = -[(h+p)F_1(x+y\ (X_{m-1}))-p+rQ_m(y\ (X_{m-1}):X_{m-1})]$$

$$+[(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y\ (X_{m-1}):X_{m-2})\{1$$

$$-F_m(\alpha y\ (X_{m-1}))\}+r\int_0^{\alpha y\ (X_{m-1})} Q_{m-1}(y\ (X_{m-1})$$

$$+x_{(m-1)}-u:X_{m-2})f_m(u)du]$$

$$= \frac{1}{l_0}\{c+l_1[(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})\{1-F_m(\alpha y(X_{m-1}))\}$$

$$+r\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f\ (u)du]\}$$

$$+[(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})\{1-F_m(\alpha y(X_{m-1}))\}$$

$$+r\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f_m(u)du]$$

$$= \frac{1}{l_0}\{c+[(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})\{1-F_m(\alpha y$$

$$(X_{m-1}))\}+\int_0^{\alpha y(X_{m-1})} Q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f_m(u)du]\} > 0 \qquad \square$$

It may be interpreted that the increase of the probability of occurring procurement leadtime 1 enlarges the partial derivative of the cost, $L(X_{m-1},y)$, with respect to the optimal ordering quantity. By direct calculation, we have inequality (3.5).

Proposition 3.

$$\frac{\partial L(X_{m-1}, y)}{\partial(1-\alpha)} > 0 \tag{3.5}$$

This proposition indicates that when $(1-\alpha)$, i.e., the rate of excess perishability under the occurrence of leadtime 1, becomes closer to 1, the total expected cost increases.

Lemma 3. ([4])

$$\int_0^y Q_m^{(i)}(u:X_{m-1})du = Q_m(y:X_{m-1}) - \Sigma_{j=1}^{i-1} Q_{m-j}(X_{m-j})H_j(y:\bar{X}_{m-j}),\ i=2,3,\ldots,m. \tag{3.6}$$

where $\bar{X}_{m-j} = (x_{(m-1)}, x_{(m-2)}, \ldots, x_{(m-j+1)})$, and

$$H_1(y) = F_1(y)$$

$$\vdots$$

$$H_j(y:\bar{X}_{m-j}) = \int_0^y F_m(y-v_{(m-1)}) \int_0^{v_{(m-1)}+x_{(m-1)}} f_{m-1}(v_{(m-1)}+x_{(m-1)}-v_{(m-2)})$$

$$\cdots \int_0^{v_{(m-j+2)}+x_{(m-j+2)}} f_{m-j+2}(v_{(m-j+2)}+x_{(m-j+2)}-v_{(m-j+1)})$$

$$f_{m-j+1}(v_{(m-j+1)}+x_{(m-j+1)})dv_{(m-j+1)} \cdots dv_{(m-1)}.$$

Now we define $L^{(i)}(X_{m-1}, y)$ as follows.

$$L^{(i)}(X_{m-1}, y) = \begin{cases} \dfrac{\partial L(X_{m-1}, y)}{\partial x_{(m-i)}} & ;\ i=1,2,\ldots,m-1 \\ \dfrac{\partial L(X_{m-1}, y)}{\partial y} & ;\ i=m \end{cases}$$

Lemma 4.

$$\begin{aligned} L^{(i)}(X_{m-1}, y) = {} & L^{(m)}(X_{m-1}, y) - c - l_0 r \Sigma_{j=1}^{i} Q_{m-j}(X_{m-j})H_j(y:\bar{X}_{m-j}) \\ & + l_1\{(h+p)F_m(x) - p - r[\Sigma_{j=1}^{i-1} Q_{m-j-1}(X_{m-j-1})H_j((1-\alpha)y + x_{(m-1)}:\bar{X}_{m-j-1}) \\ & + \int_0^{x_{(m-1)}} Q_{m-1}^{(i)}(u:X_{m-2})du - \alpha Q_{m-1}(x_{(m-1)} + (1-\alpha)y:X_{m-2}) \\ & + \Sigma_{j=2}^{i} Q_{m-j}(X'_{m-j})H_j(\alpha y:\bar{X}'_{m-j}) + \alpha Q_{m-1}(x_{(m-1)} + (1-\alpha)y:X_{m-2})F_m(\alpha y)]\} \end{aligned} \tag{3.7}$$

where $\check{X}_{m-j}$; rewriting form of X_{m-j} when X_{m-1} is replaced by $X_{m-1}+(1-\alpha)y.e_{m-1}$,

$\check{\bar{X}}_{m-j}$; rewriting form of $\bar{X}_{m-j}$ when X_{m-1} is replaced by $X_{m-1}+(1-\alpha)y.e_{m-1}$.

Proof: For $i=2,3,\ldots,m-1$, the partial differentiation of $L(X_{m-1}, y)$ with respect to $x_{(m-i)}$ is obtained as follows:

$$(3.8)\qquad L^{(i)}(X_{m-1}, y)=l_0[(h+p)F_1(x+y)-p+r\int_0^y Q_m^{(i+1)}(u: X_{m-1})du$$
$$+l_1[(h+p)F_m(x)-p+r\int_{x_{(m-1)}}^{(1-\alpha)y+x_{(m-1)}} Q^{(i)}(u:X_{m-2})du$$
$$+r\int_0^{\alpha y} Q_m^{(i+1)}(u:X_{m-1}+(1-\alpha)y.e_{m-1})du] .$$

And substituting the partial differentiation of $L(X_{m-1}, y)$ with respect to y, i.e., $L^{(m)}(X_{m-1}, y)$ which is already represented by the equation (2.10) into (3.8), the equation (3.8) is rewritten as follows:

$$L^{(i)}(X_{m-1}, y)=L^{(m)}(X_{m-1}, y)-c+l_0r[\int_0^y Q_m^{(i+1)}(u:X_{m-1})\ du-Q_m(y:X_{m-1})]$$
$$+l_1[(h+p)F_m(x)-p+r\int_{x_{(m-1)}}^{(1-\alpha)y+x_{(m-1)}} Q_{m-1}^{(i)}(u:X_{m-2})du$$
$$+r\int_0^{\alpha y} Q_m^{(i+1)}(u:X_{m-1}+(1-\alpha)y.e_{m-1})du$$
$$-(1-\alpha)rQ_{m-1}(x_{(m-1)}+(1-\alpha)y:X_{m-2})\{1-F_m(\alpha y)\}$$
$$-r\int_0^{\alpha y} Q_{m-1}(v+x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(\alpha y-v)dv$$
$$=L^{(m)}(X_{m-1}, y)-c-l_0r\Sigma_{j=1}^{i}Q_{m-j}(X_{m-j})H_j(y:\bar{X}_{m-j})$$
$$+l_1[(h+p)F_m(x)-p-r\{\Sigma_{j=1}^{i-1}Q_{m-j-1}(X_{m-j-1})H_j((1-\alpha)y+x_{(m-1)}:\bar{X}_{m-j-1})$$
$$+\int_0^{x_{(m-1)}} Q_{m-1}^{(i)}(u:X_{m-2})du-\alpha Q_{m-1}(x_{(m-1)}+(1-\alpha)y:X_{m-2})$$
$$+\Sigma_{j=2}^{i}Q_{m-j}(\check{X}_{m-j})H_j(\alpha y: \check{\bar{X}}_{m-j})+\alpha Q_{m-1}(x_{(m-1)}+(1-\alpha)y:X_{m-2})F_m(\alpha y)\}].$$

(by using Lemma 3.)

Lemma 5.

$$L^{(i,m)}(X_{m-1},\ y) \geqq 0,\ i=1,2,\ \ldots,\ m.$$

where $$L^{(i,m)}(X_{m-1},\ y) = \frac{\partial L^{(i)}(X_{m-1},\ y)}{\partial y}$$

Proof: See Appendix. □

Given the fixed ordering cost K, and $y(X_{m-1})$, there exists $s(X_{m-1})$ $(\leqq y(X_{m-1}))$ satisfying the equation (3.10).†

$$L(X_{m-1},s(X_{m-1}))= L(X_{m-1},y(X_{m-1}))+ K \tag{3.10}$$

Here, if $s(X_{m-1}) > 0$, $y(X_{m-1})$ should be ordered and if $s(X_{m-1}) \leqq 0$, the optimal policy is not to order.

Differentiating both sides of the equation (3.10) with respect to $x_{(m-i)}$ and noting

$L^{(m)}(X_{m-1},\ y(X_{m-1}))= 0$ from equation (2.10), the following equation (3.11) is obtained.

$$s^{(i)}(X_{m-1})= \frac{L^{(i)}(X_{m-1},\ y(X_{m-1}))-L^{(i)}(X_{m-1},\ s(X_{m-1}))}{L^{(m)}(X_{m-1},\ s(X_{m-1}))}, \tag{3.11}$$

where $s^{(i)}(X_{m-1})$ is defined as follows:

$$s^{(i)}(X_{m-1}) = \frac{\partial y(X_{m-1})}{\partial x_{(m-i)}},\ i=1,2,\ \ldots,\ m-1.$$

Proposition 4.

$$s^{(i)}(X_{m-1}) < s^{(i+1)}(X_{m-1}) \leqq 0,\ i=1,2,\ \ldots,\ m-1. \tag{3.12}$$

The lower bound for $s^{(1)}(X_{m-1})$ in the following case is given as follows:

$$s^{(1)}(X_{m-1}) \geqq -1 \quad ;\ \alpha = 0,1. \tag{3.13}$$

$$s^{(1)}(X_{m-1}) \geqq -2 \quad ;\ 0 < \alpha \leqq 1/2.$$

Proof: See Appendix. □

This proposition means that the increase of newer on-hand inventory reduces the optimal ordering quantity more than that of the older one. This proposition is very similar to Proposition 1 on $y^{(i)}(X_{m-1})$ and implies that the increase of

† The following discussion with respect to $s(X_{m-1})$, is almost similar to [7].

on-hand inventory by one unit leads to the decrease of $s(X_{m-1})$ by less than one unit for $\alpha=0,1$, and by less than two units for $0 < \alpha \leqq 1/2$.

Proposition 5. For $s(X_{m-1}) \geqq 0$,

(3.14) $$y(X_{m-1})-s(X_{m-1}) \geqq \frac{-K}{c+l_0\{(h+p)F_1(x)-p\}+l_1\{(1-\alpha)rQ_{m-1}(X_{m-1})\}} .$$

Proof: From the convexity of $L(X_{m-1},y)$, it is clear that the following inequality holds.

(3.15) $$\frac{-K}{y(X_{m-1})-s(X_{m-1})} \geqq L^{(m)}(X_{m-1},s(X_{m-1})).$$

Since an order is placed when $s(X_{m-1}) \geqq 0$, the following relationship is established:

(3.16) $$L(X_{m-1},0) > L(X_{m-1},y(X_{m-1}))+K \quad .$$

So an interesting relation between $y(X_{m-1})-s(X_{m-1})$ and $(1-\alpha)$ is derived as follows:

$$y(X_{m-1})-s(X_{m-1}) \geqq \frac{-K}{L^{(m)}(X_{m-1},s(X_{m-1}))} \geqq \frac{-K}{L^{(m)}(X_{m-1},0)}$$

$$= \frac{-K}{c+l_0\{(h+p)F_1(x)-p\}+l_1\{(1-\alpha)rQ_{m-1}(X_{m-1})\}} \quad .$$

The last equality comes from lemma 3. □

Inequality (3.14) clarifies the following relation between $y(X_{m-1})$ and $s(X_{m-1})$ and $(1-\alpha)$;
if the rate of excess perishability, $(1-\alpha)$, increases, the lower bound of $y(X_{m-1})-s(X_{m-1})$ also increases.

4. Numerical example

This section provides an example in order to illustrate the results of Sections 2 and 3. The cost parameters $(c, h, p, r)=(40, 10, 200, 40)$ and the maximum lifetime $m=3$. The demand distribution function is assumed to be gamma with parameters β and γ in equation (4.1).

(4.1) $$f(x) = \frac{\gamma^{\beta}}{\Gamma(\beta)} x^{\beta-1} exp(-\gamma x)$$

Table 1 represents the relations between on-hand inventory and optimal ordering quantity when $\alpha=0.5$ and $l_1=0.4$. Figure 2 illustrates the total expected cost for on-hand inventory $x=0$, 2 and 4 when $\beta=1$ and $\gamma=0.05$. Figure 3 illustrates the relation between the age distribution of inventory on hand x and optimal ordering quantity $y(X_2)$ for $x=4$, i.e., $(x_2, x_1)=(0, 4.0)$, $(1.0, 3.0)$, $(2.0, 2.0)$, $(3.0, 1.0)$, $(4.0, 0)$ when $\beta=1$ and $\gamma=0.05$. Result asserts the increase of newer inventory on hand makes more influences upon the decrease of the optimal ordering quantity than that of older one.

Table 2 represents the relation between the leadtime probability and optimal ordering quantity for $\alpha=0.5$ and $(x_2, x_1)=(2.0, 2.0)$. Figure 4 illustrates the case of $\beta=1$, $\gamma=0.05$ in Tabel 2. As l_1 increases, the total expected cost increases and optimal ordering quantity reduces.

Table 3 represents the relation between the rate of excess perishability $(1-\alpha)$ and the optimal ordering quantity for $l_1=0.4$ and $(x_2, x_1)=(2.0, 2.0)$. Figure 5 illustrates the case of $\beta=1$, $\gamma=0.05$ in Table 3. When $(1-\alpha)$ increases, total expected cost increases and optimal ordering quantity decreases, but the significant differences are not observed when $(1-\alpha)$ approaches to 0.

Table 1. The relation between the age distribution of inventory on hand (x_1, x_2) and optimal ordering quantity $y(X_2)$

x	(x_1, x_2)	$y(X_2)$ $\beta=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	$y(X_2)$ $\beta=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)
0	(0, 0)	21.793	18.855
1	(1, 0)	20.789	17.861
	(0, 1)	20.784	17.839
2	(2, 0)	19.806	16.858
	(1, 1)	19.795	16.846
	(0, 2)	19.790	16.838
3	(3, 0)	18.804	15.884
	(2, 1)	18.800	15.844
	(1, 2)	18.788	15.834
	(0, 3)	18.780	15.826
4	(4, 0)	17.810	14.914
	(3, 1)	17.804	14.873
	(2, 2)	17.804	14.845
	(1, 3)	17.796	14.833
	(0, 4)	17.796	14.812

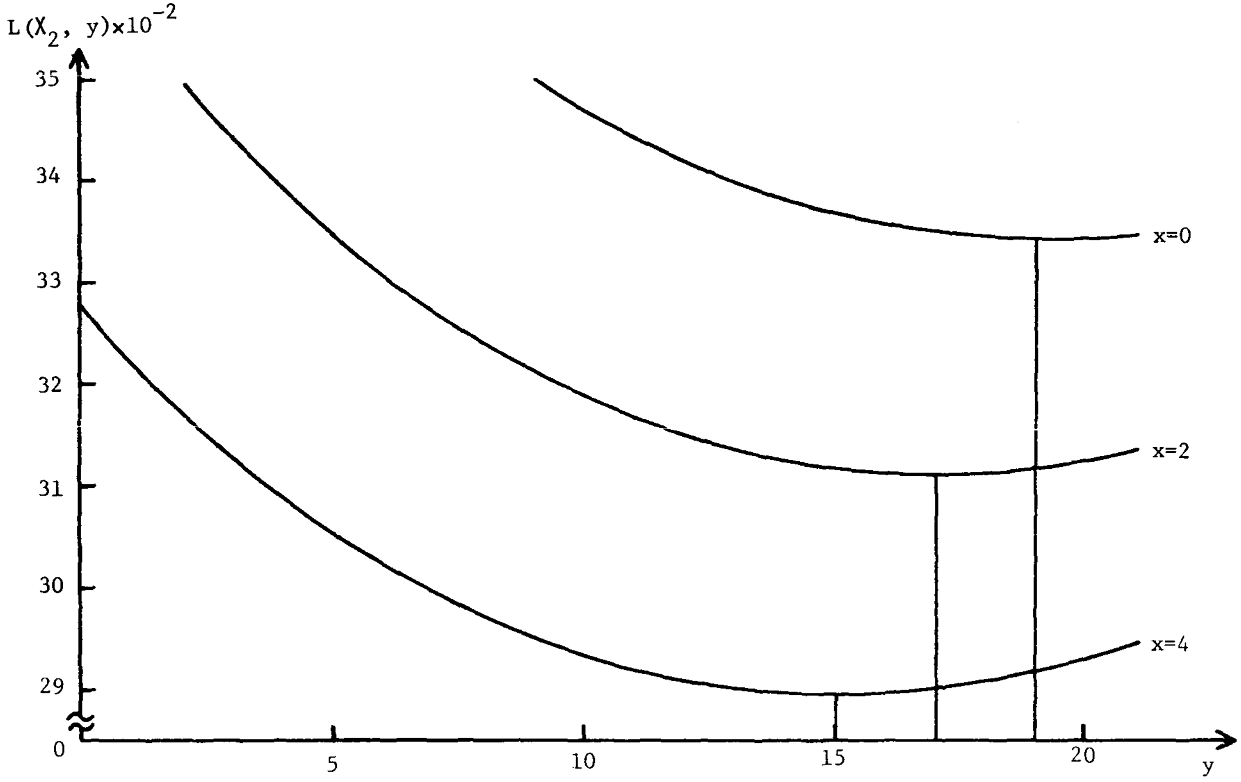

Figure 2. Total expected cost when on-hand inventory x=0, 2, 4

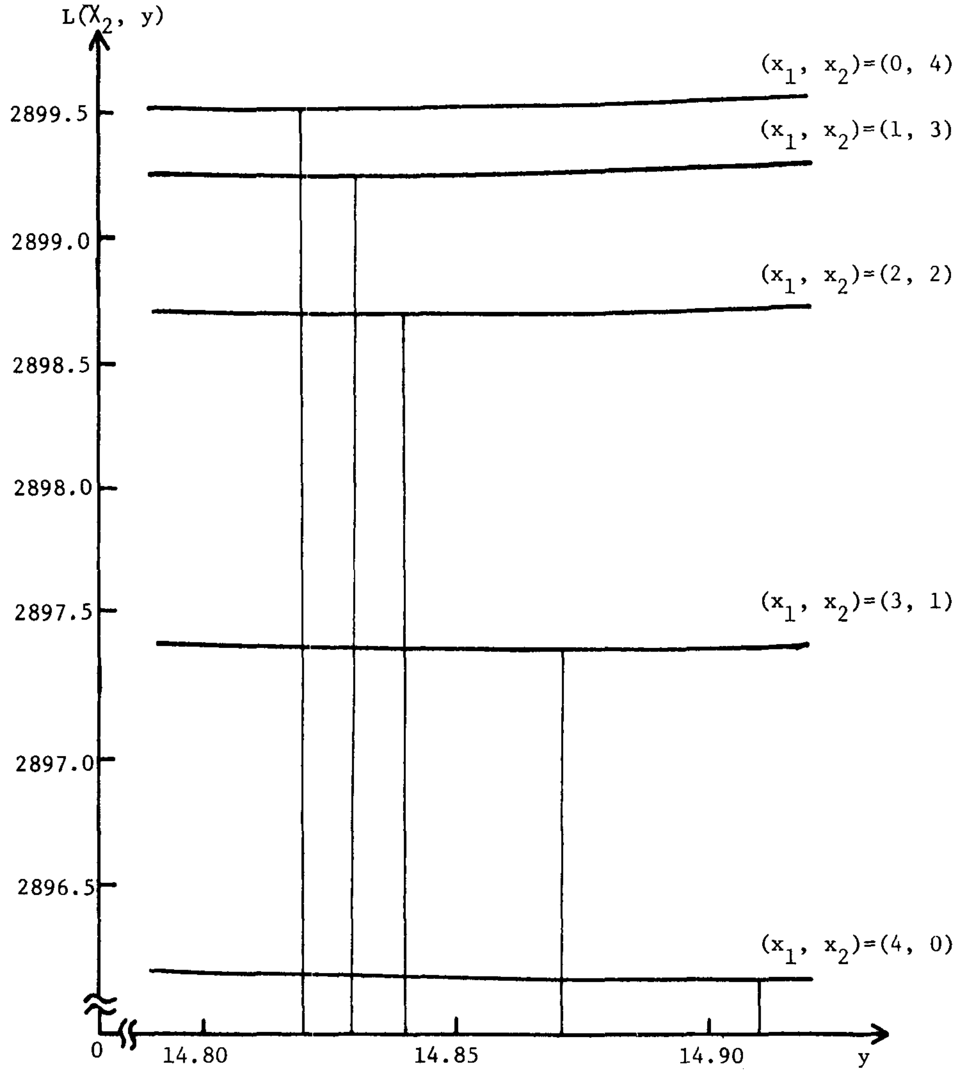

Figure 3. Total expected cost when status of inventory on hand $(x_1, x_2)=(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$

Table 2. The relation between the leadtime probability l_1 and opimal ordering quantity $y(X_2)$

	l_1	0.0	0.2	0.4	0.6
$y(X_2)$	$\beta=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	21.477	19.968	17.806	14.174
	$\beta=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)	22.486	19.228	14.851	8.321

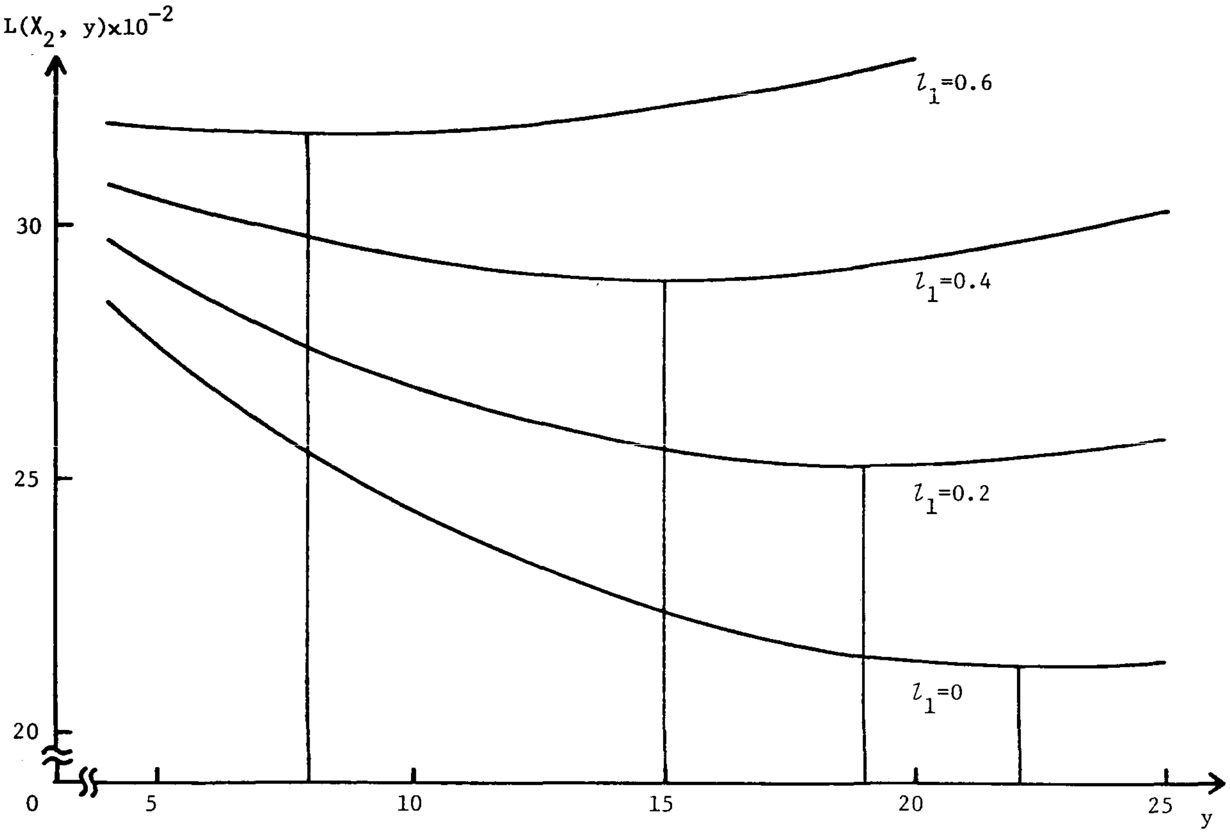

Figure 4. Total expected cost when the leadtime probability l_1=0.0, 0.2, 0.4, 0.6

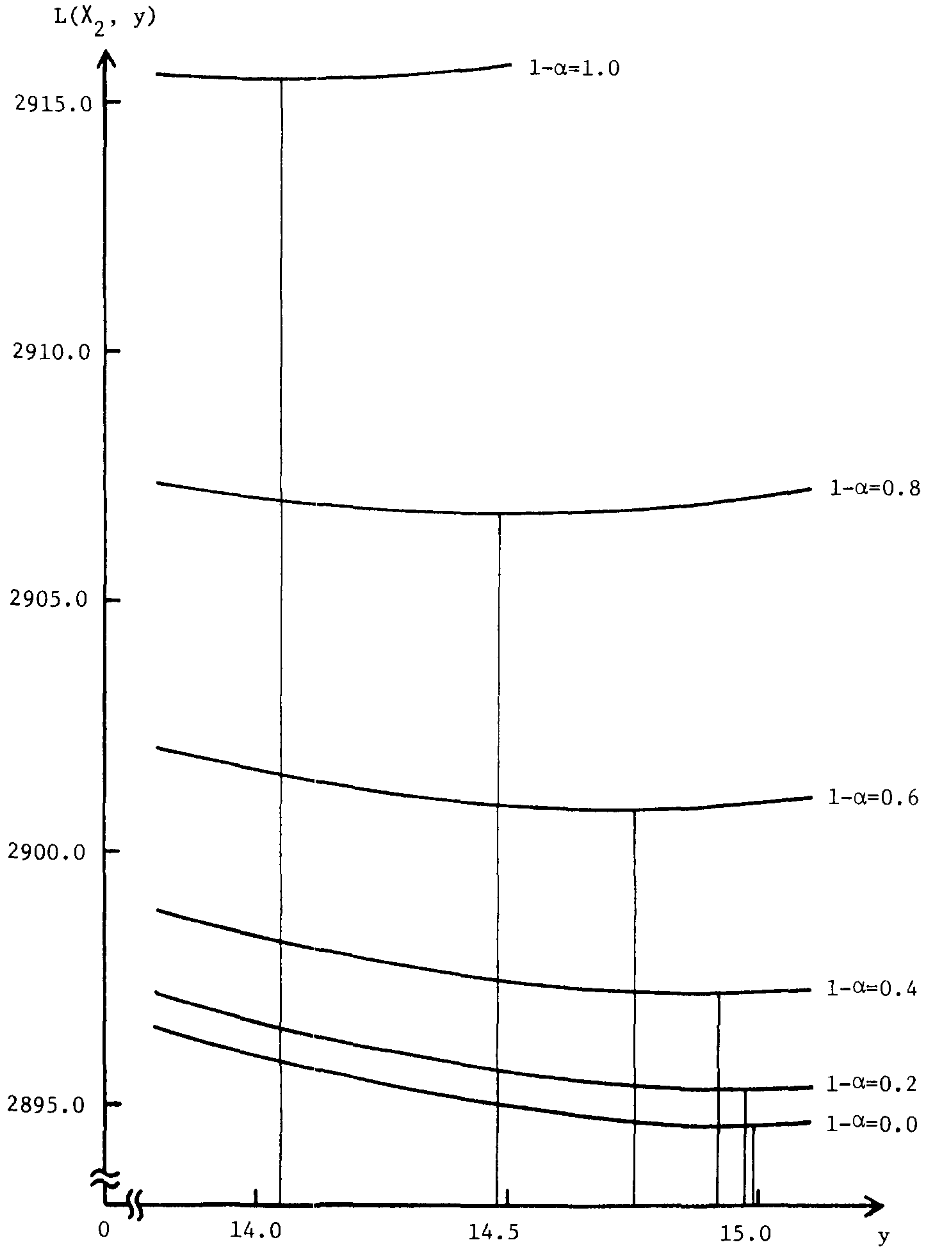

Figure 5. Total expected cost when the rate of excess perishability $1-\alpha$= 1.0, 0.8, 0.6, 0.4, 0.2, 0.0

Table 3. The relation between the rate of excess perishability occurring leadtime 1, (1-α), and optimal ordering quantity $y(X_2)$

$1-\alpha$		1.0	0.8	0.6	0.4	0.2	0.0
$y(X_2)$	$\beta=5$ $\gamma=0.25$ ($\mu=20$ $\sigma=9$)	17.644	17.754	17.802	17.806	17.806	17.806
	$\beta=1$ $\gamma=0.05$ ($\mu=20$ $\sigma=20$)	14.056	14.477	14.765	14.908	14.963	14.993

5. Conclusion

In this paper, we discussed the determination of the optimal ordering policies and their properties with respect to a fixed-life perishable commodity subject ot (*0* or *1*) procurement leadtime.

Sensitivity of some important factors such as the influences of leadtime probability, the rate of excess perishability and the status of inventory on hand upon the optimal ordering policies were analyzed. Next, set-up cost for placing an order was introduced and some characteristics were shown.

It is important to consider with respect to more general assumptions of procurement leadtime and excess perishability, though its generalization may make analysis more complicated and difficult.

Inventory depletion policy discussed in this paper had been assumed to be FIFO issuing. But in order to cope with the increase of customer service in need, customer-oriented inventory depletion policies, e.g., LIFO and etc., might be better than FIFO. From these point of view, we could develop our theme in using LIFO issuing policy for more general stochastic leadtime and stochastic excess perishability.

Acknowledgement

We are deeply indebted to Professor Sennosuke Kuriyama of Osaka Institute of Technology and associate Professor Yoshio Tabata of Osaka University for many useful suggestions.

References

[1] Bulinskaya, E. : Some Results Concerning Optimum Inventory Policies. *Theory of Probability and its Applications*, Vol.9 (1964), 389-403.

[2] Fries, B.: Optimal Ordering Policies for a Perishable Commodity with Fixed Lifetime. *Operations Research*, Vol.23, No.1(1975), 46-61.

[3] Ishii, H., Nose, T., Shiode, S. and Nishida, T.: Perishable Inventory Management subject to Stochastic Leadtime. *(to appear in European Journal of Operational Research)*

[4] Nahmias, S.: Optimal and Approximate Ordering Policies for a Perishable Product subject to Stochastic Demand. *Unpublished Ph. D. Dissertation, Northwestern University*, (1972).

[5] Nahmias, S.: Optimal Ordering Policies for Perishable Inventory-II. *Operations Research*, Vol.23, No.4 (1975), 735-749.

[6] Nahmias, S.: The Fixed-Charge Perishable Inventory Problem. *Operations Research*, Vol.26, No.3 (1978), 464-481.

[7] Nahmias, S. and Pierskalla, W.: Optimal Ordering Policies for a Product that Perishes in Two Periods subject to Stochastic Demand. *Naval Research Logistics Quarterly*, Vol.20 (1973), 207-229.

[8] Van Zyl, G.: Inventory Control for Perishable Commodities. *Unpublished Ph. D. Dissertation, University of North Carolina*, (1964).

Appendix

Proof of Proposition 1.: When $i=2, \ldots, m-1$ and $y(X_{m-1})$ is optimal quantity obtained from the equation (2.10), the partial differentiation of the equation (2.10) with respect to $x_{(m-i)}$ becomes as follows:

$$\text{(A1)} \quad \frac{\partial^2 L(X_{m-1}, y(X_{m-1}))}{\partial y \cdot \partial y_{(m-i)}} = l_0\{(h+p)f_1(x+y(X_{m-1}))(1+y^{(i)}(X_{m-1}))$$

$$+rQ_m^{(i+1)}(y(X_{m-1}):X_{m-1})+rQ_m^{(1)}(y(X_{m-1}):X_{m-1})y^{(i)}(X_{m-1})$$

$$+l_1\{(1-\alpha)r[Q_{m-1}^{(i)}((1-\alpha)y(X_{m-1})+x_{(m-1)}: X_{m-2})$$

$$+q_{m-1}((1-\alpha)y(X_{m-1})+x_{(m-1)}:X_{m-2})y^{(i)}(X_{m-1})][1$$

$$-F_m(\alpha y(X_{m-1}))]-(1-\alpha)\alpha r f_m(\alpha y(X_{m-1}))y^{(i)}(X_{m-1})Q_{m-1}((1$$

$$-\alpha)y(X_{m-1})+x_{(m-1)}:\ X_{m-2})\}+l_1\{r\alpha Q_{m-1}((1-\alpha)y(X_{m-1})$$
$$+x_{(m-1)}:\ X_{m-2})f_m(\alpha y(X_{m-1}))y^{(i)}(X_{m-1})$$
$$+r\int_0^{\alpha y(X_{m-1})}\{q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:\ X_{m-2})y^{(i)}(X_{m-1})f_m(u)$$
$$+Q^{(i)}_{m-1}(y(X_{m-1})+x_{(m-1)}-u:\ X_{m-2})f_m(u)\ du=0\ .$$

Solving the equation (A1) with respect to $y^{(i)}(X_{m-1})$,

(A2) $y^{(i)}(X_{m-1})=-\frac{E}{Z}$,

where

(A3) $$Z=l_0\{(h+p)f_1(x+y(X_{m-1}))+rq_m(y(X_{m-1}):\ X_{m-1})\}$$
$$+l_1\{(1-\alpha)rq_{m-1}((1-\alpha)y(X_{m-1})+x_{(m-1)}:\ X_{m-2})[1-F_m(\alpha y(X_{m-1}))]$$
$$+\alpha^2rf_m(\alpha y(X_{m-1}))Q_{m-1}((1-\alpha)y(X_{m-1})+x_{(m-1)}:\ X_{m-2})$$
$$+r\int_0^{\alpha y(X_{m-1})}q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:\ X_{m-2})f_m(u)du>0\ ,$$

$$E=l_0\{(h+p)f_1(x+y(X_{m-1}))+rQ_m^{(i+1)}(y(X_{m-1}):\ X_{m-1})\}$$
$$+l_1\{(1-\alpha)rQ^{(i)}_{m-1}((1-\alpha)y(X_{m-1})-x_{(m-1)}:\ X_{m-2})[1-F(\alpha y(X_{m-1}))]$$
$$+r\int_0^{\alpha y(X_{m-1})}Q^{(i)}_{m-1}(y(X_{m-1})+x_{(m-1)}-u:\ X_{m-2})f_m(u)du\}>0\ .$$

From (A2), (A3) and (A4), the inequality

(A5) $$y^{(i)}(X_{m-1})\leqq 0$$

is easily obtained.

Besides, using inequality (3.1) for each term of the equations (A3) and (A4), the inequality

(A6) $$y^{(i)}(X_{m-1})\geqq -1$$

is also obtained.

Representing the numerator of $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ by $E_{(i)}$ and $E_{(i+1)}$ respectively and using inequality (3.1), the relation between $y^{(i)}(X_{m-1})$ and $y^{(i+1)}(X_{m-1})$ is obtained as follows:

$$\text{(A7)} \qquad y^{(i)}(X_{m-1})-y^{(i+1)}(X_{m-1}) = -\frac{E_{(i)}-E_{(i+1)}}{Z} \leqq 0$$

where,

$$\begin{aligned}E_{(i)}-E_{(i+1)} = {} & l_0 r\{Q_m^{(i+1)}(y(X_{m-1}):X_{m-1}) - Q_m^{(i+2)}(y(X_{m-1}):X_{m-1})\} \\ & +l_1\{(1-\alpha)r[1-F_m(\alpha y(X_{m-1}))][Q_{m-1}^{(i)}((1-\alpha)y(X_{m-1})+x_{(m-1)}:X_{m-2}) \\ & -Q_{m-1}^{(i+1)}((1-\alpha)y(X_{m-1})+x_{(m-1)}:X_{m-2})] \\ & +r[\int_0^{\alpha y(X_{m-1})} Q_{m-1}^{(i)}(y+x_{(m-1)}-u:X_{m-2})f_m(u)du \\ & -\int_0^{\alpha y(X_{m-1})} Q_{m-1}^{(i+1)}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f_m(u)du]\} \geqq 0\end{aligned}$$

(by inequality (3.1)).

From (A5), (A6) and (A7), (3.2) can be derived.

When $i \doteq 1$, the partial differentiation, $\partial^2 L(X_{m-1}, y(X_{m-1}))/(\partial y.\partial x_{(m-i)})$, of the equation (2.10) with respect to $x_{(m-1)}$, is obtained similarly. Then $y^{(1)}(X_{m-1})$ becomes as follows:

$$\text{(A8)} \qquad y^{(1)}(X_{m-1}) = -\frac{E'}{Z'}$$

where,

$$\begin{aligned}\text{(A9)} \qquad Z' = {} & l_0\{(h+p)f_1(x+y(X_{m-1}))+rq_m(y(X_{m-1}):X_{m-1})\} \\ & +l_1\{(1-\alpha)^2 rq_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})[1-F_m(\alpha y(X_{m-1}))] \\ & +\alpha^2 rQ_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})f_m(\alpha y(X_{m-1})) \\ & +r\int_0^{\alpha y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f_m(u)du\} \geqq 0,\end{aligned}$$

$$\text{(A10)} \qquad E' = l_0\{(h+p)f_1(x+y(X_{m-1}))+rQ_m^{(2)}(y(X_{m-1}):X_{m-1})\}$$

$$+l_1\{(1-\alpha)rq_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})[1-F_m(\alpha y(X_{m-1}))]$$

$$+r\int_0^{\alpha y(X_{m-1})} q_{m-1}(y(X_{m-1})+x_{(m-1)}-u:X_{m-2})f_m(u)du\} \geqq 0 .$$

From (A8), (A9) and (A10), the same inequality as the case for $i=2,3,\ldots,$ is obtained, i.e.,

(A11) $$y^{(i)}(X_{m-1}) \leqq 0 .$$

But the similar relation to (A6) could not be obtained for $i=1$. Only for the case $\alpha=0,1$ and $0<\alpha\leqq 1/2$, (3.2) and (3.3) hold respectively.

Proof of Lemma 5.: For $i=1,2\ldots,m-1$,

(A12) $$\int_0^{\alpha y}\frac{\partial^2}{\partial x_{(m-i)}\partial y}Q_m(u:X_{m-1}+(1-\alpha)y.e_{m-1})du$$

$$=\int_0^{\alpha y}\frac{\partial^2}{\partial x_{(m-i)}\partial y}\int_0^u Q_{m-1}(v+x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(u-v)dvdu$$

(by the equation (2.7))

$$=(1-\alpha)\int_0^{\alpha y}\frac{\partial}{\partial x_{(m-i)}}\int_0^u q_{m-1}(v+x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(u-v)dvdu$$

$$=(1-\alpha)\int_0^{y}\frac{\partial}{\partial x_{(m-i)}}\int_0^{\alpha-t} q_{m-1}(s+x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(t)dsdt$$

$$=(1-\alpha)\int_0^{\alpha y}Q_{m-1}^{(i)}(y+x_{(m-1)}-t:X_{m-2})f_m(t)dt-(1-\alpha)Q_{m-1}^{(i)}(x_{(m-1)}+(1-\alpha)y:X_{m-2})F_m(\alpha y)$$

(setting $t=u-v$, $s=v$).

Differentitating both sides of the equation (3.9) with respect to y,

(A13) $$L^{(i,m)}(X_{m-1},y) = l_0[hf_1(x+y)+pf_1(x+y)+rQ_m^{(i+1)}(y:X_{m-1})]$$

$$+l_1[(1-\alpha)rQ_{m-1}^{(i)}((1-\alpha)y+x_{(m-1)}:X_{m-2})$$

$$+\alpha rQ_m^{(i+1)}(\alpha y:X_{m-1}+(1-\alpha)y.e_{m-1})$$

$$+r\int_0^{\alpha y}\frac{\partial^2}{\partial x_{(m-i)}\partial y}Q_m(u:X_{m-1}+(1-\alpha)y.X_{m-1})du]\quad .$$

Substituting the equation (A12) into the last term of the equation (A13) ,

$$L^{(i,m)}(X_{m-1},\ y)=\ l_0[hf_1(x+y)+pf_1(x+y)+rQ_m^{(i+1)}(y:X_{m-1})]$$
$$+l_1[(1-\alpha)rQ_{m-1}^{(i)}((1-\alpha)y+x_{(m-1)}:X_{m-2})\{1-F_m(\alpha y)\}$$
$$+\alpha rQ_m^{(i+1)}(\alpha y:X_{m-1}+(1-\alpha)y.e_{m-1})$$
$$+(1-\alpha)r\int_0^{\alpha y}Q_{m-1}^{(i)}(y+x_{(m-1)}-t:X_{m-2})f_m(t)dt]\geqq 0\ .$$

For $i=m$, differentiating both sides of the equation (2.10) with respect to y,

$$L^{(m,m)}(X_{m-1},y)=\ l_0[(h-p)f_1(x+y)+rQ_m^{(1)}(y:X_{m-1})]$$
$$+l_1[(1-\alpha)^2rQ_{m-1}^{(1)}(x_{(m-1)}+(1-\alpha)y:X_{m-2})\{1-F_m(\alpha y)\}$$
$$+\alpha^2rQ_{m-1}(x_{(m-1)}+(1-\alpha)y:X_{m-2})f_m(\alpha y)$$
$$+r\int_0^{\alpha y}Q_{m-1}^{(1)}(y+x_{(m-1)}-u:X_{m-2})f_m(u)du]\geqq 0$$

□

Proof of Proposition 4.: Since $y(X_{m-1})>s(X_{m-1})$ from the equation (3.10) and $L^{(i)}(X_{m-1},\ y)$ is increasing function from Lemma 3, the numerator of the equation (3.11) is larger than or equal to 0. Since $L^{(m)}(X_{m-1},\ y)$ is also increasing function from Lemma 5, considering $L^{(m)}(X_{m-1},\ y(X_{m-1}))\ =\ 0$, the denominator of the equation (3.11) is less than or equal to 0. Thus,

(A14) $$s^{(i)}(X_{m-1})\leqq 0$$

holds.

Further the relation between $s^{(i)}(X_{m-1})$ and $s^{(i-1)}(X_{m-1})$ is obtained as follows:

(A15) $$s^{(i)}(X_{m-1})-s^{(i-1)}(X_{m-1})=\frac{A}{L^{(m)}(X_{m-1},\ s(X_{m-1}))}\ ,$$

where

$$A = L^{(i)}(X_{m-1},\ y(X_{m-1})) - L^{(i-1)}(X_{m-1},\ y(X_{m-1})) - L^{(i)}(X_{m-1},\ s(X_{m-1}))$$

$$+L^{(i-1)}(X_{m-1},\ s(X_{m-1}))$$

$$= l_0 rQ_{m-i}(X_{m-i})\{H_i(s(X_{m-1}):\bar{X}_{m-i}) - H_i(y(X_{m-1}):\bar{X}_{m-i})\}$$

$$+l_1 rQ_{m-i}(X_{m-i})\{H_{i-1}((1-\alpha)s(X_{m-1})+x_{(m-1)}:\bar{X}_{m-i})$$

$$-H_{i-1}((1-\alpha)y(X_{m-1})+x_{(m-1)}:\bar{X}_{m-i}))\}$$

$$+l_1 rQ_{m-i}(X_{m-1})\{H_i(\alpha s(X_{m-1}):\bar{X}_{m-i}) - H_i(\alpha y(X_{m-1}):\bar{X}_{m-i})\} < 0$$

(using Lemma 3 and Lemma 4).

Taking $L^{(m)}(X_{m-1},\ s(X_{m-1})) \leqq 0$ into consideration, the following relations are are obtained:

(A16) $$s^{(i-1)}(X_{m-1}) < s^{(i)}(X_{m-1}) \quad .$$

In order to obtain the lower bound of $s^{(i)}(X_{m-1})$, it is sufficient to examine the lower bound of $s^{(1)}(X_{m-1})$.

(A17) $$s^{(1)}(X_{m-1}) = \frac{W}{W}$$

where $$V = -L^{(m)}(X_{m-1}, s(X_{m-1})) - l_0 r\ Q_{m-1}(X_{m-1})\{F_1(y(X_{m-1}) - F\ (s(X_{m-1}))\}$$

$$+l_1 r\alpha\{(1-F_m(\alpha y(X_{m-1}))Q_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})$$

$$-(1-F_m(\alpha s(X_{m-1}))))Q_{m-1}(x_{(m-1)}+(1-\alpha)s(X_{m-1}):X_{m-2})\}\ ,$$

$$W = L^{(m)}(X_{m-1}, s(X_{m-1}))$$

$$= l_0[(p+h)\{F_1(x+s(X_{m-1})) - F_1(x+y(X_{m-1}))\}$$

$$+r\{Q_{m-1}(X_{m-1})F_m(s(X_{m-1})) - Q_{m-1}(X_{m-1})F_m(y(X_{m-1}))\}$$

$$+r\{\int_0^{s(X_{m-1})} Q^{(1)}_{m-1}(v+x_{(m-1)}:X_{m-2})F_m(s(X_{m-1})-v)dv$$

$$-\int_0^{y(X_{m-1})} Q_{m-1}^{(1)}(v+x_{(m-1)}:X_{m-2})F_m(y(X_{m-1})-v)dv\}]$$

$$+l_1[r(1-\alpha)\{Q_{m-1}(x_{(m-1)}+(1-\alpha)s(X_{m-1}):X_{m-2})(1-F_m(\alpha s(X_{m-1})))$$

$$-Q_{m-1}(x_{(m-1)}+(1-\alpha)y(X_{m-1}):X_{m-2})(1-F_m(\alpha y(X_{m-1})))\}].$$

It is easily shown that in equation (A17),

(A18) $\quad s^{(i)}(X_{m-1}) > -1 \quad$, if $\alpha = 0$ or 1,

and

(A19) $\quad s^{(i)}(X_{m-1}) > -2 \quad$, if $\alpha \leqq \frac{1}{2}$.

But for $\frac{1}{2} < \alpha < 1$, the lower bound of $s^{(i)}(X_{m-1})$ could not be obtained. Inequalities (A14), (A16), (A18) and (A19) prove Proposition 4. □

Toyokazu NOSE: Department of Industrial Engineering, Osaka Institute of Technology, 5-16-1, Omiya, Asahi-ku, Osaka, 535, Japan.