Journal of the Operations Research Society of Japan
Vol. 23, No. 4, December 1980

DIFFUSION APPROXIMATION METHOD FOR MULTI-SERVER QUEUEING SYSTEM WITH BALKING

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(Received May 13, 1980)

Abstract This paper deals with the diffusion approximation technique for solving multi-server queueing problems with balking having Erlangian inter-arrival time and Erlangian service time distributions. Probability of joining of a new customer to the system is assumed to vary as $e^{-\gamma y}$ where γ is a positive parameter and y is the queue length. The approximation technique is based on the theory of diffusion, considering only means and variances of arrival and departure processes. Approximate formulas for $P(n)$, probability of finding n customers in the system, and L, mean number of customers in the system, at steady state, are given. Finally, comparisons of approximate and exact or simulated values of mean number L of customers in the system are made for some $E_l/E_k/s\ (\infty)$ systems with balking to show the effectiveness of the approximation technique and graphs of approximate values of L for several systems are drawn which can be used in practice.

1. Introduction

In practical queueing problems, arriving customers do not always join the queue. if it is long, to receive service. That is, a newly arriving customer gets increasingly discouraged to join the queue with the increase of queue size and sometimes balks (refuses to join) if he thinks that he may not receive service within the time he can wait. He tries either to find out another empty service station or one with comparatively shorter queue length, or leaves permanently without receiving service, after balking. Besides this phenomenon, other types of customer impatience such as reneging, jockeying, etc., do also occur in real life queueing problems but we consider here only $E_\ell/E_k/s(\infty)$ queueing problems with balking. Although probability of joining of a new customer to the already existing queue can be considered as a function of queue length in general, it is rather difficult to find out an exact rule which represents the actual balking phenomenon, as this phenomenon depends solely on the thinking of the individual arriving customer. Balking probability function may be assumed to take different forms as have been reported in literatures. Here, we consider, a newly arrived customer joins the queue with probability $e^{-\gamma y}$, where γ is a positive parameter and y is the queue length. A considerable number of literatures have been devoted on the subject, although, most of the earlier works are on single server Markovian models [1],[2],[4],[5],[7] ~ [9]. However, some results on M/M/s(∞)[7],[9],M/G/1(∞) [8], etc.,systems with balking are also available. As far as the authors know, exact results for GI/G/s(∞) system with balking are not known yet. Here, therefore, we proposed a diffusion approximation method for solving $E_\ell/E_k/s(\infty)$ systems with balking, which can be used to solve a wide variety of practical problems falling under this group. This work is an extension of the earlier works by the authors [10],[11].

Assumptions and considerations regarding the applicability of diffusion approximation technique to queueing problems have been discussed in detail in [3],[6],[10]. Those can be applicable in this case too with modifications in mean and variance functions for instanteneous change rate of customer number, which is discussed in the next section. However, an outline of the diffusion approximation technique for multi-server queueing problems without balking [10] is given bellow as it is essential for the development of the approximation method for the present model. The following notations are used in this paper:

λ = mean rate of arrival

μ = mean service rate of each server

σ_a^2 = variance of inter-arrival time

σ_s^2 = variance of service time

$\ell = 1/(\lambda^2 \sigma_a^2) \geqq 1$

$k = 1/(\mu^2 \sigma_s^2) \geqq 1$

s = number of parallel servers

$\alpha = \lambda/\mu$

$\rho = \alpha/s$

$a = 2(1-\rho)/(\rho/\ell + 1/k)$

The diffusion approximation method for $E_\ell/E_k/s(\infty)$ system without balking is as follows:

The steady state diffusion equations for the probability density function (p.d.f.) $f(x)$ of the customer number x in the system are [10],

i) for $x < s$

(1.1) $$0 = -(\lambda - \mu x) f(x) + \frac{1}{2}\frac{d}{dx}\{(\lambda/\ell + \mu x/k) f(x)\}$$

ii) for $x \geqq s$

(1.2) $$0 = -(\lambda - \mu s) f(x) + \frac{1}{2}\frac{d}{dx}\{(\lambda/\ell + \mu s/k) f(x)\}$$

Equations (1.1) and (1.2) can be written in the following general form,

(1.3) $$0 = - F(x) f(x) + \frac{1}{2}\frac{d}{dx}\{D(x) f(x)\}$$

and its solution is

(1.4) $$f(x) = \frac{c}{D(x)}\, e^{2\int^{x}\{F(x)/D(x)\}dx}$$

Hence, the solution of (1.1) is

(1.5) $$f_1(x) = c_1 (x + \alpha k/\ell)^{2\alpha k(k/\ell + 1) - 1}\, e^{-2kx} \equiv c_1 g_1(x)$$

and that of (1.2) is

(1.6) $$f_2(x) = c_2 e^{-ax}$$

The two functions $f_1(x)$ and $f_2(x)$ are continuously connected at $x = s$, i.e., $f_1(s) = f_2(s)$. Therefore, (1.6) becomes

(1.7) $$f_2(x) = c_1 g_1(s)\, e^{-a(x-s)}$$

Here the constant of integration c_1 is determined by the normalization criterion that the integrated value of $f(x)$ in the region $(0,\infty)$ is unity.

Once the probability density functions $f_1(x)$ and $f_2(x)$ are known, various other system parameters may then be calculated. In reference [3], the steady state probability of finding n customers in the system is calculated first by discretizing $f(x)$ and other system parameters are calculated thereafter. In reference [10], the approximate formula for queue length is given directly.

2. Assumptions and Approximate Solution Method

We assume that customers arrive from an infinite source to a multi-server service station, with inter-arrival time having Erlangian distribution, at a mean rate of λ. The system capacity is considered as infinite. If the queue length is zero, then the arriving customer immediately enters the system. But, if the arriving customer finds a queue, he decides whether to join or to leave. Probability of joining in that case is

$$P(y) = e^{-\gamma y} = \beta^{y} = \beta^{(x-s)} \tag{2.1}$$

where β is a positive parameter less than 1. So, with probability

$$R(y) = 1 - P(y) \tag{2.2}$$

the arriving customer refuses to join the queue and thus becomes a lost customer. Once the arrived customer joins the queue, he is not allowed to leave the system until the service completes, i.e., the reneging probability is considered as zero here. Queue forms in one line and service is on 'first come - first serve' basis with no interruption or failure in the service mechanism when customers are present in the system.

Fundamental considerations and boundary conditions (a reflecting boundary at the origin and also that $f(\infty) = 0$, $df(\infty)/dx = 0$) for the diffusion equation, as those in the case with GI/G/s(∞) system [3],[10], remain the same in this model too. The system model is as follows.

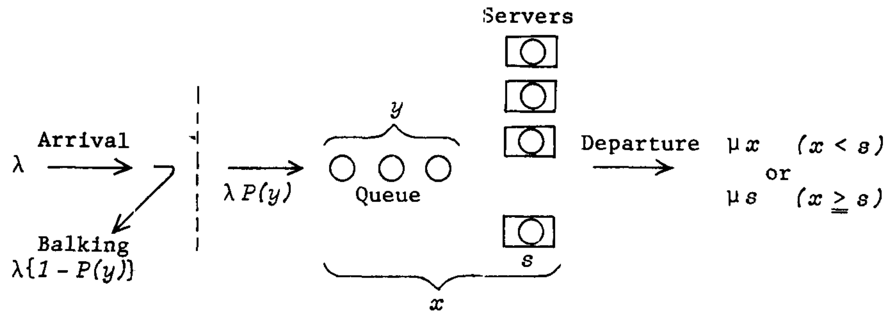


Fig.1 The system model

i) When $x < s$:

The system remains partially loaded and there is no queue formation. So, the steady state diffusion equation for the p.d.f. $f(x)$ of x, the number of customers in the system, remains the same as (1.1) for this model too, as there is no balking. Its solution is given in (1.5).

ii) When $x \geqq s$:

Queue forms because the system becomes fully loaded. So, balking of customers occurs according to the probability rule of (2.2) . Therefore, arriving customers join the queue with the modified arrival rate of $\lambda P(y)$, whereas the mean service rate of μs remains the same as that in the case without balking. Hence, the mean of instanteneous change rate of x , the number of customers in the system, is

$$(2.3) \qquad F(x) = \lambda P(y) - \mu s = \lambda e^{-\gamma(x-s)} - \mu s$$

and the variance of that is calculated as

$$(2.4) \qquad D(x) = (\lambda/\ell) P^2(y) + \lambda P(y)\{1 - P(y)\} + \mu s/k = \lambda(1/\ell - 1) e^{-2\gamma(x-s)} + \lambda e^{-\gamma(x-s)} + \mu s/k$$

which is derived from the following considerations.

Let us consider that N_t customers arrive in the interval of time t, where t is assumed as sufficiently large. In Chapter 4 of the reference [6] , it is derived that N_t is approximately normally distributed with

$$E[N_t] \simeq \lambda t \qquad \text{and} \qquad Var[N_t] \simeq \lambda t(\lambda^2 \sigma_a^2)$$

where $(\lambda^2 \sigma_a^2) = 1/\ell$ by our definition. So, N_t can be expressed as

$$(2.5) \qquad N_t \simeq \lambda t(1 + \frac{\xi}{\sqrt{\lambda t}})$$

with

$$E[\xi] = 0 \qquad \text{and} \qquad Var[\xi] = 1/\ell$$

Next, let us consider that out of total arrival of N_t customers only N_t' ones join the queue and the rest balk. On the assumption that queue length y remains constant during the arrival process, N_t' is binomially distributed with

$$E[N_t'] = N_t P(y) \quad \text{and} \quad Var[N_t'] = N_t P(y)\{1 - P(y)\}$$

when N_t is considered as constant. N_t' is also approximately normal. So, N_t' can be expressed as

$$(2.6) \qquad N_t' \simeq N_t\{P(y) + \frac{\eta}{\sqrt{N_t}}\} \simeq N_t\{P(y) + \frac{\eta}{\sqrt{\lambda t}}\}$$

with

$$E[\eta] = 0 \qquad \text{and} \qquad Var[\eta] = P(y)\{1 - P(y)\}$$

From (2.5) and (2.6), we get

$$N_t' \simeq \lambda t\,\{1+\xi/\sqrt{\lambda t}\}\{P(y)+\eta/\sqrt{\lambda t}\} \qquad (2.7)$$
$$\simeq \lambda t\,\{P(y)+\xi P(y)/\sqrt{\lambda t}+\eta/\sqrt{\lambda t}\}$$

Therefore, the variance of N_t' can be calculated as

$$Var[N_t'] \simeq Var[\lambda t\,\{P(y)+\xi P(y)/\sqrt{\lambda t}+\eta/\sqrt{\lambda t}\}] \qquad (2.8)$$
$$= Var[\sqrt{\lambda t}\;P(y)\,\xi+\sqrt{\lambda t}\;\eta]$$
$$= \lambda t\;P^2(y)\;Var[\xi]+\lambda t\;Var[\eta]$$
$$= \lambda t[P^2(y)/\ell+P(y)\{1-P(y)\}]$$

Finally, the variance of instanteneous change rate in the number of customers who join the queue is calculated as

$$\lim_{t\to\infty}\frac{Var[N_t']}{t} = \lambda P^2(y)/\ell+\lambda P(y)\{1-P(y)\} \qquad (2.9)$$

Since the variance of instanteneous change rate in x is expressed as the summation of variance of input and that of output processes, we get $D(x)$ of (2.4) by adding variance of departure rate $\mu s/k$ with that for joining process from (2.9) [10].

Using (2.3) and (2.4), the solution (1.4) becomes

$$f_2(x) = \{\frac{c}{D(x)}\}\,e^{2\int^x \{F(x)/D(x)\}dx} \qquad (2.10)$$

$$= \frac{const.}{(Az^2+Bz+C)}\;e^{-\frac{2}{\gamma}\left[\int^z \frac{B}{Az^2+Bz+C}\,dz\;-\;\int^z \frac{1}{z(Az^2+Bz+C)}\,dz\right]}$$

where z is substituted for $P(y)$, i.e., $z=P(y)=e^{-\gamma y}=e^{-\gamma(x-s)}$ and

$$A=\rho(1/\ell-1),\quad B=\rho,\quad C=1/k$$

Depending on the values of ℓ and k, $f_2(x)$ of (2.10) takes the following different forms which are obtained from (2.10) after simplification.

a) For $\ell>1, k\neq\infty$ or $(A<0,\ C>0)$, e.g., for D/M/s, $E_2/E_2/s$, $E_2/M/s$, $D/E_2/s$ systems with balking

$$f_2(x) = const.\;(Az^2+Bz+C)^{-1-\frac{1}{\gamma C}}\left(\left|\frac{2Az+B-H}{2Az+B+H}\right|\right)^{-\frac{2B}{\gamma H}(1+\frac{1}{2C})}\;z^{\frac{2}{\gamma C}} \qquad (2.11)$$

where $H\equiv\sqrt{B^2-4AC}$. Moreover, $Az^2+Bz+C\geq 0$ because this is the variance term which is always positive.

b) For $\ell > 1$, $k = \infty$ or $(A < 0, C = 0)$, e.g., for $E_2/D/s$ system with balking

$$(2.12) \qquad f_2(x) = const.\ (Az+B)^{-1+\frac{2}{\gamma}+\frac{2A}{\gamma B^2}}\ z^{-1-\frac{2}{\gamma}-\frac{2A}{\gamma B^2}}\ e^{-\frac{2}{\gamma B z}}$$

where $Az+B \geqq 0$ is considered.

c) For $\ell = 1$, $k \neq \infty$ or $(A = 0, C > 0)$, e.g., for $M/M/s$, $M/E_2/s$ systems with balking

$$(2.13) \qquad f_2(x) = const.\ (Bz+C)^{-1-\frac{2}{\gamma}-\frac{2}{\gamma C}}\ z^{\frac{2}{\gamma C}}$$

where $Bz+C \geqq 0$ is considered.

d) For $\ell = 1$, $k = \infty$ or $(A = 0, C = 0)$, e.g., for $M/D/s$ system with balking

$$(2.14) \qquad f_2(x) = const.\ z^{-1-\frac{2}{\gamma}}\ e^{-\frac{2}{\gamma B z}}$$

The above equations (2.11) ~ (2.14) can be written in the following general form

$$(2.15) \qquad f_2(x) = const.\ g_2(x)$$

where $g_2(x)$ can be obtained by the substitution $z = e^{-\gamma(x-s)}$ in the right hand sides of the above equations (2.11) ~ (2.14).

We consider that the two functions $f_1(x)$ and $f_2(x)$ are continuously connected at $x = s$, i.e., $f_1(s) = f_2(s)$, like that in the case with $GI/G/s(\infty)$ system without balking [10]. Therefore, $f_2(x)$ of (2.15) becomes

$$(2.16) \qquad f_2(x) = c_1\{\frac{g_1(s)}{g_2(s)}\}\, g_2(x)$$

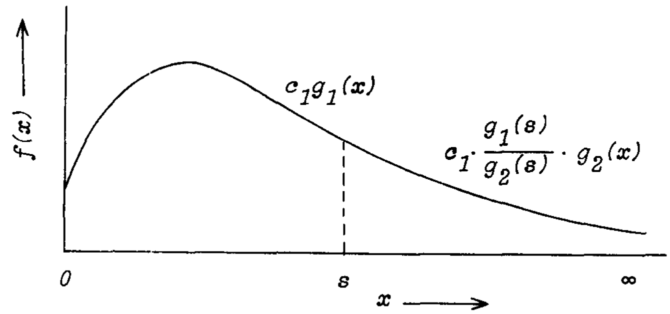


Fig. 2 Continuity of functions

The constant of integration c_1 of (1.5) and (2.16) is determined by the normalization criterion, i.e.,

$$1 = \int_0^s c_1\, g_1(x)\, dx + \int_s^\infty c_1 \{ \frac{g_1(s)}{g_2(s)} \} g_2(x)\, dx \tag{2.17}$$

Probability of finding n customers in the system in the steady state condition can be obtained by discretizing $f(x)$ as follows

$$P(n) = \int_{n-.5}^{n+.5} f(x)\, dx \qquad (n \geq 1) \tag{2.18}$$

taking into consideration that the probability mass is centered at $x = n$ [3]. Therefore, using (1.5), (2.15) ~ (2.18), we get the approximate formula for the mean number of customers in the system as

$$L_1 = \sum_{n=1}^{\infty} n\, P(n) = \sum_{n=1}^{\infty} n \int_{n-.5}^{n+.5} f(x)\, dx \tag{2.19}$$

$$= \frac{\sum_{n=1}^{s-1} n \int_{n-.5}^{n+.5} \frac{g_1(x)}{g_1(s)}\, dx + s \left[\int_{s-.5}^{s} \frac{g_1(x)}{g_1(s)}\, dx + \int_{s}^{s+.5} \frac{g_2(x)}{g_2(s)}\, dx \right] + \sum_{n=s+1}^{\infty} n \int_{n-.5}^{n+.5} \frac{g_2(x)}{g_2(s)}\, dx}{\int_0^s \frac{g_1(x)}{g_1(s)}\, dx + \int_s^\infty \frac{g_2(x)}{g_2(s)}\, dx}$$

The discretization as has been made in (2.18) can not be applied for $n = 0$, if a reflecting boundary at origin is assumed by the reason that the continuous customer number x should be positive as described in references [3], [6] and [10]. It seems, however, that such correspondance as

$$P(0) = \int_{-.5}^{.5} f(x)\, dx$$

is natural and moreover it gives better approximate results for smaller values of ρ (traffic intensity). Hence, we propose that the reflecting boundary may be shifted to -0.5 to the left of origin. For higher values of ρ, this shift of boundary has got almost no effect as the p.d.f. $f_1(x)$ takes very negligible value in that case in the region left of origin. On the other hand, for smaller values of ρ, this shift of boundary has remarkable effect as $f_1(x)$ takes appreciable real values in this region.

For practical calculation, we take the lower boundary at x_0, where $x_0 = \max(-\alpha k/\ell, -0.5)$, because $f_1(x)$ of (1.5) does not take rational values beyond $-\alpha k/\ell$ and it happens that $-0.5 < -\alpha k/\ell$, in some cases for some combination of α, ℓ, k values.

Hence, the modified approximate formula L_2 for the mean number of customers in the system is obtained from (2.19) by replacing zero, the lower limit of integration, of the first term in the denominator by x_0 as defined above.

3. Numerical Examples

The approximation method, as proposed in this paper, has been tested for eleven different $E_\ell/E_k/s$ systems with balking. Numerical calculations for L_1 and L_2 values for all of the systems are made with

$$s = 1,\ 2,\ 5,\ 10$$
$$\rho = 0.3,\ 0.5,\ 0.7,\ 1.0,\ 1.5,\ 2.0$$
$$\beta = e^{-\gamma} = 0.1,\ 0.3,\ 0.5,\ 0.7\ 0.9$$

and the calculated approximate values are compared with exact or simulated values. Some of them are shown in Tables 1 ~ 4. Only in Table 1, values of both L_1 and L_2 are compared with exact values obtained by solving system state equations for M/M/s system with balking [7] . From this Table, it is seen that L_2 gives better approximation especially for smaller values of ρ , although for higher values of ρ the approximate values are not much different from those obtained by using (2.19). In Table 2 ~ 4, comparisons of simulation results and only those of L_2 are shown. In the Tables 'Sim' stands for simulation results.

Numerical integration of (2.19) is done by using Gauss's integration formula which is available in FACOM M200 computer. Overflow in the calculation of (2.19) can generally be avoided if (2.19) is written in that particular form as the values of the probability density functions are not so large. However, in the calculation specially with higher values of k (~10), we need to rewrite the expressions for $\{ g_1(x)/g_1(s) \}$ and $\{ g_2(x)/g_2(s) \}$ of (2.19), by taking their natural logarithms and then making them as exponents with exponential bases to get the original values, so as to avoid overflows. Calculation of the last term in the numerator of (2.19) can be made as the series converges. Calculation of the second term in the denominator of (2.19) can be made by the substitution $z = e^{-\gamma(x-s)}$ or $x = s - (1/\gamma) \log_e(z)$, which changes the interval of integration from ($s \sim \infty$) to ($1 \sim 0$). The calculation time for the approximate values is extremely short in comparison with that for simulation values.

Table 5 shows the upper bounds of percentage absolute errors of L_2 with exact or simulated values for all of the eleven systems considered here. Each upper bound is the maximum value among all the percentage absolute error values for all values of ρ and β , which are listed above, for a particular value of s .

In Figures 3 ~ 10, only the approximate values of the mean number L_2 of customers in the system for eight different $E_\ell/E_k/s$ systems with balking are plotted by using X - Y plotter of the computer.

Table 1. Mean number of customers in system in M/M/s(∞) system with balking

ρ	Type of result	s = 2			s = 5		
		β = 0.1	0.5	0.9	0.1	0.5	0.9
0.3	Exact	0.615	0.628	0.650	1.499	1.502	1.507
	L_2	0.639	0.660	0.687	1.539	1.545	1.552
	L_1	0.824	0.849	0.882	1.652	1.658	1.665
0.7	Exact	1.364	1.519	1.982	3.282	3.409	3.784
	L_2	1.380	1.562	2.044	3.267	3.418	3.808
	L_1	1.494	1.682	2.178	3.299	3.450	3.841
1.0	Exact	1.779	2.107	3.500	4.177	4.504	5.891
	L_2	1.771	2.134	3.544	4.127	4.493	5.902
	L_1	1.848	2.212	3.625	4.140	4.505	5.913
1.5	Exact	2.235	2.848	6.466	4.995	5.635	9.356
	L_2	2.203	2.854	6.480	4.925	5.612	9.358
	L_1	2.245	2.891	6.500	4.928	5.615	9.359
2.0	Exact	2.514	3.363	9.089	5.404	6.280	12.076
	L_2	2.475	3.359	9.092	5.337	6.257	12.076
	L_1	2.501	3.378	9.096	5.338	6.258	12.076

Table 2 . Mean number of customers in system in $E_2/E_2/s(\infty)$ system with balking

ρ	Type of result	s = 2			s = 5		
		β = 0.1	0.5	0.9	0.1	0.5	0.9
0.3	Sim	0.611	0.613	0.617	1.491	1.491	1.503
	L_2	0.587	0.591	0.595	1.502	1.502	1.503
0.7	Sim	1.429	1.515	1.777	3.420	3.435	3.652
	L_2	1.339	1.448	1.678	3.367	3.445	3.606
1.0	Sim	1.892	2.159	3.271	4.418	4.672	5.757
	L_2	1.744	2.025	3.058	4.292	4.572	5.602
1.5	Sim	2.369	2.914	6.411	5.204	5.801	9.205
	L_2	2.167	2.748	6.177	5.030	5.636	9.142
2.0	Sim	2.625	3.402	8.951	5.562	6.372	11.960
	L_2	2.416	3.240	8.889	5.363	6.207	11.888

Table 3. Mean number of customers in system in $M/D/s(\infty)$ system with balking

ρ	Type of result	s = 2			s = 5		
		β = 0.1	0.5	0.9	0.1	0.5	0.9
0.3	Sim	0.617	0.613	0.631	1.500	1.490	1.509
	L_2	0.637	0.638	0.638	1.527	1.527	1.527
0.7	Sim	1.373	1.469	1.751	3.326	3.423	3.644
	L_2	1.341	1.455	1.654	3.381	3.455	3.587
1.0	Sim	1.793	2.067	3.134	4.215	4.544	5.641
	L_2	1.709	2.011	3.037	4.258	4.556	5.580
1.5	Sim	2.270	2.752	6.226	5.078	5.657	9.074
	L_2	2.115	2.700	6.142	4.960	5.574	9.101
2.0	Sim	2.535	3.267	8.811	5.475	6.241	11.812
	L_2	2.363	3.171	8.829	5.293	6.128	11.826

Table 4. Mean number of customers in system in $D/M/s(\infty)$ system with balking

ρ	Type of result	s = 2			s = 5		
		β = 0.1	0.5	0.9	0.1	0.5	0.9
0.3	Sim	0.602	0.599	0.601	1.500	1.497	1.499
	L_2	0.543	0.551	0.560	1.504	1.505	1.507
0.7	Sim	1.470	1.529	1.692	3.484	3.522	3.627
	L_2	1.363	1.470	1.722	3.364	3.445	3.630
1.0	Sim	1.994	2.204	3.283	4.593	4.764	5.812
	L_2	1.797	2.060	3.084	4.336	4.599	5.624
1.5	Sim	2.491	3.025	6.514	5.361	5.961	9.559
	L_2	2.262	2.809	6.210	5.142	5.710	9.182
2.0	Sim	2.739	3.532	9.237	5.697	6.487	12.097
	L_2	2.532	3.318	8.950	5.494	6.295	11.949

Table 5. Upper bounds of percentage absolute errors for different systems for L_2

Type of result	s = 2	s = 5	s = 10
$M/M/s$	6	3	1
$M/E_2/s$	5	4	2
$E_2/M/s$	6	4	2
$E_2/E_2/s$	9	4	2
$M/D/s$	7	4	3
$D/M/s$	10	6	3
$E_2/D/s$	13	6	3
$D/E_2/s$	13	6	3
$E_{10}/E_{10}/s$	15	7	3
$E_{10}/D/s$	20	9	5
$D/E_{10}/s$	17	7	4

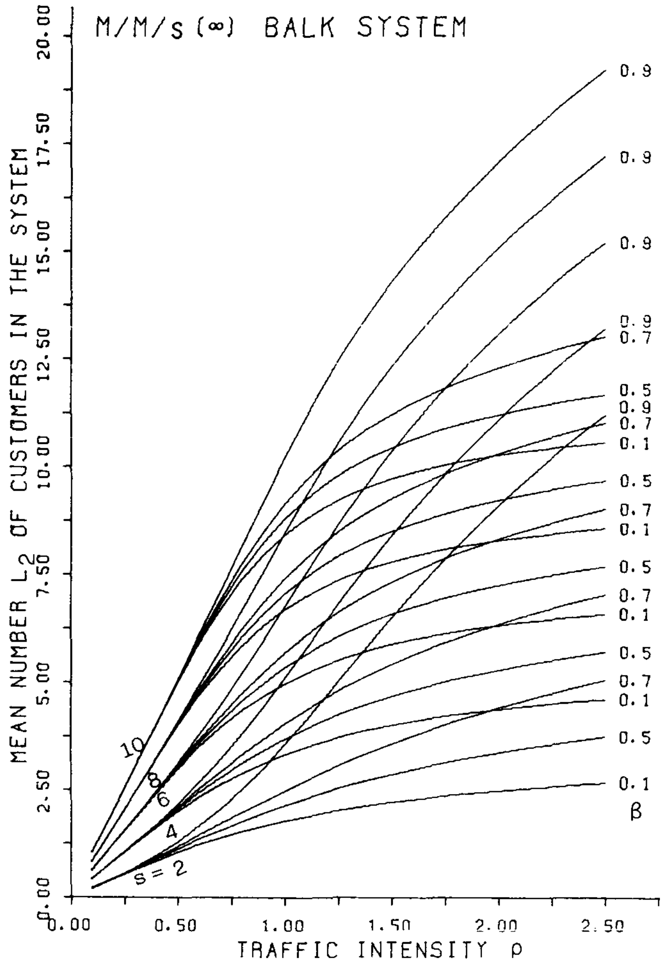
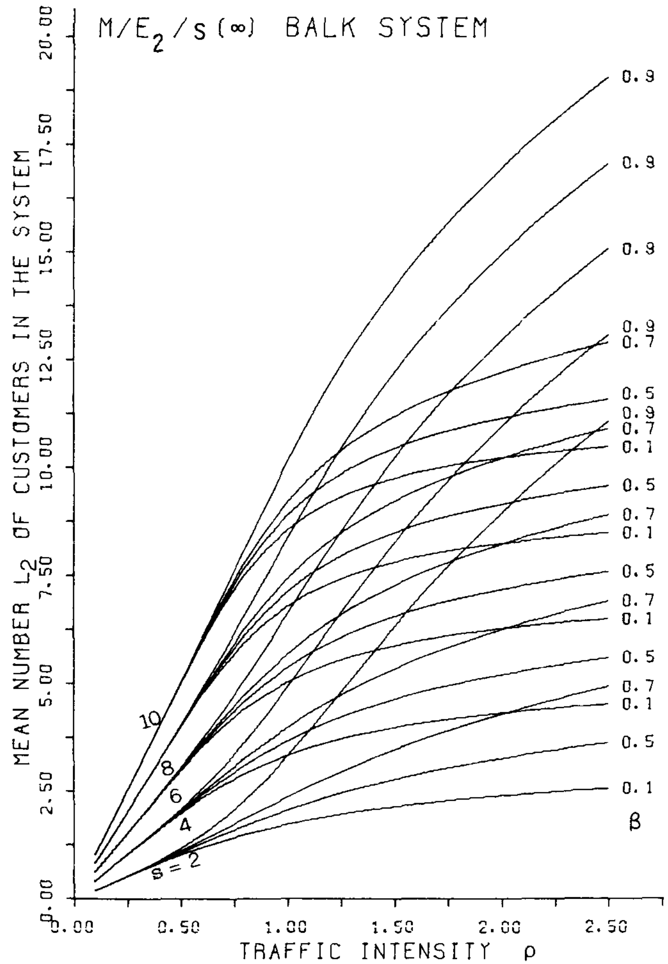

M/M/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
s = 2
4
6
8
10
β
0.9
0.7
0.5
0.1
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00

Fig. 3


M/E2/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
s = 2
4
6
8
10
β
0.9
0.7
0.5
0.1
0.00
0.50
1.00
1.50
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2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00

Fig. 4

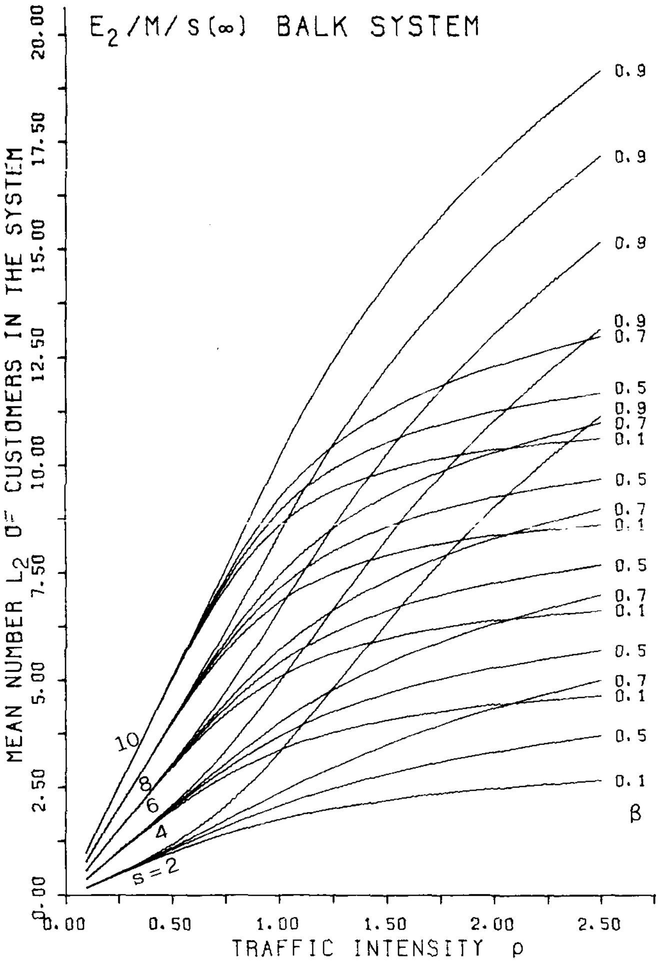

E2/M/s(∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
s=2
4
6
8
10
β
0.9
0.7
0.5
0.1

Fig. 5

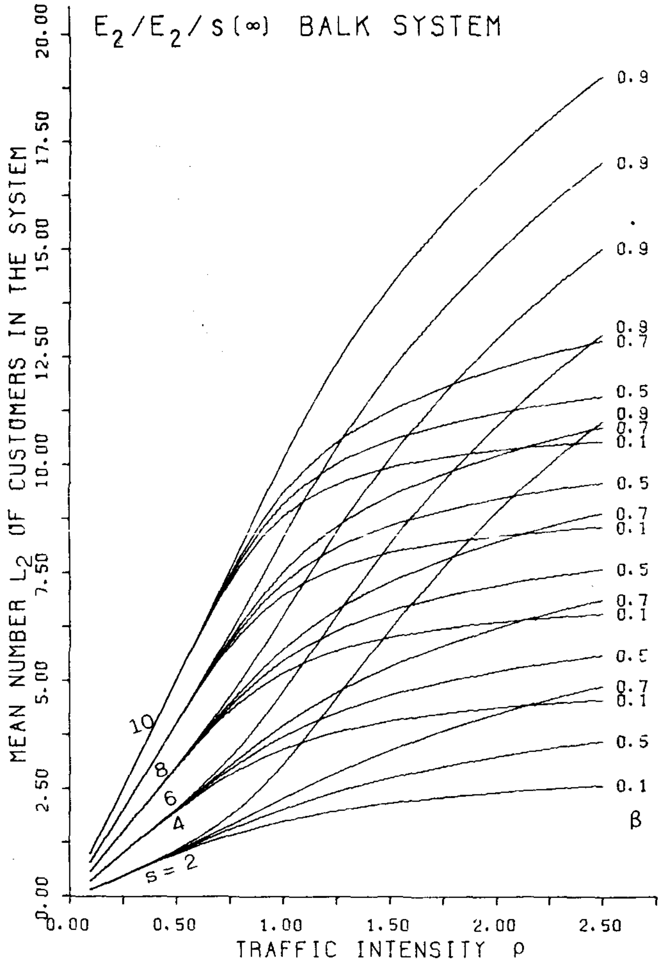

E2/E2/s(∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
s=2
4
6
8
10
β
0.9
0.7
0.5
0.1

Fig. 6

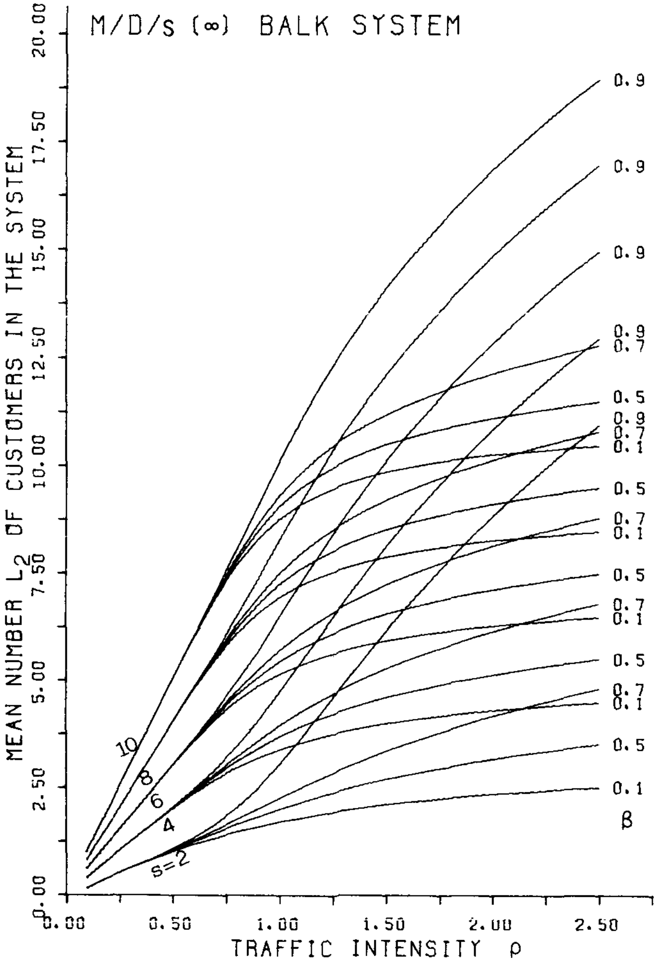
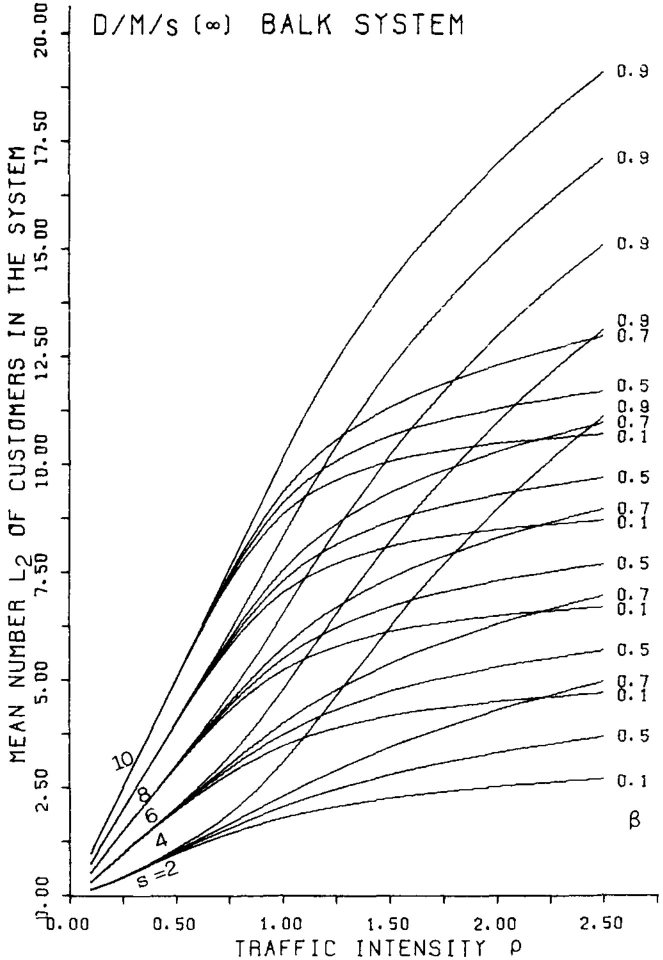

M/D/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
10
8
6
4
s=2
0.9
0.7
0.5
0.1
β

Fig. 7


D/M/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
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0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
10
8
6
4
s=2
0.9
0.7
0.5
0.1
β

Fig. 8

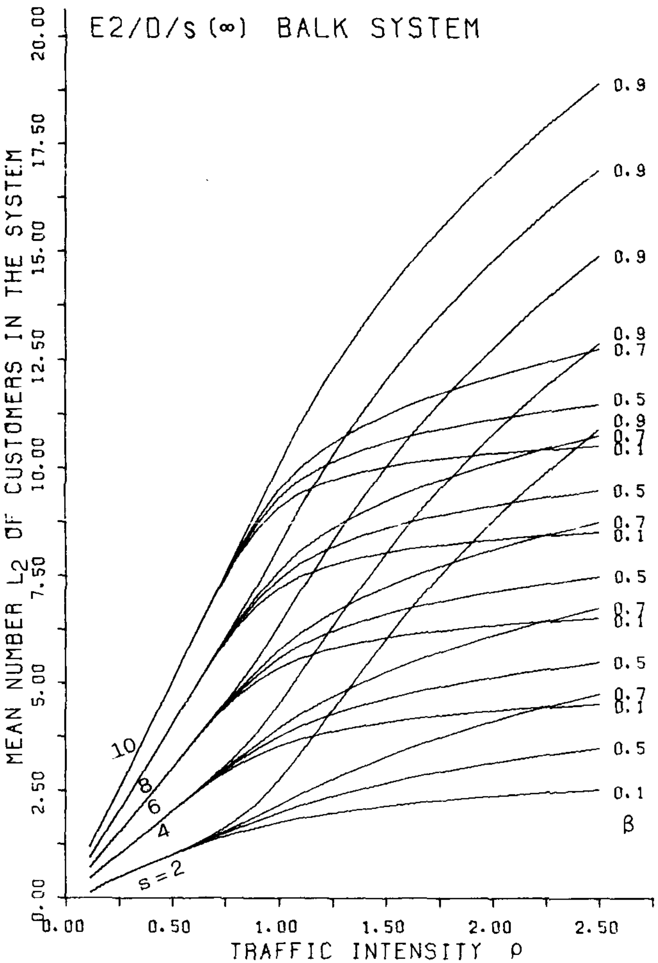
E2/D/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
s=2
4
6
8
10
β
0.9
0.7
0.5
0.1

Fig. 9

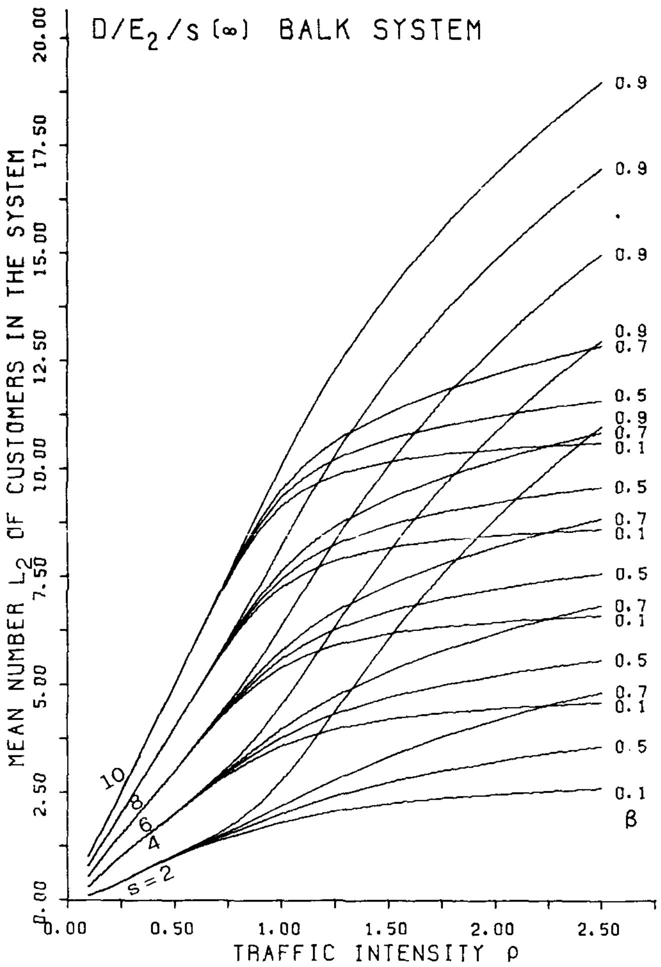
D/E2/s (∞) BALK SYSTEM
MEAN NUMBER L2 OF CUSTOMERS IN THE SYSTEM
TRAFFIC INTENSITY ρ
0.00
0.50
1.00
1.50
2.00
2.50
2.50
5.00
7.50
10.00
12.50
15.00
17.50
20.00
s=2
4
6
8
10
β
0.9
0.7
0.5
0.1

Fig. 10

4. Conclusion

In this paper, an approximation method for the solution of multi-server queueing problems with balking using diffusion theory has been shown. Effectiveness of the proposed method for $E_\ell/E_k/s(\infty)$ system with balking can be evaluated from the values of percentage absolute errors which are tabulated in Table 5. This method gives good approximations for higher values of $1/\ell$, $1/k$, s and β. For smaller values of ρ, the approximate formula L_2 gives good results although (2.19) does not do so, which is clear from Table 1.

For $s = 1$ also, good idea about the system performance can be made although errors become comparatively larger. This observation is made from the calculated values, obtained by using the proposed formulas, which are not shown in the Tables. It may be said that the proposed formula L_2 is applicable to the practical cases of $E_\ell/E_k/s(\infty)$ systems with balking for $s \geqq 2$ and the graphs of L_2 can be used in practical field.

This approximation method for $E_\ell/E_k/s(\infty)$ system with balking may possibly be extended to cover more general cases, in a way similar to that for $GI/G/s(\infty)$ system without balking, as shown in reference [3].

Acknowledgement

We are grateful to the referees for their valuable comments and suggestions to improve this paper.

We should also say thanks for the help of Management Science Development Foundation.

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