Journal of the Operations Research
Society of Japan
Vol. 23, No. 4, December 1980



# A SEQUENTIAL UNIT ALLOCATION FOR PARALLEL REDUNDANT SYSTEM

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(Received November 26, 1979; Revised May 12, 1980)

*Abstract* The present paper deals with a continuous-time sequential allocation problem in the following context. Consider an m-unit parallel system in which each unit stochastically deteriorates with age. When the system breaks down, all or some failed units are replaced with new units or the failed system is left as it is until the end of planning horizon. As a result of this action, an idle time occurs and the cost is incurred during the idle time. We investigate a sequential unit allocation procedure that minimizes the total expected cost under the conditions that the planning horizon is given and a finite number of units are available for replacement. Some important properties of optimal policy are obtained for $m$=2, and a simple example is presented to illustrate our model.

## 1. Introduction

In [4,5,6] continuous-time sequential allocation problems are considered where the probability distribution of elapsed time between two successive decision points in time is independent of the decision just made. In [3] it is dependent on the decision just made, but an infinite number of units are available.

The present paper discusses a continuous-time sequential allocation problem where the probability distribution of elapsed time between two successive decision points in time is dependent on the decision just made in the following context. An $m$-unit parallel system must be operated under the conditions that the planning horizon is given and a finite number of units are available for replacement. Note that the system may be left in a failed state because of the limitation of units on hand. Therefore an idle time occurs and the opportunity cost is incurred during the idle time. Crookes [1] considers a replacement strategy that when a system (one-unit system) breaks down a replacement is made or not according as the remaining time to a planned replacement is larger or less than the given number. Modifying it to our $m$-unit parallel system, we restrict admissible policies in the following way. When the system breaks down, all or some failed units are replaced with new or the failed system is left as it is until the end of planning horizon. It should be noted that a decision is made only just after system-down, therefore the system is idle until the end of planning horizon if 'leave' decision is employed. Our problem is to find a sequential unit allocation procedure that minimizes the total expected cost under the conditions that the planning horizon is given and a finite number of units are available for replacement.

In Section 2, we formulate this problem by dynamic programming. In Section 3, some properties of optimal policies for $m=2$ are derived. In Section 4, we discuss a simple example.

## 2. Model and Formulation

Consider an $m$-unit parallel system in which each unit stochastically deteriorates with age. It should be noted that the system-down occurs if and only if all of the $m$ units fails. It is assumed that when the system breaks down, $i$ failed units $(1 \leq i \leq m)$ are replaced with new or the failed system is left as it is until the end of planning horizon. Note that a decision can be made only just after system-down. As a result the system is idle until the end of planning horizon if 'leave' decision is selected. The opportunity cost is incurred during the idle time. Our problem is to find a sequential unit allocation procedure that minimizes the total expected cost under the conditions that the planning horizon is $X$ and $N$ units are available for replacement at the beginning of planning period.

Concentrating on our model, we define the following notations:

$f(t)$=p.d.f. of lifetime $T$ of each unit,

$F(t)$=c.d.f. of $T$ with finite mean,

$C$=idle cost per unit time $(C>0)$,

$L$=fixed cost of replacement $(L \geq 0)$,

$K$=replacement cost per unit $(K>0)$,

$V_n(x)$=the total expected cost when there are $n$ units left on hand, a remaining time just after system-down is given by $x$, and an optimal policy is followed.

Using the Principle of Optimality, we have the following recursive relation for $0 \leq x \leq X$ and $n=1,\ldots,N$;

$$V_n(x)=\min\begin{cases} Cx, \\ \min\limits_{j=1,\ldots,\min\{m,n\}} \{L+jK+\int_0^x V_{n-j}(x-t)j(F(t))^{j-1}f(t)dt\}, \end{cases} \tag{2.1}$$

$$V_0(x)=Cx. \tag{2.2}$$

Equation (2.1) is obtained as follows: If the system is left, the opportunity cost $Cx$ is incurred since there is no chance of making decision until the end of planning horizon in our model. If $j$ units $(j=1,\ldots,\min\{m,n\})$ are replaced, then the immediate cost $L+jK$ and the future cost $\int_0^x V_{n-j}(x-t)j(F(t))^{j-1}f(t)dt$ are incurred since the p.d.f. of lifetime of $j$-unit parallel system is given by $j(F(t))^{j-1}f(t)$. Equation (2.2) follows since there is no unit on hand and the system must be left until the end of planning horizon.

Unfortunately it is difficult to solve this recursive relation explicitly. But some properties are derived for $m=2$ (2-unit system) as we show in the subsequent sections.

## 3. Properties of Optimal Policy for m=2

It should be noted that the optimal policy can be derived from equations (2.1) and (2.2). Unfortunately, we can not solve them explicitly. So we investigate some properties of the optimal policy.

First we determine the region where it is optimal to leave the system.

**Theorem 1.** If $CE(T)>L+K$, $CE(\max\{T_1,T_2\})>L+2K$, and there exist $x_1^*$ and $x_2^*$ satisfying $L+K-CE(T)+CT_F(x_1^*)=0$ and $L+2K-CE(\max\{T_1,T_2\})+CT_{F_2}(x_2^*)=0$,

then

$$V_1(x)=\begin{cases} Cx & \text{for } x \leq x_1^*, \\ L+K+\int_0^x V_0(x-t)f(t)dt & \text{for } x>x_1^*, \end{cases}$$

and

$$V_n(x)\begin{cases} =Cx & \text{for } x\leq \min\{x_1^*,x_2^*\}, \\ <Cx & \text{for } x>\min\{x_1^*,x_2^*\}, \end{cases} \quad \text{for } n=2,\ldots,N,$$

where $E(T)=\int_0^\infty tdF(t)$, $T_F(x)=\int_x^\infty (t-x)dF(t)$, $F_2(t)=(F(t))^2$, and $T_1$ and $T_2$ are independent and identically distributed random variables with c.d.f. $F$.

Proof: From (1) with $m=2$ and $n=1$,

$$V_1(x)=\min\{Cx,\ L+K+\int_0^x C(x-t)f(t)dt\}$$

$$=Cx+\min\{0,\ L+K-CE(T)+CT_F(x)\}.$$

As was shown in DeGroot [2], the transform $T_F(x)$ is a nonnegative convex and strictly decreasing function of $x$. Thus the form of $V_1(x)$ is expressed in Theorem 1.

Suppose that $V_k(x)=Cx$ for $x\leqq\min\{x_1^*,x_2^*\}$ and $k=n-1,\ n$.

Note that

$$V_{n+1}(x)=\min\{Cx,\ L+K\int_0^x V_n(x-t)f(t)dt,\ L+2K+\int_0^x V_{n-1}(x-t)2F(t)f(t)dt\}.$$

For $x\leqq\min\{x_1^*,x_2^*\}$, the second and the third term in braces is expressed respectively as follows:

$$L+K+\int_0^x V_n(x-t)f(t)dt=L+K+\int_0^x C(x-t)f(t)dt$$

$$=Cx+L+K-CE(T)+CT_F(x)$$

$$\geqq Cx \qquad \text{(equality holds if and only if } x=x_1^*\text{)},$$

and

$$L+2K+\int_0^x V_{n-1}(x-t)2F(t)f(t)dt=L+2K+\int_0^x C(x-t)2F(t)f(t)dt$$

$$=Cx+L+2K-CE(\max\{T_1,T_2\})+CT_{F_2}(x)$$

$$\geqq Cx \qquad \text{(equality holds if and only if } x=x_2^*\text{)}.$$

These results imply that $V_k(x)=Cx$ for $x\leqq\min\{x_1^*,x_2^*\}$ and $k=n+1$. Obviously, $V_0(x)=Cx$ and $V_1(x)=Cx$ for $x\leqq x_1^*$. Thus, by induction, $V_n(x)=Cx$ for $x\leqq\min\{x_1^*,x_2^*\}$ and $n=2,\dots,N$ is proved.

We next show $V_n(x)<Cx$ for $x>\min\{x_1^*,x_2^*\}$ and $n=2,\dots,N$. It should be noted that $V_n(x)$ is continuous with respect to $x$ and differentiable except at most two points. From the fact mentioned above,

$$Cx=\begin{cases} L+K+\int_0^x V_{n-1}(x-t)f(t)dt & \text{for } x=x_1^*, \\ L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt & \text{for } x=x_2^*. \end{cases}$$

For $n=2,\dots,N$, if we show

$$\frac{d}{dx}(L+K+\int_0^x V_{n-1}(x-t)f(t)dt)<(\leqq)C \qquad \text{for } x=(>)x_1^*,$$

$$\frac{d}{dx}(L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt)<(\leqq)C \qquad \text{for } x=(>)x_2^*,$$

the proof is complete. This is easily verified by induction if we note that

$$\frac{d}{dx}(L+K+\int_0^x V_{n-1}(x-t)f(t)dt)=\int_0^x \frac{d}{dx} V_{n-1}(x-t)f(t)dt,$$

$$\frac{d}{dx}(L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt)=\int_0^x \frac{d}{dx} V_{n-2}(x-t)2F(t)f(t)dt,$$

since $V_{n-1}(0)=V_{n-2}(0)=0$. Q.E.D.

Remark. Similar result for $m>2$ is easily verified.

From Theorem 1, when there is only one unit on hand, then it is optimal to leave the system if the remaining time is less than or equal to $x_1^*$ and replace just one unit if the remaining time is more than $x_1^*$. When there are more than or equal to two units on hand, then it is optimal to leave the system if the remaining time is less than or equal to $min\{x_1^*,x_2^*\}$ and replace one or two units if the remaining time is more than $min\{x_1^*,x_2^*\}$.

For a while, we consider the special case of $L=0$. It is intuitively clear that there is no advantage of replacing two units. This is shown in the next theorem.

Theorem 2. If $CE(T)>K$ and there exists $x_1^*$ satisfying $K-CE(T)+CT_F(x_1^*)=0$, then

$$V_n(x)=\begin{cases} Cx & \text{for } x\leq x_1^*, \\ K+\int_0^x V_{n-1}(x-t)f(t)dt & \text{for } x>x_1^*, \end{cases}$$

for $n=1,\ldots,N$.

Proof: From Theorem 1, it is sufficient to show that $x_1^*<x_2^*$ and $K+\int_0^x V_{n-1}(x-t)f(t)dt<2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt$ for $x>x_1^*$. Since $T_{F_2}(x)=T_F(x)+F(x)T_F(x)+\int_x^\infty T_F(t)f(t)dt$ and $E(max\{T_1,T_2\})=E(T)+\int_0^\infty T_F(t)f(t)dt$,

$$\begin{aligned}2K-CE(max\{T_1,T_2\})+CT_{F_2}(x_1^*)&=K-CE(T)+CT_F(x_1^*)+K-C\int_0^\infty T_F(t)f(t)dt\\ &\quad+CF(x_1^*)T_F(x_1^*)+C\int_{x_1^*}^\infty T_F(t)f(t)dt\\ &=K(1-F(x_1^*))+C(T_F(0)F(x_1^*)\\ &\quad-\int_0^{x_1^*} T_F(t)f(t)dt)>0.\end{aligned}$$

Thus $x_1^*<x_2^*$.

To show it is not optimal to replace two units for $x>x_1^*$, we compare the expected cost incurred when two units are replaced and then the optimal policy is followed with the expected cost incurred when two units are replaced one by one and then the optimal policy is followed. The former is equal to $2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt(=A)$ and the latter to $K+\int_0^x(K+\int_0^{x-t} V_{n-2}(x-t-u)f(u)du)f(t)dt(=B)$. Then

$$A-B=2K+\iint_{D1} V_{n-2}(x-\max\{t,u\})f(t)f(u)dtdu$$
$$-\{K(1+F(x))+\iint_{D2} V_{n-2}(x-t-u)f(t)f(u)dtdu\}$$
$$\geqq K(1-F(x))+\iint_{D2}(V_{n-2}(x-\max\{t,u\})-V_{n-2}(x-t-u))f(t)f(u)dtdu\geqq 0,$$

since $\max\{t,u\}\leqq t+u$ for $(t,u)\in D2$ and $V_{n-2}(\cdot)$ is nondecreasing, where $D1=\{(t,u);\ 0<t<x,\ 0<u<x\}$ and $D2=\{(t,u);0<t<x,\ 0<u<x,\ 0<t+u<x\}$. Q.E.D.

Remark. Similar result for $m>2$ is easily verified.

Theorem 2 specifies the structure of optimal policy for $L=0$. The same method gives a sufficient condition that it is optimal to replace one unit for $L>0$.

Theorem 3. If $x>x_1^*$ and $K/(L+K)\geqq F(x)$, then it is optimal to replace one unit when there is still a time $x$ to go and there are more than or equal to two units on hand.

Remark. For $m>2$, the following is verified:
If $x>x_1^*$ and $K/(L+K)\geqq(F(x)+\cdots+F^{(\min\{m,n\}-1)}(x))$, then it is optimal to replace one unit when there is still a time $x$ to go and there are $n$ units on hand, where $F^{(i)}$ denotes the $i$-fold convolution of $F$.

The following Theorems 4 and 5 are concerned with sufficient conditions that the identical alternative is optimal even if the remaining time increases a little.

Theorem 4. If $n\geqq 2$, $V_n(y)=L+K+\int_0^y V_{n-1}(y-t)f(t)dt$,

$$0<x-y\leqq\begin{cases}\infty & \text{for } n=2,\\ x_1^* & \text{for } n=3,\\ \min\{x_1^*,x_2^*\} & \text{for } n=4,\ldots,\end{cases}$$

and $xF(x)-yF(y)-\int_y^x tf(t)dt-x((F(x))^2-(F(y))^2)+\int_y^x t2F(t)f(t)dt\leqq -d(y)/C$,

then $V_n(x)=L+K+\int_0^x V_{n-1}(x-t)f(t)dt$,

where $d(y)=(L+K+\int_0^y V_{n-1}(y-t)f(t)dt)-(L+2K+\int_0^y V_{n-2}(y-t)2F(t)f(t)dt)$.

Proof: It is sufficient to show $d(x)\leqq 0$. It should be noted that

$$\int_0^x V_{n-1}(x-t)f(t)dt \leqq \int_0^y (V_{n-1}(y-t)+C(x-y))f(t)dt + \int_y^x C(x-t)f(t)dt.$$

Thus
$$L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt+d(x) \leqq L+2K+\int_0^y V_{n-2}(y-t)2F(t)f(t)dt+d(y)$$
$$+\int_0^y C(x-y)f(t)dt+\int_y^x C(x-t)f(t)dt.$$

Since $\int_y^x V_{n-2}(x-t)2F(t)f(t)dt=\int_y^x C(x-t)2F(t)f(t)dt$ from the assumption,

$$d(x) \leqq d(y)+C\{xF(x)-yF(y)-\int_y^x tf(t)dt-x((F(x))^2-(F(y))^2)$$
$$+\int_y^x t2F(t)f(t)dt\} \leqq d(y)-d(y) \leqq 0.$$ Q.E.D.

Theorem 5. If $n \geqq 2$, $V_n(y)=L+2K+\int_0^y V_{n-2}(y-t)2F(t)f(t)dt$,

$$0<x-y \leqq \begin{cases} x_1^* & \text{for } n=2, \\ \min\{x_1^*, x_2^*\} & \text{for } n=3,\ldots, \end{cases}$$

and
$$x(F(x))^2-y(F(y))^2-\int_y^x t2F(t)f(t)dt-x(F(x)-F(y))+\int_y^x tf(t)dt<d(y)/C,$$

then
$$V_n(x)=L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt.$$

Proof: Using the relation,

$$\int_0^x V_{n-2}(x-t)2F(t)f(t)dt \leqq \int_0^y (V_{n-2}(y-t)+C(x-y))2F(t)f(t)dt$$
$$+\int_y^x C(x-t)2F(t)f(t)dt,$$

we can show $d(x)>0$ similarly to Theorem 4. Q.E.D.

We discussed the optimal policy in Theorem 1 or Theorems 3,4,5 according as the remaining time is short or not short. Theorem 6 deals with the case that the remaining time is sufficiently long.

Theorem 6. Suppose $CE(T)>L+K$. Then the following hold for sufficiently large $x$:

(i) If $L<C\int_0^\infty (1-F(t))^2 dt$, then

$$V_n(x)=L+K+\int_0^x V_{n-1}(x-t)f(t)dt \qquad \text{for } n=2,3,\ldots,N.$$

(ii) If $L>C\int_0^\infty (1-F(t))^2 dt$, then

$$V_n(x)=L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt \qquad \text{for } n=2,4,6,\ldots .$$

Proof: Though $\lim_{x\to\infty} V_n(x)$ does not exist, $\lim_{x\to\infty}(V_n(x)-Cx)=v_n$ does exist and satisfies the following relation, as shown in Appendix.

$$v_0=0,$$
$$v_1=min\{0,\ L+K-CE(T)\},$$
$$v_n=min\{0,\ L+K-CE(T)+v_{n-1},\ L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-2}\}$$
for $n=2,3,\ldots,N$.

It is easily shown that

if $L+C\int_0^\infty T_F(t)f(t)dt-CE(T)=L-C\int_0^\infty(1-F(t))^2dt<0$, then

$$L+K-CE(T)+v_{n-1}<L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-2} \quad \text{for } n=2,3,\ldots,N,$$

and if $L-C\int_0^\infty(1-F(t))^2dt>0$, then

$$L+K-CE(T)+v_{n-1}>L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-2} \quad \text{for } n=2,4,\ldots .$$

Since $\lim_{x\to\infty}(L+K+\int_0^x V_{n-1}(x-t)f(t)dt-Cx)=L+K-CE(T)+v_{n-1}$ and

$$\lim_{x\to\infty}(L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt-Cx)=L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-2},$$

the proof is complete. Q.E.D.

Theorem 6 says that when the remaining time is sufficiently long, it is optimal to replace one unit if $L<C\int_0^\infty(1-F(t))^2dt$, and it is optimal to replace two units if $L>C\int_0^\infty(1-F(t))^2dt$ and even number of units are on hand. This result is intuitively clear since $\{n(L+K)+C(x-nE(T))\}-\{(n/2)L+nK+C(x-(n/2)E(max\{T_1,T_2\}))\}=(n/2)\{L-C(2E(T)-E(max\{T_1,T_2\}))\}=(n/2)\{L-C\int_0^\infty(1-F(t))^2dt\}$ for even $n$ and $\{n(L+K)+C(x-nE(T))\}-\{((n-1)/2+1)L+nK+C(x-((n-1)/2)E(max\{T_1,T_2\})-E(T))\}=((n-1)/2)\{L-C\int_0^\infty(1-F(t))^2dt\}$ for odd $n$.

## 4. Example for Exponential Life Time

In the previous section we discussed some properties of optimal policy. Here we obtain the optimal policy explicitly for $f(t)=\lambda e^{-\lambda t}$ and $n=2$.

From Theorem 1, if $CE(T)>L+K$, $CE(max\{T_1,T_2\})>L+2K$, then

$$V_2(x)=\begin{cases} Cx & \text{for } x\leqq min\{x_1^*,x_2^*\}, \\ min\{L+K+\int_0^x V_1(x-t)f(t)dt,\ L+2K+\int_0^x V_0(x-t)2F(t)f(t)dt\} & \text{for } x>min\{x_1^*,x_2^*\}, \end{cases}$$

and

$$V_1(x)=\begin{cases} Cx & \text{for } x\leqq x_1^*, \\ Cx+CT_F(x)+L+K-CE(T) & \text{for } x>x_1^*. \end{cases}$$

Define $h(x)$ as follows:

$$h(x):=(L+K+\int_0^x V_1(x-t)f(t)dt)-(L+2K+\int_0^x V_0(x-t)2F(t)f(t)dt) \quad \text{for } x\geqq x_1^*.$$

A simple calculation yields

$$h(x)=C\{\int_0^x(1-F(t))F(t)dt-\int_0^{x-x_1^*}(1-F(x-t))F(t)dt-K/C\},$$

$$\lim_{x\to x_1^*+0} h'(x)=CF(x_1^*)(1-F(x_1^*))>0, \quad \text{and} \quad \lim_{x\to\infty} h(x)=C\{L/C-\int_0^\infty(1-F(t))^2dt\}.$$

It should be noted that the regions in which it is optimal to leave the system $D_0$, to replace one unit $D_1$, and to replace two units $D_2$ are expressed as follows: If $x_1^*>x_2^*$, then $D_0=\{x;\ x\leqq x_2^*\}$, $D_2=\{x;\ x_2^*<x<x_1^* \text{ or } h(x)>0\}$, and $D_1=\{x;\ h(x)\leqq 0\}$. If $x_1^*<x_2^*$, then $D_0=\{x;\ x\leqq x_1^*\}$, $D_1=\{x;\ h(x)\leqq 0 \text{ and } x>x_1^*\}$, and $D_2=\{x;\ h(x)>0\}$. Substituting $f(x)=\lambda e^{-\lambda x}$ into $h(x)$, we obtain

$$h(x)=C\{L/C+e^{-\lambda x}(x-x_1^*)-(1-e^{-2\lambda x})/(2\lambda)\},$$

where $x_1^*=(1/\lambda)ln\{(1-\lambda(L+K)/C)^{-1}\}$.

Note that we assume $L+K<C/\lambda$ and $L+2K<3C/(2\lambda)$. A tedious calculation shows that if $(L+K)^2<2CK/\lambda$ $(x_1^*<x_2^*)$, then $h(x)\leqq 0$ for $x>x_1^*$, if $(L+K)^2>2CK/\lambda$ $(x_1^*>x_2^*)$ and $L>C/(2\lambda)$, then $h(x)>0$ for $x>x_1^*$, and if $(L+K)^2>2CK/\lambda$ and $L<C/(2\lambda)$, then $h(x)>0$ for $x_1^*<x<x_{12}^*$ and $h(x)<0$ for $x>x_{12}^*$ where $x_{12}^*$ satisfies $C(1-e^{-2\lambda x_{12}^*})/(2\lambda)-L-Ce^{-\lambda x_{12}^*}(x_{12}^*-x_1^*)=0$, and $x_2^*=(1/\lambda)ln\{(2-\sqrt{1+2\lambda(L+2K)/C})^{-1}\}$.

Thus, when there are two units on hand, the optimal policy is as follows:
In Region 1 (see Fig. 1),

$$\text{it is optimal to} \left\{\begin{array}{l}\text{leave the system}\\ \\ \text{replace one unit}\end{array}\right\} \text{if} \left\{\begin{array}{l}x\leqq x_1^*\\ \\ x>x_1^*\end{array}\right\}.$$

In Region 2,

$$\text{it is optimal to} \left\{\begin{array}{l}\text{leave the system}\\ \text{replace two units}\\ \text{replace one unit}\end{array}\right\} \text{if} \left\{\begin{array}{l}x\leqq x_2^*\\ x_2^*<x<x_{12}^*\\ x\geqq x_{12}^*\end{array}\right\}.$$

In Region 3,

$$\text{it is optimal to} \left\{\begin{array}{l}\text{leave the system}\\ \\ \text{replace two units}\end{array}\right\} \text{if} \left\{\begin{array}{l}x\leqq x_2^*\\ \\ x>x_2^*\end{array}\right\}.$$

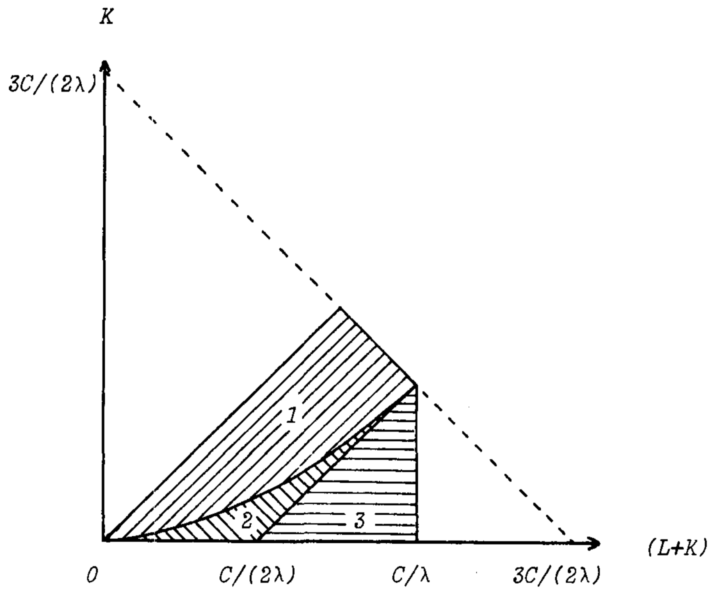


Fig. 1.

From the structure of optimal policy for $n=2$ derived above, one may expect that the longer is the remaining time, the less units are replaced for $n\geq 2$. But this conjecture is not true.

**Lemma 1.** The nonnegative sequence $\{V_n(x)\}$ is monotone decreasing in $n$ for any $x$.

**Proof:** $V_1(x)=\min\{Cx,\ L+K+\int_0^x V_0(x-t)f(t)dt\}\leq Cx=V_0(x).$

$$V_2(x)=\min\{Cx,\ L+K+\int_0^x V_1(x-t)f(t)dt,\ L+2K+\int_0^x V_0(x-t)2F(t)f(t)dt\}$$

$$\leq\min\{Cx,\ L+K+\int_0^x V_0(x-t)f(t)dt\}=V_1(x).$$

Suppose $V_{k-1}(x)\geq V_k(x)$ for $k=n-1$ and $n$.

$$V_{n+1}(x)=\min\{Cx,\ L+K+\int_0^x V_n(x-t)f(t)dt,\ L+2K+\int_0^x V_{n-1}(x-t)2F(t)f(t)dt\}$$

$$\leq\min\{Cx,\ L+K+\int_0^x V_{n-1}(x-t)f(t)dt,\ L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt\}$$

$$=V_n(x).$$

Q.E.D.

By Lemma 1, we can define $V(x)=\lim_{n\to\infty} V_n(x)$, which satisfies the following functional equation:

$$V(x)=\min\{Cx,\ L+K+\int_0^x V(x-t)f(t)dt,\ L+2K+\int_0^x V(x-t)2F(t)f(t)dt\},$$

since $\lim_{n\to\infty}\int_0^x V_n(x-t)f(t)dt=\int_0^x \lim_{n\to\infty} V_n(x-t)f(t)dt$ from $\int_0^x V_n(x-t)f(t)dt\leq$

$\int_0^x C(x-t)f(t)dt<\infty$. From Theorem 1 and the definition of $x_1{}^*$ and $x_2{}^*$,

$$V(x)\begin{cases} =Cx & \text{for } x\leqq min\{x_1{}^*,x_2{}^*\}, \\ <Cx & \text{for } x>min\{x_1{}^*,x_2{}^*\}. \end{cases}$$

For the case of $f(t)=\lambda e^{-\lambda t}$, we can solve the above functional equation. Suppose $L+K<C/\lambda$ and $L+2K<3C/(2\lambda)$.

If $K\geqq L$, then

$$V(x)=\begin{cases} Cx & \text{for } x\leqq x_1{}^*=(ln\{(1-\lambda(L+K)/C)^{-1}\})/\lambda, \\ \lambda(L+K)(x-x_1{}^*)+Cx_1{}^* & \text{for } x>x_1{}^*. \end{cases}$$

If $K<L$ and $(L+K)^2<2CK/\lambda$, then

$$V(x)=\begin{cases} Cx & \text{for } x\leqq x_1{}^*, \\ \lambda(L+K)(x-x_1{}^*)+Cx_1{}^* & \text{for } x_1{}^*<x\leqq x_{12}{}^{**}, \\ 2\lambda(L+2K)x/3+a_1e^{-3\lambda x}+a_2 & \text{for } x>x_{12}{}^{**}, \end{cases}$$

where $x_{12}{}^{**}=(ln\{(L+K)/((L-K)(1-\lambda(L+K)/C))\})/(2\lambda)$, $a_1=2(L-K)e^{3\lambda x_{12}{}^{**}}/9$ and $a_2=C(1-\lambda(L+K)/C)x_1{}^*+\lambda(L-K)x_{12}{}^{**}/3-2(L-K)/9$.

If $(L+K)^2>2CK/\lambda$, then

$$V(x)=\begin{cases} Cx & \text{for } x\leqq x_2{}^*=(ln\{(2-\sqrt{1+2C(L+2K)/\lambda})^{-1}\})/\lambda, \\ 2\lambda(L+2K)x/3+a_3e^{-3\lambda x}+a_4 & \text{for } x>x_2{}^*, \end{cases}$$

where $a_3=(3C/(2\lambda)-(L+2K))e^{3\lambda x_2{}^*}/9-Ce^{\lambda x_2{}^*}/(6\lambda)$ and

$a_4=2\lambda(3C/(2\lambda)-(L+2K))x_2{}^*/3-(3C/(2\lambda)-(L+2K))/9+Ce^{-2\lambda x_2{}^*}/(6\lambda)$.

Thus, when there are infinite number of units on hand, the optimal policy is as follows:

In Region A (see Fig. 2.),

$$\text{it is optimal to}\begin{Bmatrix}\text{leave the system}\\ \text{replace one unit}\end{Bmatrix}\text{ if }\begin{Bmatrix}x\leqq x_1{}^*\\ x>x_1{}^*\end{Bmatrix}.$$

In Region B,

$$\text{it is optimal to}\begin{Bmatrix}\text{leave the system}\\ \text{replace one unit}\\ \text{replace two units}\end{Bmatrix}\text{ if }\begin{Bmatrix}x\leqq x_1{}^*\\ x_1{}^*<x\leqq x_{12}{}^{**}\\ x>x_{12}{}^{**}\end{Bmatrix}.$$

In Region C,

$$\text{it is optimal to} \begin{Bmatrix} \text{leave the system} \\ \text{replace two units} \end{Bmatrix} \text{ if } \begin{Bmatrix} x \leqq x_2^* \\ x > x_2^* \end{Bmatrix}.$$

Since $V(x)=\lim_{n\to\infty} V_n(x)$, the conjecture that the longer is the remaining time, the less units are replaced is not true.

Critical numbers $x_1^*$, $x_2^*$, $x_{12}^*$, and $x_{12}^{**}$ are listed in Table 1 for various values of $L$ and $K$. It is assumed that $\lambda=C=1$.

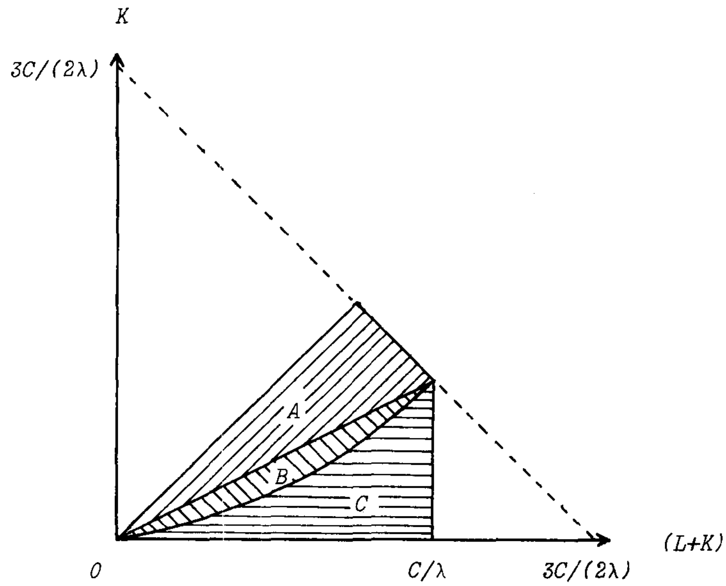


Fig. 2.

Table 1. Critical numbers $x_1^*$, $x_2^*$, $x_{12}^*$, and $x_{12}^{**}$

| $K$ | $L$ | Region | $x_1^*$ | $x_2^*$ | $x_{12}^*$ | $x_{12}^{**}$ |
|---|---|---|---|---|---|---|
| 0.01563 | 0.2344 | 2,C | — | 0.2711 | 1.782 | — |
| 0.0625 | 0.1875 | 1,B | 0.2877 | — | — | 0.4904 |
| 0.1875 | 0.0625 | 1,A | 0.2877 | — | — | — |
| 0.0625 | 0.4375 | 2,C | — | 0.6120 | 3.959 | — |
| 0.1875 | 0.3125 | 1,B | 0.6931 | — | — | 1.0397 |
| 0.3750 | 0.1250 | 1,A | 0.6931 | — | — | — |
| 0.1250 | 0.6250 | 3,C | — | 1.0739 | — | — |
| 0.2656 | 0.4844 | 2,C | — | 1.3511 | 5.597 | — |
| 0.3125 | 0.4375 | 1,B | 1.3863 | — | — | 1.5890 |
| 0.5000 | 0.2500 | 1,A | 1.3863 | — | — | — |

## 5. Conclusion

The present paper considers a sequential unit allocation problem for parallel sysrem. Some properties of optimal unit allocation procedure which minimizes the total expected cost are developed. It is optimal to leave the system for a 'short' remaining time until the end of planning horizon. Some sufficient conditions that specify the optimal policy are obtained when the remaining time is 'long', that is, sufficiently long. If the remaining time is 'intermediate', the structure of optimal policy is not clear. A simple example is presented to illustrate our model.

## Appendix

If $V_n(x)$ satisfies the following recursive relations:

$$V_0(x)=Cx,$$

$$V_1(x)=min\{Cx,\ L+K+\int_0^x V_0(x-t)f(t)dt\},$$

$$V_n(x)=min\{Cx,\ L+K+\int_0^x V_{n-1}(x-t)f(t)dt,\ L+2K+\int_0^x V_{n-2}(x-t)2F(t)f(t)dt\}$$

for $n=2,3,\ldots,N$,

then $\lim_{x\to\infty}(V_n(x)-Cx)=v_n$ exists and satisfies

$$v_0=0,$$

$$v_1=min\{0,\ L+K-CE(T)\},$$

$$v_n=min\{0,\ L+K-CE(T)+v_{n-1},\ L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-2}\}$$

for $n=2,3,\ldots,N$.

**Proof:** Since $V_0(x)-Cx=Cx-Cx=0$, then $v_0=0$.
$V_1(x)-Cx=min\{0,\ L+K-CE(T)+CT_F(x)\}$. So $v_1=min\{0,\ L+K-CE(T)\}$.
Suppose $\lim_{x\to\infty}(V_k(x)-Cx)=v_k$ exists for $k\leqq n$ $(n\geqq 2)$.

$$V_{n+1}(x)-Cx=min\{0,\ L+K+\int_0^x(V_n(x-t)-C(x-t))f(t)dt-C\int_0^\infty min\{x,t\}f(t)dt,$$
$$L+2K+\int_0^x(V_{n-1}(x-t)-C(x-t))2F(t)f(t)dt$$
$$-C\int_0^\infty min\{x,t\}2F(t)f(t)dt\}.$$

Since $min\{x,t\}\leqq t$ and $\int_0^\infty tf(t)dt=E(T)<\infty$, $\lim_{x\to\infty}\int_0^\infty min\{x,t\}f(t)dt$
$=\int_0^\infty \lim_{x\to\infty} min\{x,t\}f(t)dt=E(T)$. Furthurmore it can be verified that

$\lim_{x\to\infty} \int_0^x (V_n(x-t)-C(x-t))f(t)dt=v_n$ and $\lim_{x\to\infty} \int_0^x (V_{n-1}(x-t)-C(x-t))2F(t)f(t)dt=v_{n-1}$.

Thus $v_{n+1}=min\{0,\ L+K-CE(T)+v_n,\ L+2K-CE(T)-C\int_0^\infty T_F(t)f(t)dt+v_{n-1}\}$. Q.E.D.

## References

[1] Crookes, P. C. I. : Replacement Strategies. *Operational Research Quarterly*, Vol. 14(1963), 167-184.

[2] DeGroot, M. H. : *Optimal Statistical Decisions*. McGraw-Hill, New York, 1970.

[3] Derman, C., Lieberman, G. J. and Ross, S. M. : A Renewal Decision Problem. *Management Science*, Vol. 24(1978), 554-561.

[4] ______ : A Stochastic Sequential Allocation Model. *Operations Research*, Vol. 23(1975), 1120-1130.

[5] Donis, J. M. and Pollock, S. M. : Allocation of Resources to Randomly Occurring Opportunities, *Naval Research Logistics Quarterly*, Vol. 14 (1967), 513-527.

[6] Sakaguchi, M. : A Sequential Allocation Problem for Randomly Appearing Targets, *Mathematica Japonica*, Vol. 21(1976), 89-103.

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