ON A THREE-PERSON SILENT MARKSMANSHIP CONTEST

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Abstract   In this paper, we consider a three-person silent marksmanship contest. Each of three players 1, 2 and 3 has a gun with exactly one bullet which may be fired at any time on [0, 1] aiming at his own target. The accuracy function $A_i(x)$ for player i is strictly increasing and differentiable with $A_i(0) = 0$ and $A_i(1) = 1$. The first player hitting his target gets payoff +1 and other two players get payoff zero. For the game, we get a Nash equilibrium point and equilibrium payoff for each player. The form of the Nash equilibrium point differs whether $A_1(x)/A_2(x)A_3(x)$, $A_2(x)/A_1(x)A_3(x)$ and $A_3(x)/A_1(x)A_2(x)$ decrease or one of these increases. Some examples are given to illustrate the results.

1. Introduction

An m-person marksmanship contest is a non-zero-sum game with the follow-
ing structure: There are m players (i.e., contestants) 1, 2, ... , m and m
targets, and each target is associated with one player. Every player has a
gun with exactly one bullet which may be fired at any time in [0, 1] aiming at
his own target. Starting at time 0, each player walks toward his own target;
he will reach his target at time 1. If player i fires his bullet at time x,
the probability that he hits his target is $A_i(x)$. The function $A_i(x)$ is called
the accuracy function for player i and is strictly increasing and differenti-
able on [0, 1] with $A_i(0) = 0$ and $A_i(1) = 1$. The accuracy functions are fixed
and known beforehand to all players. As soon as one of the players hits his
target, the contest ends and the first player hitting his target gets payoff +1
and other players get payoff zero. If none of the players hit their targets or
some players hit their targets at the same time, then the payoff is zero for
every player. In this situation, each player wishes to delay firing as long as
possible to increase his accuracy, while at the same time he does not wish to
delay so long that his opponents can precede him with effectiveness.

A marksmanship contest is noisy if each player knows when the others have fired and silent if neither player knows when the others have fired. Thus a noisy contest is the game with complete information in which each contestant is informed of his opponents' action, firing bullet, as soon as it takes place. On the other hand, in a silent contest each contestant is ignorant of any firing by the others.

The model is evidently m-person non-zero-sum game version of the duels. Silent duels were solved by Restrepo [4] and noisy duels were solved by Fox and Kimeldorf [1]. A more complete bibliography and a discussion of more general games of timing are found in Karlin [2]. Two-person noisy and silent marksmanship contests and m-person silent contest with equal accuracy functions were solved by Sakaguchi [5].

A Nash equilibrium point which is the most basic concept for non-cooperative m-person games is a set of strategies of the m players, with the property that no player can improve his expectation by changing his own strategy, if the strategies of the others are held fixed (Nash [3]). For example, the set of strategies (F*, G*, H*) is a Nash equilibrium point for a three-person game if and only if

$$M_1(F^*, G^*, H^*) \geq M_1(F, G^*, H^*),$$
$$M_2(F^*, G^*, H^*) \geq M_2(F^*, G, H^*)$$

and

$$M_3(F^*, G^*, H^*) \geq M_3(F^*, G^*, H)$$

for any strategies F, G and H of players 1, 2 and 3, respectively, where $M_i(F, G, H)$ denotes the expected payoff for player i when player 1 plays F, player 2 plays G and player 3 plays H. Thus, each player's strategy is optimal against those of the others. We note that (F*, G*, H*) is a Nash equilibrium point if and only if

$$M_1(F^*, G^*, H^*) \geq M_1(x, G^*, H^*),$$
$$M_2(F^*, G^*, H^*) \geq M_2(F^*, y, H^*)$$

and

$$M_3(F^*, G^*, H^*) \geq M_3(F^*, G^*, z)$$

for any pure strategies x, y and z of players 1, 2 and 3, respectively. We further note that

$$M_1(x, G^*, H^*) = M_1(F^*, G^*, H^*) \qquad \text{for any } x \text{ in } [a, b]$$

if F* has a density part over $[a, b]$ and similarly for $M_2(F^*, y, H^*)$ and $M_3(F^*, G^*, z)$.

2. Formulation

We consider the three-person silent marksmanship contest. There are three players 1, 2 and 3, and each player has a gun with exactly one bullet which may be fired at any time on $[0, 1]$ aiming at his own target. The accuracy function for player i is $A_i(x)$ with $A_i(0) = 0$ and $A_i(1) = 1$. As soon as one of the players hits his target, the contest ends and the first player hitting his target gets payoff $+1$ and other players get payoff zero. If none of the players hit their targets or some players hit their targets at the same time, then the payoff is zero for every player.

Since the contest is silent, none of the players can know whether the other players have fired their bullets or not. Furthermore, every player has only one byllet, and thus, a pure strategy for each player is determining his firing time. A mixed strategy for each player is a distribution function over $[0, 1]$. Let $\bar{M}_i(x, y, z)$ be the expected payoff for player i when players 1, 2 and 3 fire their bullets at times x, y and z, respectively. Then we have

$$\bar{M}_1(x, y, z) = \begin{cases} A_1(x), & x < \min(y, z), \\ A_1(x)\{1 - A_2(y)\}, & y < x < z, \\ A_1(x)\{1 - A_3(z)\}, & z < x < y, \\ A_1(x)\{1 - A_2(y)\}\{1 - A_3(z)\}, & \max(y, z) < x, \end{cases}$$

and similarly for $\bar{M}_2(x, y, z)$ and $\bar{M}_3(x, y, z)$.

In this paper, we assume that $A_1(x)/A_2(x)A_3(x)$, $A_2(x)/A_1(x)A_3(x)$ and $A_3(x)/A_1(x)A_2(x)$ are monotone. Then, it can be easily shown that at most one of these is monotone increasing, and so, we suppose that $A_1(x)/A_2(x)A_3(x)$ and $A_2(x)/A_1(x)A_3(x)$ are monotone decreasing.

3. The Case with Decreasing $A_3(x)/A_1(x)A_2(x)$

In this section, we assume that $A_1(x)/A_2(x)A_3(x)$, $A_2(x)/A_1(x)A_3(x)$ and $A_3(x)/A_1(x)A_2(x)$ are strictly decreasing functions. Let us define a class of mixed strategies for three players in which we shall search for equilibrium strategies. We seek for a strategy for player i (i = 1, 2 and 3) consisting of a density part $f_i(x)$ over the interval $[a, 1]$ and a mass part α_i at $x = 1$. We have with the normalization condition

$$\int_a^1 f_i(x)\,dx + \alpha_i = 1 \qquad \text{for } i = 1, 2 \text{ and } 3.$$

We note that the function $f_i(x)$ may be zero on some interval in $[a, 1]$. Let V_i (i = 1, 2 and 3) be the equilibrium payoff of the game for player i. If player 1 fires his bullet at time x in $[a, b]$ (b is determined later), then he gets

payoff +1 if and only if

(i) each of players 2 and 3 does not hit his target before time x

and

(ii) the shoot of player 1 hits his target.

Thus, we have

(1) $$V_1 = A_1(x)\{1 - \int_a^x A_2(y)f_2(y)dy\}\{1 - \int_a^x A_3(y)f_3(y)dy\}$$

for all x in $[a, b]$. Similarly, we get

(2) $$V_2 = A_2(x)\{1 - \int_a^x A_1(y)f_1(y)dy\}\{1 - \int_a^x A_3(y)f_3(y)dy\}$$

and

(3) $$V_3 = A_3(x)\{1 - \int_a^x A_1(y)f_1(y)dy\}\{1 - \int_a^x A_2(y)f_2(y)dy\}$$

for all x in $[a, b]$. From (1), (2) and (3), we get

(4) $$V_i = A_i(a),$$

and thus,

(5) $$1 - \int_a^x A_i(y)f_i(y)dy = \phi_i(x)/\phi_i(a)$$

for all x in $[a, b]$, where

$$\phi_1(x) = \{A_1(x)/A_2(x)A_3(x)\}^{1/2},$$

$$\phi_2(x) = \{A_2(x)/A_1(x)A_3(x)\}^{1/2}$$

and

$$\phi_3(x) = \{A_3(x)/A_1(x)A_2(x)\}^{1/2}.$$

By differentiating (5) with respect to x, we obtain

(6) $$f_i(x) = - \frac{\phi_i'(x)}{A_i(x)\phi_i(a)} \qquad \text{for all } x \text{ in } [a, b].$$

We have assumed that $A_1(x)/A_2(x)A_3(x)$, $A_2(x)/A_1(x)A_3(x)$ and $A_3(x)/A_1(x)A_2(x)$ are strictly decreasing, or equivalently, $\phi_i'(x) < 0$. Thus, we get $f_i(x) > 0$ for all x in $[a, b]$. It is easy to show that the equation

$$- \int_x^b \frac{\phi_i'(y)}{A_i(y)} \, dy = \phi_i(x)$$

has a unique root in $[0, b]$ for any value of b in $(0, 1)$. Let a_i be the unique root a of the equation

$$\int_a^1 f_i(x)dx = 1.$$

Without loss of generality, we assume $a_1 \geq a_2 \geq a_3$ and set $a = a_2$ and let b be the unique root of the equation

$$\int_a^b f_1(x)dx = 1.$$

Then as player 1 has fired his bullet before time b, we have

(7) $V_2 = A_2(x)\{1 - \int_a^b A_1(y)f_1(y)dy\}\{1 - \int_a^x A_3(y)f_3(y)dy\}$

and

(8) $V_3 = A_3(x)\{1 - \int_a^b A_1(y)f_1(y)dy\}\{1 - \int_a^x A_2(y)f_2(y)dy\}$

for all x in $(b, 1)$. It follows, from (4), (5) and (8), that

$$f_2(x) = \frac{1}{\phi_1(b)\phi_2(a)} \frac{A_3'(x)}{A_2(x)A_3(x)^2} \qquad \text{for } b < x < 1.$$

Similarly, we get

$$f_3(x) = \frac{1}{\phi_1(b)\phi_3(a)} \frac{A_2'(x)}{A_2(x)^2 A_3(x)} \qquad \text{for } b < x < 1.$$

In this case, $\alpha_1 = 0$, and α_2, α_3, a and b are determined as follows:

$$\int_a^b f_1(x)dx = 1,$$

$$\alpha_i = 1 - \int_a^1 f_i(x)dx \qquad i = 2 \text{ and } 3$$

and

$$\alpha_2\alpha_3 = 0.$$

Note that the simultaneous equations

$$\begin{cases} \int_a^b f_1(x)dx = 1 \\ \int_a^1 f_i(x)dx = 1 \qquad (i = 2 \text{ or } 3) \end{cases}$$

have unique roots a and b if $a_1 \geq a_2$ and $a_1 \geq a_3$.

Theorem 1. Let us suppose that $A_1(x)/A_2(x)A_3(x)$, $A_2(x)/A_1(x)A_3(x)$ and $A_3(x)/A_1(x)A_2(x)$ are strictly decreasing on $[0, 1]$. Let a_{ij} and b_{ij} be the roots of the simultaneous equations

$$\begin{cases} \int_{a_{ij}}^{b_{ij}} g_i(x)dx = \phi_i(a_{ij}) \\[2em] \int_{a_{ij}}^{b_{ij}} g_j(x)dx + \dfrac{1}{\phi_i(b_{ij})} \int_{b_{ij}}^{1} g_{jk}(x)dx = \phi_j(a_{ij}) \end{cases}$$

$(1 \leq i, j, k \leq 3,\ i \neq j,\ i \neq k,\ j \neq k)$, where

$$g_i(x) = - \phi_i'(x)/A_i(x) \qquad\qquad (i = 1, 2, 3)$$

and

$$g_{jk}(x) = A_k'(x)/A_j(x)A_k(x)^2 \qquad (1 \leq j, k \leq 3,\ j \neq k).$$

Let a be the maximum value of a_{ij} such that $0 < a_{ij} < b_{ij} \leq 1$ and let b be the corresponding value of b_{ij}. Then a set of equilibrium strategies is given as follows:

(i) If $a = a_{1j}$, then

$$f_1^*(x) = \begin{cases} g_1(x)/\phi_1(a), & \text{for} \quad a \leq x \leq b, \\[1em] 0, & \text{for} \quad b < x \leq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} g_2(x)/\phi_2(x), & \text{for} \quad a \leq x \leq b, \\[1em] g_{23}(x)/\phi_2(a)\phi_1(b), & \text{for} \quad b < x \leq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} g_3(x)/\phi_3(x) & \text{for} \quad a \leq x \leq b, \\[1em] g_{32}(x)/\phi_3(a)\phi_1(b), & \text{for} \quad b < x \leq 1, \end{cases}$$

and $\alpha_1^* = 0$. Moreover, if $a = a_{12}$, then $\alpha_2^* = 0$ and $\alpha_3^* \geq 0$ and if $a = a_{13}$, then $\alpha_2^* \geq 0$ and $\alpha_3^* = 0$.

(ii) If $a = a_{2j}$, then

$$f_1^*(x) = \begin{cases} g_1(x)/\phi_1(a), & \text{for} \quad a \leq x \leq b, \\[1em] g_{13}(x)/\phi_1(a)\phi_2(b), & \text{for} \quad b < x \leq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} g_2(x)/\phi_2(a), & \text{for} \quad a \leq x \leq b, \\[1em] 0, & \text{for} \quad b < x \leq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} g_3(x)/\phi_3(a), & \text{for} \quad a \leq x \leq b, \\[1em] g_{31}(x)/\phi_3(a)\phi_2(b), & \text{for} \quad b < x \leq 1, \end{cases}$$

and $\alpha_2^* = 0$. Moreover, if $a = a_{21}$, then $\alpha_1^* = 0$ and $\alpha_3^* \geq 0$ and if $a = a_{23}$, then $\alpha_1^* \geq 0$ and $\alpha_3^* = 0$.

(iii) If $a = a_{3j}$, then

$$f_1^*(x) = \begin{cases} g_1(x)/\phi_1(a), & \text{for} \quad a \leq x \leq b, \\[1em] g_{12}(x)/\phi_1(a)\phi_3(b), & \text{for} \quad b < x \leq 1, \end{cases}$$

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$$f_2^*(x) = \begin{cases} g_2(x)/\phi_2(a), & \text{for } a \leqq x \leqq b, \\ g_{21}(x)/\phi_2(a)\phi_3(b), & \text{for } b < x \leqq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} g_3(x)/\phi_3(a), & \text{for } a \leqq x \leqq b, \\ 0, & \text{for } b < x \leqq 1, \end{cases}$$

and $\alpha_3^* = 0$. Moreover, if $a = a_{31}$, then $\alpha_1^* = 0$ and $\alpha_2^* \geqq 0$ and if $a = a_{32}$, then $\alpha_1^* \geqq 0$ and $\alpha_2^* = 0$.

In all these cases, the equilibrium payoff V_i for player i is $A_i(a)$.

Proof: Without loss of generality, we assume $a = a_{12}$. Let $M_i(x)$ be the expected payoff for player i when he plays a pure strategy x and other two players play the strategies described in Theorem 1. We have already shown

$$M_1(x) = A_1(a) = V_1 \qquad \text{for } a \leqq x \leqq b,$$
$$M_2(x) = A_2(a) = V_2 \qquad \text{for } a \leqq x < 1$$

and

$$M_3(x) = A_3(a) = V_3 \qquad \text{for } a \leqq x < 1.$$

It is easy to see, for $i = 1$, 2 and 3, that

$$M_i(x) = A_i(x) < A_i(a) = V_i \qquad \text{for all } x \text{ in } [0, a).$$

Furthermore, we have

$$M_1(1) = \{1 - \int_a^1 A_2(y)f_2^*(y)dy\}\{1 - \int_a^1 A_3(z)f_3^*(z)dz - \alpha_3^*\} \leqq V_1,$$

$$M_2(1) = \{1 - \int_a^1 A_1(x)f_1^*(x)dx\}\{1 - \int_a^1 A_3(z)f_3^*(z)dz - \alpha_3^*\} \leqq V_2$$

and

$$M_3(1) = \{1 - \int_a^1 A_1(x)f_1^*(x)dx\}\{1 - \int_a^1 A_2(y)f_2^*(y)dy\} = V_3.$$

For $b < x < 1$, we get

$$M_1(x) = A_1(x)\{1 - \int_a^x A_2(y)f_2^*(y)dy\}\{1 - \int_a^x A_3(z)f_3^*(z)dz\}$$

$$= A_1(x) \, \frac{1}{\phi_2(a)\phi_1(b)} \, \frac{1}{A_3(x)} \, \frac{1}{\phi_3(a)\phi_1(b)} \, \frac{1}{A_2(x)}$$

$$= \frac{A_1(x)}{A_2(x)A_3(x)} \, \frac{A_2(b)A_3(b)}{A_1(b)} \, A_1(a) < A_1(a) = V_1$$

since $A_1(x)/A_2(x)A_3(x)$ is decreasing. Thus, for $i = 1$, 2 and 3, we have

$$M_i(x) \leqq A_i(a) = V_i \qquad \text{for all } x \text{ in } [0, 1].$$

This terminates our proof.

From Theorem 1, it is seen that, if three players play the equilibrium strategies, one of the players fires his bullet over the time interval $[a, b]$ and another player fires over the time interval $[a, 1]$ and the remaining player fires over the same interval $[a, 1]$ and at time 1 with some probability which may be zero. This is a generalization of the results for the two-person silent contest.

4. The Case with Increasing $A_3(x)/A_1(x)A_2(x)$

In this section, we assume that $A_3(x)/A_1(x)A_2(x)$ is monotone increasing. We shall seek for a Nash equilibrium point in the following class of strategies:

(i) Player i (i = 1 and 2) fires his bullet at time x belonging to the time interval $[a, 1]$ according to the density function $f_i(x)$ and at time 1 with probability α_i.

(ii) Player 3 fires his bullet at time x belonging to the time interval $[b, 1]$ where $a < b$ according to the density function $f_3(x)$ and at time 1 with probability α_3.

We have with the normalizing conditions

$$\int_a^1 f_i(x)\,dx + \alpha_i = 1 \qquad \text{for } i = 1 \text{ and } 2$$

and

$$\int_b^1 f_3(x)\,dx + \alpha_3 = 1.$$

As before, we set V_i (i = 1, 2 and 3) be the equilibrium payoff of the game for player i. Then, since player 3 does not fire until time b, we have

(9) $$V_1 = A_1(x)\{1 - \int_a^x A_2(y)f_2(y)\,dy\}$$

and

(10) $$V_2 = A_2(x)\{1 - \int_a^x A_1(y)f_1(y)\,dy\}$$

for $a \leqq x \leqq b$. Therefore, we get

(11) $$V_i = A_i(a) \qquad \text{for } i = 1 \text{ and 2,}$$

and thus,

(12) $$f_1(x) = \frac{A_2(a)A_2'(x)}{A_1(x)A_2(x)^2} \qquad \text{for } a \leqq x \leqq b$$

and

(13) $f_2(x) = \dfrac{A_1(a)A_1'(x)}{A_1(x)^2 A_2(x)}$ for $a \leqq x \leqq b$.

Let a_1 and a_2 be the unique roots of the equations

$$\int_a^b f_1(x)\,dx = 1 \qquad \text{and} \qquad \int_a^b f_2(x)\,dx = 1,$$

respectively. Without loss of generality, we assume $a_1 \geqq a_2$ and we set $a = a_1$ and $\alpha_1 = 0$. Since player 1 has fired his bullet before time b, we have

(14) $V_2 = A_2(x)\{1 - \displaystyle\int_a^b A_1(y)f_1(y)\,dy\}\{1 - \int_b^x A_3(y)f_3(y)\,dy\}$

and

(15) $V_3 = A_3(x)\{1 - \displaystyle\int_a^b A_1(y)f_1(y)\,dy\}\{1 - \int_a^x A_2(y)f_2(y)\,dy\}$

for $b < x < 1$. By means of (11), (12) and (13), (14) and (15) become as follows:

(16) $A_2(b) = A_2(x)\{1 - \displaystyle\int_b^x A_3(y)f_3(y)\,dy\}$

(17) $V_3 = \dfrac{A_2(a)}{A_2(b)} A_3(x)\{\dfrac{A_1(a)}{A_1(b)} - \displaystyle\int_b^x A_2(y)f_2(y)\,dy\}$.

From (16), we have

$$f_3(x) = \frac{A_2(b)A_2'(x)}{A_2(x)^2 A_3(x)} \qquad \text{for } b \leqq x < 1$$

and from (17), we get

$$V_3 = \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)}$$

and

$$f_2(x) = \frac{A_1(a)A_3(b)}{A_1(b)} \frac{A_3'(x)}{A_2(x)A_3(x)^2} \qquad \text{for } b < x < 1.$$

In this case, α_2, α_3, a and b are determined as follows:

$$\int_a^b f_1(x)\,dx = 1,$$

$$\alpha_2 = 1 - \int_a^1 f_2(x)\,dx,$$

$$\alpha_3 = 1 - \int_b^1 f_3(x)\,dx,$$

$$\alpha_2\alpha_3 = 0.$$

We note that the simultaneous equations

$$\begin{cases} \int_a^b f_1(x)\,dx = 1 \\ \int_a^1 f_2(x)\,dx = 1 \end{cases} \quad \text{and} \quad \begin{cases} \int_a^b f_1(x)\,dx = 1 \\ \int_b^1 f_3(x)\,dx = 1 \end{cases}$$

have unique roots a and b.

Theorem 2. Let us suppose that $A_3(x)/A_1(x)A_2(x)$ is increasing over $[0, 1]$. Let a_{ij} and b_{ij} $(1 \leqq i, j \leqq 2, i \neq j)$ be the roots of the simultaneous equation

$$\begin{cases} \int_{a_{ij}}^{b_{ij}} g_{ij}(x)\,dx = \dfrac{1}{A_j(a_{ij})} \\[2mm] \int_{a_{ij}}^{b_{ij}} g_{ji}(x)\,dx + \dfrac{A_3(b_{ij})}{A_i(b_{ij})} \int_{b_{ij}}^1 g_{j3}(x)\,dx = \dfrac{1}{A_i(a_{ij})} \end{cases}$$

and let a_{i3} and b_{i3} $(1 \leqq i \leqq 2)$ be the roots of the simultaneous equation

$$\begin{cases} \int_{a_{i3}}^{b_{i3}} g_{i3-i}(x)\,dx = \dfrac{1}{A_{3-i}(a_{i3})} \\[2mm] \int_{b_{i3}}^1 g_{33-i}(x)\,dx = \dfrac{1}{A_{3-i}(b_{i3})} \end{cases},$$

where

$$g_{ij}(x) = A'_j(x)/A_i(x)A_j(x)^2.$$

Let a be the maximum value of a_{ij} such that $0 < a_{ij} < b_{ij} \leqq 1$ and let b be the corresponding value of b_{ij}. Then a set of equilibrium strategies is given as follows:

(i) If $a = a_{1i}$, then

$$f_1^*(x) = \begin{cases} A_2(a)g_{12}(x), & \text{for } a \leqq x \leqq b, \\ 0, & \text{for } b < x \leqq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} A_1(a)g_{21}(x), & \text{for } a \leqq x \leqq b, \\ \dfrac{A_1(a)A_3(b)}{A_1(b)}\,g_{23}(x), & \text{for } b < x \leqq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} 0, & \text{for } a \leq x \leq b, \\ A_2(b)g_{32}(x), & \text{for } b < x \leq 1, \end{cases}$$

and $\alpha_1^* = 0$. Moreover, if $a = a_{12}$, then $\alpha_2^* = 0$ and $\alpha_3^* \geq 0$ and if $a = a_{13}$, then $\alpha_2^* \geq 0$ and $\alpha_3^* = 0$.

(ii) If $a = a_{2i}$, then

$$f_1^*(x) = \begin{cases} A_2(a)g_{12}(x), & \text{for } a \leq x \leq b, \\ \dfrac{A_2(a)A_3(b)}{A_2(b)} g_{13}(x), & \text{for } b < x \leq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} A_1(a)g_{21}(x), & \text{for } a \leq x \leq b, \\ 0, & \text{for } b < x \leq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} 0, & \text{for } a \leq x \leq b, \\ A_1(b)g_{31}(x), & \text{for } b < x \leq 1, \end{cases}$$

and $\alpha_2^* = 0$. Moreover, if $a = a_{21}$, then $\alpha_1^* = 0$ and $\alpha_3^* \geq 0$ and if $a = a_{23}$, then $\alpha_1^* \geq 0$ and $\alpha_3^* = 0$.

In these two cases, the equilibrium payoff V_i for player i is given as follows:

$$V_1 = A_1(a), \qquad V_2 = A_2(a), \qquad V_3 = \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} .$$

Proof: Without loss of generality, we assume $a = a_{12}$. Let $M_i(x)$ be the expected payoff for player i when he plays a pure strategy x and other two players play the strategies described in Theorem 2. It is easy to see, for $i = 1$ and 2, that

$$M_i(x) = A_i(x) < A_i(a) = V_i \qquad \text{for } 0 \leq x < a.$$

We have shown

$$M_1(x) = A_1(a) = V_1 \qquad \text{for } a \leq x \leq b$$

and

$$M_2(x) = A_2(a) = V_2 \qquad \text{for } a \leq x < 1.$$

Since $A_1(x)/A_2(x)A_3(x)$ is decreasing, we have

$$M_1(x) = A_1(x)\{1 - \int_a^x A_2(y)f_2^*(y)dy\}\{1 - \int_b^x A_3(y)f_3^*(y)dy\}$$

$$= \frac{A_1(x)}{A_2(x)A_3(x)} \frac{A_2(b)A_3(b)}{A_1(b)} A_1(a) \leq A_1(a) = V_1 \qquad \text{for } b < x < 1.$$

Moreover,

$$M_1(1) = \{1 - \int_a^1 A_2(y)f_2^*(y)dy\}\{1 - \int_b^1 A_3(y)f_3^*(y)dy - \alpha_3^*\} \leq A_1(a) = V_1$$

and

$$M_2(1) = \{1 - \int_a^b A_1(y)f_1^*(y)dy\}\{1 - \int_b^1 A_3(y)f_3^*(y)dy - \alpha_3^*\} \leqq A_2(a) = V_2.$$

Thus, we get

$$M_i(x) \leqq A_i(a) = V_i \qquad \text{for } i = 1 \text{ and } 2, \text{ and } 0 \leqq x \leqq 1.$$

We further obtain

$$M_3(x) = A_3(x) \leqq A_3(a) \leqq \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} = V_3 \qquad \text{for } 0 \leqq x \leqq a,$$

and

$$M_3(x) = A_3(x)\{1 - \int_a^x A_1(y)f_1^*(y)dy\}\{1 - \int_a^x A_2(y)f_2^*(y)dy\}$$

$$= A_3(x)\frac{A_1(a)A_2(a)}{A_1(x)A_2(x)} \leqq \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} = V_3 \qquad \text{for } a \leqq x < b$$

since $A_3(x)/A_1(x)A_2(x)$ is increasing. We have shown

$$M_3(x) = \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} = V_3 \qquad \text{for } b \leqq x < 1$$

and it is easy to see

$$M_3(1) = \{1 - \int_a^b A_1(y)f_1^*(y)dy\}\{1 - \int_a^1 A_2(y)f_2^*(y)dy\}$$

$$= \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} = V_3.$$

Thus, we get

$$M_3(x) \leqq \frac{A_1(a)A_2(a)A_3(b)}{A_1(b)A_2(b)} = V_3 \qquad \text{for } 0 \leqq x \leqq 1.$$

This completes our proof.

5. Examples

We give some examples of three-person silent marksmanship contests.

Example 1. Let $A_1(x) = A_2(x) = A_3(x) = x$. Then

$$f_1^*(x) = f_2^*(x) = f_3^*(x) = \frac{1}{2} a^{1/2} x^{-5/2} \qquad \text{for } a \leqq x \leqq 1,$$

where $a = 0.2831$. Moreover, $V_1 = V_2 = V_3 = 0.2831$ and $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$. This coincides with the result by Sakaguchi [5].

Example 2. Let $A_1(x) = A_2(x) = x$ and $A_3(x) = x^{3/2}$. Then

$$f_1^*(x) = f_2^*(x) = \frac{3}{4} a^{3/4} x^{-11/4} \qquad \text{for } a \leqq x \leqq 1$$

and

$$f_3^*(x) = \frac{1}{4} a^{1/4} x^{-11/4} \qquad \text{for } a \leqq x \leqq 1,$$

where $a = 0.3577$. Moreover, $V_1 = V_2 = 0.3577$, $V_3 = 0.2139$, $\alpha_1^* = \alpha_2^* = 0$ and $\alpha_3^* = 0.4427$. (Note: the value of b is 1 in Examples 1 and 2.)

Example 3. If $A_1(x) = x$ and $A_2(x) = A_3(x) = x^{3/2}$, then

$$f_1^*(x) = \begin{cases} ax^{-3}, & a \leqq x \leqq b, \\ 0, & b < x \leqq 1, \end{cases}$$

and

$$f_2^*(x) = f_3^*(x) = \begin{cases} \frac{1}{2} a^{1/2} x^{-3}, & a \leqq x \leqq b, \\ \frac{3}{2} a^{1/2} b x^{-4}, & b < x \leqq 1, \end{cases}$$

where $a = 0.3892$ and $b = 0.8268$. Furthermore, $\alpha_1^* = \alpha_2^* = \alpha_3^* = 0$, $V_1 = 0.3892$ and $V_2 = V_3 = 0.2428$.

Example 4. If $A_1(x) = x$, $A_2(x) = x^{3/2}$ and $A_3(x) = x^2$, then we have

$$f_1^*(x) = \begin{cases} \frac{5}{4} a^{5/4} x^{-13/4}, & a \leqq x \leqq b, \\ 0, & b < x \leqq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} \frac{3}{4} a^{3/4} x^{-13/4}, & a \leqq x \leqq b, \\ 2a^{3/4} b^{5/4} x^{-9/2}, & b < x \leqq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} \frac{1}{4} a^{1/4} x^{-13/4}, & a \leqq x \leqq b, \\ \frac{3}{2} a^{1/4} b^{5/4} x^{-9/2}, & b < x \leqq 1, \end{cases}$$

where $a = 0.4456$ and $b = 0.9154$. Furthermore, we get $\alpha_1^* = \alpha_2^* = 0$ and $\alpha_3^* = 0.4375$, $V_1 = 0.4456$, $V_2 = 0.2975$ and $V_3 = 0.1986$.

Example 5. Let $A_1(x) = x^{1/2}$, $A_2(x) = x$ and $A_3(x) = x^2$. Then

$$f_1^*(x) = \begin{cases} ax^{-5/2}, & a \leqq x \leqq b, \\ 0, & b < x \leqq 1, \end{cases}$$

$$f_2^*(x) = \begin{cases} \frac{1}{2} a^{1/2} x^{-5/2}, & a \leqq x \leqq b, \\ 2a^{1/2} b^{3/2} x^{-4}, & b < x \leqq 1, \end{cases}$$

$$f_3^*(x) = \begin{cases} 0, & a \leqq x \leqq b, \\ bx^{-4}, & b < x \leqq 1, \end{cases}$$

where $a = 0.2975$ and $b = 0.9269$. Moreover, $\alpha_1^* = \alpha_2^* = 0$, $\alpha_3^* = 0.9210$, $V_1 = 0.5454$, $V_2 = 0.2975$ and $V_3 = 0.1562$.

Acknowledgements

References

[1] Fox, M. and Kimeldorf, G.: Noisy Duels. *SIAM Journal on Applied Mathematics*. Vol. 17, No. 2 (1969), 353-361.

[2] Karlin, S.: *Mathematical Methods and Theory in Games, Programming and Economics*, II, Addison-Wesley, Reading, Massachusetts, 1959.

[3] Nash, J.: Non-Cooperative Games. *Annals of Mathematics*. Vol. 54, No. 2 (1951), 286-295.

[4] Restrepo, R.: Tactical Problems Involving Several Actions. *Annals of Mathematics Studies*, No. 39: *Contributions to the Theory of Games*, III (*Eds. Dresher, M., Tucker, A. W. and Wolfe, P.*). Princeton University Press, Princeton, 1957, 313-335.

[5] Sakaguchi, M.: Marksmanship Contests —— Non-Zero-Sum Game of Timing. *Mathematica Japonica*. Vol. 22, No. 5 (1978), 585-596.

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