Journal of the Operations Research
Society of Japan
Vol. 23, No. 3, September 1980

NON-ZERO-SUM GAMES RELATED TO THE SECRETARY PROBLEM

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(Received April 12, 1980)

Abstract This note reanalyzes a non-zero-sum game version of the secretary problem recently treated by Kurano, Nakagami and Yasuda [3], under a modified formulation of the problem. The equilibrium derived differs from the former one and has an interesting assymptotic behavior which reconfirms a main theorem formerly proved by Presman and Sonin [4]. The equilibrium value in the limit is a positive number which is a unique root in (0, 1/2) of a transcendental equation.

1. Introduction and Purpose

In their paper [3], Kurano, Nakagami and Yasuda analysed a two-person game version of the so-called secretary problem in which the situation can be represented as follows: There are two competitors and two identical sets of N applicants, and each player considers his set in the conditions of the well-known secretary problem, taking decisions independently of the other player. As soon as one of the players stops, the other player is informed of this information and he drops out of the game. The goal of each player is to maximize the probability of win, i.e. stopping at the best one in his set of objects. In [3] the authors give a non-zero-sum game formulation and shows that the unique Nash-equibilium strategy for each player has a threshold character, i.e. each player must stop at the arrival of the first candidate (the object which is best among those preceding it) after the m*-th, where m* is the smallest positive integer satisfying $m^*\sum_{j=m^*}^{N-1} j^{-2}<1$.

This result is clearly a non-interesting one in opportunity analysis since both of the threshold point and the equilibrium value tend to zero in the limit as $N \to \infty$.

The purpose of this note is to reanalyze the above problem, showing first that the Nash-equilibrium strategy, which gives an interesting asymptotic result, is obtained under a modified setting of the game. Next we illustrate, for purposes of comparison, with the solutions of other related problems as well as the one in [3]. It is assumed that the reader is familiar with the simplest form of the secretary problem and its solution (see, *e.g.*, Gilbert and Mosteller [2]).

2. A Non-Zero-Sum Game Related to the Secretary Problem

We first state the problem as follows: (1) There are two competitors and two identical sets of N objects, and each player considers his set in the conditions of the well-known secretary problem, taking decisions (*i.e.* continue or stop observing) independently of the other player. (2) If one of the players stops, the other player is *not* informed of this fact and continues playing. We call a "win" for each player the event in which he gets to be the first to stop at the best one in his set of objects. If the two player stop simultaneously at the best one in each player's set of objects, both of them are the winners. (3)The goal of each player in the game is to maximize the probability that he becomes a single winner.

A strategy which prescribes stopping at the first candidate that appears (if any) after the m-th will be called m-level strategy and will be denoted by σ_m. The importance of the m-level strategies for a class of the secretary problems we are discussing about is widely known (see, *e.g.* Presman and Sonin [4], Gaver [1], and Smith [5]).

Suppose that players I and II employ the strategy σ_{m_1} and σ_{m_2} with $m_1 \leqq m_2$. respectively. Let us call the event in which a player stops at the best one in his set of objects, as his sub-win. Then under the above strategy-pair $\sigma_{m_1} - \sigma_{m_2}$, with $m_1 \leqq m_2$, we find that

$$\Pr.\{\text{I's sub-win}\} = \sum_{j=m_1+1}^{N} \frac{m_1}{(j-1)j} \cdot \frac{j}{N} ,$$

and

$$\Pr.\{ (\text{I's sub-win}) \ \& \ (\text{II stops earlier \& subwins}) \}$$

$$= \sum_{j=m_2+1}^{N} \frac{m_1}{(j-1)j} \cdot \frac{j}{N} \sum_{k=m_2+1}^{j-1} \frac{m_2}{(k-1)k} \cdot \frac{k}{N} ,$$

since the probability that the j-th is the first candidate after the m_1-th is $m_1/(j-1)j$. Hence we have

(2.1) $$\Pr.\{\text{I's win}\} = \Pr.\{\text{I's sub-win}\} - \Pr.\{(\text{I's sub-win}) \ \& \ (\text{II stops earlier \& subwins})\}$$

$$= \frac{m_1}{N}\sum_{j=m_1}^{N-1} j^{-1} - \frac{m_1 m_2}{N^2}\sum_{j=m_2}^{N-1} j^{-1}\sum_{k=m_2}^{j-1} k^{-1}.$$

Similarly we can get

(2.2) Pr.{II's win} = Pr.{II's sub-win}

- Pr.{(II's sub-win) & (I stops earlier & subwins)}

$$= \frac{m_2}{N}\sum_{k=m_2}^{N-1} k^{-1} - \frac{m_1 m_2}{N^2}\sum_{k=m_2}^{N-1} k^{-1}\sum_{j=m_1}^{k-1} j^{-1}.$$

Note that the second terms in the r.h.s.'s of (2.1) and (2.2) are not identical.

In order to derive the payoffs of the game we use the following simple procedure: We consider, instead of the time points 1, 2, ..., N, the points 1/N, 2/N, ..., N/N, and take the limit N, m_1, $m_2 \to \infty$, with $m_1/N \to x$, and $m_2/N \to y$. Then we find, in the limit of (2.1) and (2.2),

(2.3) $\Pr.\{\text{I's win}\} = -x \log x - (1/2)xy(\log y)^2$,

(2.4) $\Pr.\{\text{II's win}\} = -y \log y + (1/2)xy(\log y)^2 - (x \log x)(y \log y)$,

both for $0 < x \leqq y < 1$. This result, together with consideration of symmetry in the problem, lead to the payoff functions of the non-zero-sum game on the unit square:

$$(2.5)\quad M_1(\sigma_x,\sigma_y) = \begin{cases} -x \log x - (1/2)xy(\log y)^2, & \text{if } x \leqq y \\ -x \log x + (1/2)xy(\log x)^2 - (x \log x)(y \log y), & \text{if } x > y. \end{cases}$$

$$(2.6)\quad M_2(\sigma_x,\sigma_y) = \begin{cases} -y \log y + (1/2)xy(\log y)^2 - (x \log x)(y \log y), & \text{if } x \leqq y \\ -y \log y - (1/2)xy(\log x)^2, & \text{if } x > y. \end{cases}$$

where σ_x denotes the x-level strategy for player I, and σ_y, for II.

Elementary calculus which we shall omit describing since it is easy but lengthy and not interesting gives the result that (1) there exits a unique equilibrium strategy-pair consisting of the identical level strategies σ_{z*} , where $z* \fallingdotseq 0.29533$ is a unique root of the equation $-\log x = 1 + \frac{1}{2}x(\log x)^2$, or equivalently

(2.7) $-x \log x = 1 - (1 - 2x)^{1/2}$.

and (2) the equilibrium value is equal to z*. This result coincides with an observation deductable from the main theorem proved by Presman and Sonin [4 ; Theorem 1].

3. Remarks

In the final section we shall illustrate, for purposes of comparison, with the solutions of other related problems as well as the one in [3].

3.1. We shall make a revisit to the example treated in [3 ; Example 4.3]. The problem was already described in Section 1. Here note that any player loses his possibility of win as soon as his opponent makes a stop earlier than him. Instead of (2.1) and (2.2), we now get for a strategy-pair $\sigma_{m_1} - \sigma_{m_2}$, with $m_1 \leqq m_2$,

$$\Pr.\{\text{I's win}\} = \Pr.\{\text{I stops earlier \& subwins}\}$$
$$= \sum_{j=m_1+1}^{m_2} \frac{m_1}{(j-1)j}\frac{j}{N} + \sum_{j=m_2+1}^{N} \frac{m_1}{(j-1)j}\frac{j}{N}\frac{m_2}{j-1}$$
$$= \frac{m_1}{N}\sum_{j=m_1}^{m_2-1} j^{-1} + \frac{m_1 m_2}{N}\sum_{j=m_2}^{N-1} j^{-2} ,$$

the sum in the first term being interpreted as zero if $m_1 = m_2$, and

$$\Pr.\{\text{II's win}\} = \Pr.\{\text{II stops earlier \& subwins}\}$$
$$= \sum_{k=m_2+1}^{N} \frac{m_2}{(k-1)k}\frac{k}{N}\frac{m_1}{k-1} = \frac{m_1 m_2}{N}\sum_{k=m_2}^{N-1} k^{-2} .$$

These probabilities become in the limit of N, m_1 $m_2 \to \infty$, with $m_1/N \to x$, and $m_2/N \to y$,

$$\Pr.\{\text{I's win}\} = x \log(y/x) + x(1 - y),$$
$$\Pr.\{\text{II's win}\} = x(1 - y) .$$

both for $0 < x \leqq y < 1$. Thus we have reached to a pair of payoff functions parallel to (2.5) and (2.6),

$$M_1(\sigma_x,\sigma_y) = \begin{cases} x \log(y/x) + x(1-y), & \text{if } x \leqq y \\ y(1-x), & \text{if } x > y \end{cases}$$

$$M_2(\sigma_x,\sigma_y) = \begin{cases} x(1-y), & \text{if } x \leqq y \\ y \log(x/y) + y(1-x), & \text{if } x > y, \end{cases}$$

and differentiation shows that there exists a unique equilibrium point $x^* = y^* = 0$, and the equilibrium value is equal to zero.

Since there remains no possibility of a win for a player if the other player has stopped earlier than him, both of the players compete with the opponent in making the earliest stop, and thus the consequence is the above non-interesting behavior.

3.2. Similar analysis as made in Section 2 can be made for another class of secretary problems. We show one example in which the problem is represented as follows: (1) There are two competitors each of whom faces the identical stopping problem over the Poisson process with the identical arrival rate. For player I the stopping problem is: (1a) Let X_i, i=1,2,..., be a sequence of iid r,v.'s, each with uniform distribution over the unit interval [0, 1]. (1b) X_1, X_2,... appear according to a Poisson process with rate λ, and the decision about the stopping must be made before some fixed moment, unity, say. (1c) The objective is to maximize the probability of stopping at the maximum of r.v.'s. (1d) The decision processes are independent, i.e., the decisions for a player are made independently of the other player. (2) and (3) are the same as those in Section 2. This problem corresponds to a non-zero-sum game version of the problem studied by Gaver [1].

Suppose player I sets aside a time T_1 to look over the field, leaving himself $U_1 = 1 - T_1$ for choice. Let his strategy be to examine all X-values to occur in T_1 and then to stop at the first candidate (i.e., maximum relative value), if any thereafter. This strategy will be called T_1-level strategy and will be denoted by σ_{T_1}. The T_2-level strategy for player II is similarly defined with X-values replaced by Y-values. Let X_0, for player I, be the maximum value appeared in time $[0, T_1]$ and its cdf be denoted by $F_0(x)$. Then

$$F_0(x) = P_r\,\{\text{no arrival greater than x, in } [0, T_1]\} = e^{-\lambda T_1(1-x)}.$$

Y_0 and its cdf $G_0(y) = e^{-\lambda T_2(1-y)}$ can similarly be defined.

Under the strategy-pair $\sigma_{T_1} - \sigma_{T_2}$, with $T_1 \leqq T_2$ we can find that

$$\text{Pr. }\{\text{I's subwin} \mid X_0=x\} = \int_0^{U_1} e^{-\lambda t(1-x)}\lambda dt \int_x^1 e^{-\lambda(U_1-t)(1-z)}dz$$
$$= \int_0^{U_1} \frac{e^{-\lambda t(1-x)} - e^{-\lambda U_1(1-x)}}{U_1 - t}\,dt$$

the integrand being Pr.{I's sub-win by stopping at $T_1+t \mid X_0=x$}, and

(3.1) $$\text{Pr.}\{(\text{I's subwin})\&(\text{II stops earlier \& subwins}) \mid X_0=x, Y_0=y\}$$
$$= \int_{T_2-T_1}^{U_1} \frac{e^{-\lambda s(1-x)} - e^{-\lambda U_1(1-x)}}{U_1 - s}\,ds \int_0^{s-T_2+T_1} \frac{e^{-\lambda u(1-y)} - e^{-\lambda U_2(1-y)}}{U_2 - u}\,du.$$

Taking expectations with respect to X_0 and Y_0, gives

$$\text{Pr.}\{\text{I's subwin}\} = T_1\int_0^{U_1} \{\frac{1}{T_1+t}(1-e^{-\lambda(T_1+t)}) - (1-e^{-\lambda})\}\,\frac{dt}{U_1-t},$$

and

$$\text{Pr.}\{(\text{I's subwin})\&(\text{II stops earlier \& subwins})\}$$
$$= T_1\int_{T_2-T_1}^{U_1} \{\frac{1}{T_1+s}(1-e^{-\lambda(T_1+s)}) - (1-e^{-\lambda})\}\,\frac{ds}{U_1-s}$$

$$\cdot\ T_2\int_0^{s-T_2+T_1}\{\frac{1}{T_2+u}(1-e^{-\lambda(T_2+u)})-(1-e^{-\lambda})\}\frac{du}{U_2-u}\ .$$

In the event that opportunities come thick and fast, i.e., as $\lambda\to\infty$,

these probabilities become $T_1\int_0^{U_1}\frac{dt}{T_1+t} = -T_1\log T_1$ and

(3.2) $$T_1T_2\int_{T_2-T_1}^{U_1}\frac{ds}{T_1+s}\int_0^{s-T_2+T_1}\frac{du}{T_2+u} = (\tfrac{1}{2})T_1T_2(\log T_2)^2,$$

respectively, and hence we have, for $0 \leqq T_1 \leqq T_2 \leqq 1$,

(3.3) $$\lim_{\lambda\to\infty}\Pr\{\text{I's win}\} = -T_1\log T_1 - (1/2)T_1T_2(\log T_2)^2.$$

Also we can similarly show that

$\Pr.\{(\text{II's subwin})\&(\text{I stops earlier \& subwins})\mid X_0=x,\ Y_0=y\}$

$$=\int_0^{U_2}\frac{e^{-\lambda t(1-y)}-e^{-\lambda U_2(1-y)}}{U_2-t}\,dt\int_0^{t+T_2-T_1}\frac{e^{-\lambda u(1-x)}-e^{-\lambda U_1(1-x)}}{U_1-u}\,du,$$

$$\lim_{\lambda\to\infty}\Pr.\{(\text{II's subwin})\&(\text{I stops earlier \& subwins})\} = -(1/2)T_1T_2(\log T_2)^2+(T_1\log T_1)(T_2\log T_2),$$

and

(3.4) $$\lim_{\lambda\to\infty}\Pr.(\text{II's win})=-T_2\log T_2+(1/2)T_1T_2(\log T_2)^2-(T_1\log T_1)(T_2\log T_2),$$

instead of (3.1), (3.2), and (3.3) respectively.

Equations (3.3) and (3.4) are the reproductions of (2.3) and (2.4), with x and y, replaced by T_1 and T_2, respectively, so that we again have the pair of payoff functions (2.5)-(2.6), for the non-zero-sum game we have considered.

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