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FINDING THE WEIGHTED MINIMAX FLOW IN A POLYNOMIAL TIME

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Abstract We give a procedure of solving the weighted minimax flow problem in a polynomial time. The procedure utilizes the capacity modification technique and the binary search method.

1. INTRODUCTION

We have considered variants of the maximum flow problems [5,6]. [5] discusses a minimax flow, i.e., the flow which minimizes the maximum arc-flow value among all flows of maximum flow value. [6] treats a weighted minimax flow (minimax cost flow). A minimax flow is obtained in $O(m_0 n^3)$ time where n is the number of nodes in a network and m_0 is the number of arcs across the minimum cut in the network [5]. However the algorithm developed by the authors is not polynomial-bounded for the weighted minimax flow problem [6].

The objective in this paper is to develop a polynomial-bounded algorithm for the weighted minimax flow problem.

The (weighted) minimax flow is desirable in the sense that the flow runs through the network fairly or equitably with respect to arcs. For the application of the weighted minimax flow to the sharing problem, see [7].

With respect to other network flow problems related to minimax flows we have the time transportation problem [1, 3, 4, 11] and the storage management problem [10].

However the time transportation problem is discussed only in the context of a bipartite graph and the storage management problem is only in the context of a linear graph (i.e., one which can be drawn in a one-dimentional space), in addition both objective functions are optimized considering only arc weights. On the other hand, our minimax flow problem is generalized in the following two points.

The first one is that a network considered is general. The second is that we take into account not only the arc weight but also the arc-flow value in the objective function.

2. WEIGHTED MINIMAX FLOW PROBLEM

Let $G=(N,A)$ be a network where N is the set of nodes and A is the set of directed arcs connecting nodes. Let $s\in N$ be a source and $t\in N$ be a sink. Each $(i,j)\in A$ has a positive capacity $c(i,j)$ and a positive weight $w(i,j)$. A flow is denoted by f. Given a flow f, we refer to $f(i,j)$ as the arc flow $f(i,j)$ or the flow in arc (i,j). We assume for simplicity that source s has no incoming arcs and that sink t has no outgoing arcs in G. In addition we assume that all $c(i,j)$, $w(i,j)$ and $f(i,j)$ are integers.

The weighted minimax flow problem is:

$$\min[\max_{(i,j)\in A} w(i,j)f(i,j)] \tag{1}$$

s.t.

$$\sum_{(i,j)\in A} f(i,j)=\sum_{(j,i)\in A} f(j,i) \quad , j\neq s,t \tag{2}$$

$$\sum_{(s,i)\in A} f(s,i)=v^* \tag{3}$$

$$\sum_{(i,t)\in A} f(i,t)=v^* \tag{4}$$

$$0\le f(i,j)\le c(i,j) \quad , (i,j)\in A \tag{5}$$

where v^* is the value of the maximum flow from source s to sink t.

For solving the above problem we use the capacity modification technique proposed in [5] instead of the minimax cost path in [6].

For a nonnegative variable D, we define the new capacity $c'(i,j)$ of arc (i,j) as follows:

$$c'(i,j)=\min(\lfloor D/w(i,j)\rfloor, c(i,j)), \quad (i,j)\in A \tag{6}$$

where $\lfloor x\rfloor$ denotes the largest integer y that satisfies $y\le x$. The capacity of a cut $(X,\bar{X})$, denoted by $c(X,\bar{X})$, where X denotes a subset of N and $\bar{X}$ denotes the complement of X in N, is the sum of the capacities of the arcs across the cut $(X,\bar{X})$;

$$c(X,\overline{X})=\sum_{(i,j)\in(X,\overline{X})} c(i,j).$$

By introducing the new capacity we have the new value of the capacity of the cut $(X,\overline{X})$ as follows:

$$c(X,\overline{X})=\sum_{(i,j)\in(X,\overline{X})} c'(i,j).$$

For any cut $(X,\overline{X})$ we have $c(X,\overline{X})\geq c'(X,\overline{X})$ and from the famous <u>max-flow min-cut theorem</u> [1] we also have $v^*=\min c(X,\overline{X})$. We set $F(D)=\min_{(X,\overline{X}) \text{ in all cuts}} c'(X,\overline{X})$. Let D^* be the minimum value of D satisfying $F(D)=v^*$. Since we have $D^*\geq f(i,j)w(i,j)$ for any arc $(i,j)\in A$, D^* is the optimal value of (1), or of the weighted minimax flow problem. Thus our task is to find the value of D^*. We will find D^* by the binary search [9].

3. DETERMINING THE OPTIMAL VALUE

Consider the network with capacities $c'(i,j)$ instead of $c(i,j)$. The new network is denoted by $G(D)$. Note, if D is infinite, then $G(D)$ and G are identical. Let $v(D)$ be the value of the maximum flow in $G(D)$. If the value $v(D)$ is strictly less than v^*, then the value D is smaller than D^*. Otherwise, D is equal to or larger than D^*. Define c and w such that $c(i,j)\leq c$ and $w(i,j)\leq w$ for any arc (i,j) in A. Then we have $D^*\leq\max_{(i,j)\in A} c(i,j)w(i,j)\leq cw$. Fow two real-valued D' and D'' such that $D'<D^*\leq D''$, where we note $v(D')<v^*$ and $v(D'')=v^*$, define the interval $I=D''-D'$. If $I\leq 1$, then we have $D^*=\lfloor D''\rfloor$ from the definition of $c'(i,j)$ of expression (6). At the same time the maximum flow in $G(D'')$ gives the weighted minimax flow in G. Let I_p denote the interval after applying the binary search p times. Then we have $I_p\leq cw/2^p$. If $p\geq\log(cw)$, then we have $I_p\leq 1$. Define $p=\lceil\log(cw)\rceil$ and $n=|N|$ where $\lceil x\rceil$ denotes the smallest integer y that satisfies $y\geq x$. After the maximum flows are found p times, D^* is obtained. Since one maximum flow in $G(D)$ is found in $O(n^3)$ time [8], D^* is obtained in $O(Pn^3)$time. Hence (see [9, p.7, p.157]) the weighted minimax flow problem, as well as the unweighted minimax flow problem, is polynomial-bounded.

If we redefine the value of P as follows:

$$P=\lceil\log(\max_{(i,j)\in A} c(i,j)w(i,j)\rceil,$$

then we can obtaine D^* faster.

REFERENCES

[1] Burkard, R.E., : A General Hungarian Method for the Algebraic Transportation Problem. *Discrete Mathematics* , 22 (1978), 219-232.

[2] Ford, L.R.Jr., and Fulkerson, D.R. : *Flows in Networks* , Princeton University Press, Princeton, 1962.

[3] Garfinkel, R.S. and Rao, M.R., : The Bottleneck Transportation Problem, *Naval Res. Logist. Quart.*, 18 (1971), 465-472.

[4] Hammer, P.L., : Time Minimizing Transportation Problems, *Naval Res. Logist. Quart.*, 16 (1969), 345-357.

[5] Ichimori, T., Ishii, H., and Nishida, T., : A Minimax Flow Problem, *Math. Japonica*, 24 (1979), 1, 65-71.

[6] Ichimori, T., Murata, M., Ishii, H., and Nishida, T., : Minimax Cost Flow Problem, *Technol. Repts. of the Osaka Univ.*, 30 (1980), 39-44.

[7] Ichimori, T., Ishii, H., and Nishida, T., : *Optimal Sharing*, (Submitted).

[8] Karzanov, A.V., : Determining the Maximal Flow in a Network by the Method of Preflow. *Soviet Math. Dokl.*, 15 (1974), 2, 434-437.

[9] Lawler, E.L., : *Combinatorial Optimization;networks and matroids*, Holt, Rinehart and Winston, New York, 1970.

[10] Stanat, D.F. and Magó, G.A., : Minimizing Maximum Flows in Linear Graphs, *Networks*, 9 (1979), 333-361.

[11] Swarc, W., : Some Remarks on the Time Transportation Problem, *Naval Res. Logist. Quart.*, 18 (1971), 475-485.

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