Journal of the Operations Research
Society of Japan
Vol. 23, No. 3, September 1980

A SIMPLEX PROCEDURE FOR A FIXED CHARGE PROBLEM

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(Received August 16, 1979; Final May 6, 1980)

Abstract An approximate solution method for solving the optimization problem which contains semi-fixed costs represented as a lower semi-continuous step function is developed. The fundamental idea of the algorithm is based on the simplex procedure of linear programming. We define the decrease in the objective function considering twice pivot calculations, and preparing two kinds of simplex tableau we propose the computational procedure to systematically obtain the approximate solution. Also some properties of the pivot calculations are theoretically analyzed. Finally some numerical examples are solved to illustrate the procedure and to test the effectiveness of the algorithm.

1. Introduction

In this paper, we develop an algorithm for solving the optimization problem which contains semi-variable costs represented as a piecewise linear function shown in Figure 1 and semi-fixed costs represented as a lower semi-continuous step function shown in Figure 2.

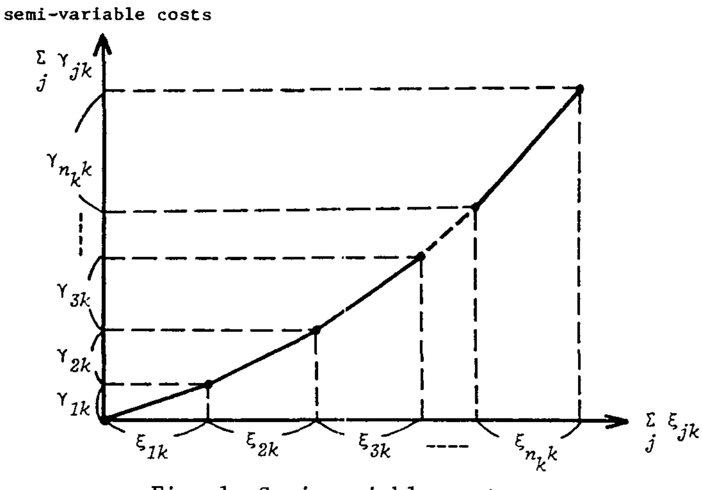

Fig. 1 Semi-variable costs

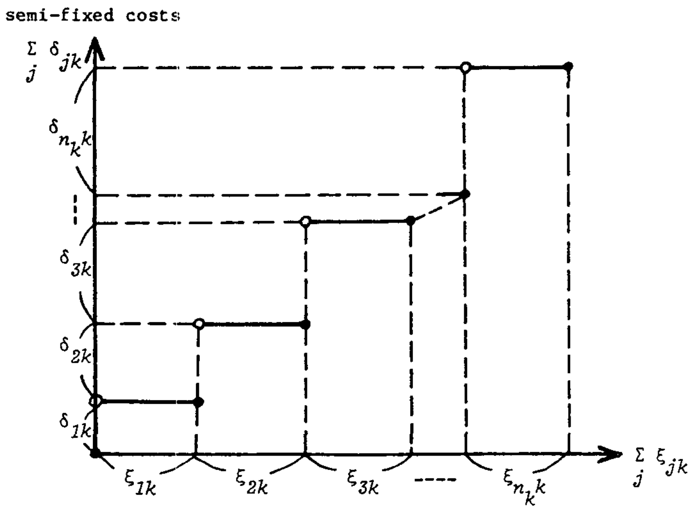

Fig. 2 Semi-fixed costs

Consider the problem of minimizing

(1) $$F = \sum_{k=1}^{l} c_k' x_k + \sum_{k=1}^{l} d_k' [x_k]$$

subject to

(2) $$\sum_{k=1}^{l} A_k x_k \geqq b, \qquad 0 \leqq x_k \leqq u \qquad (k=1,---,l), \text{ and}$$

(3) $$\begin{cases} \text{if } \xi_{jk} > 1, \text{ then } \xi_{ik} = 1 \text{ for } i=1,---,j-1 \\ \text{if } \xi_{jk} = 0, \text{ then } \xi_{ik} = 0 \text{ for } i=j+1,---,n_k \end{cases} \qquad (k=1,---,l),$$

where

$$A_k = \begin{pmatrix} a_{1k} & --- & a_{n_k k} \end{pmatrix}, \; a_{jk} = \begin{pmatrix} \alpha_{1jk} \\ \vdots \\ \alpha_{mjk} \end{pmatrix}, \; b = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}, \; c_k = \begin{pmatrix} \gamma_{1k} \\ \vdots \\ \gamma_{n_k k} \end{pmatrix}, \; d_k = \begin{pmatrix} \delta_{1k} \\ \vdots \\ \delta_{n_k k} \end{pmatrix}, \; u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$x_k = \begin{bmatrix} \xi_{1k} \\ \vdots \\ \xi_{n_k k} \end{bmatrix}, \quad [x_k] = \begin{bmatrix} [\xi_{1k}] \\ \vdots \\ [\xi_{n_k k}] \end{bmatrix}, \text{ and } [\xi_{jk}] = \begin{cases} 1 & \text{if } \xi_{jk} > 0 \\ 0 & \text{if } \xi_{jk} = 0 \end{cases} \quad (j=1,---,n_k;\ k=1,---,l).$$

The constraint (3) means that variable ξ_{ik} $(i=1,---,j-1)$ have to take the value 1 if the variable ξ_{jk} takes a positive value and ξ_{ik} $(i=j+1,---,n_k)$ have to take the value 0 if ξ_{jk} takes 0. Also it is assumed that it holds

$$\begin{aligned} &\alpha_{ijk} \geqq \alpha_{ij-1k} && (i=1,---,m\ ;\ j=2,---,n_k;\ k=1,---,l) \\ (4)\qquad &\gamma_{jk} \geqq \gamma_{j-1k} && (j=2,---,n_k;\ k=1,---,l) \\ &\delta_{jk} \geqq \delta_{j-1k} && (j=2,---,n_k;\ k=1,---,l). \end{aligned}$$

The prime represents the transposition of vectors.

Originally this problem appeared when determining the production planning for the mixed-model assembly line production system [1]. In [1], the problem is formulated as a kind of separable programming which minimizing the objective function constructed from the sum of a convex function and a kind of step function under the constraints of linear inequalities. Approximating the convex function as a piecewise linear function and generalizing the problem, we have relations from (1) to (4). The problem of minimizing (1) subject to (2) and (3) is considered to be a kind of the fixed charge problem and, introducing 0-1 variables, we can treat this problem as a mixed-integer programming problem [3]. Also an algorithm which is based upon a branch and bound method is presented for the general fixed charge problem [4].

In this paper, we will attempt to solve the problem (1)-(3) by means of the simplex method. Though some approximate solution methods using the simplex method have been proposed for the fixed charge problem [2,6,7], we will derive an another approximate algorithm from the different point of view making use of following properties of the problem, that is,

(a) From the assumption (4), for the problem of minimizing only the first term of the objective function (1) subject to (2) and (3), we can carry out the ordinary calculations of the simplex algorithm without considering the restriction (3) and, in optimal state, the restriction (3) is automatically satisfied [5].

(b) From the restriction (3), for the problem (1)-(3), we know that if the variable ξ_{jk} is a basis and it holds $0 < \xi_{jk} < 1$, then ξ_{j+1k} is the only candidate variable which enters into the basis and ξ_{ik} $(i=j+2,---,n_k)$ must not enter into the basis before ξ_{j+1k}. Also, if $0 < \xi_{jk} < 1$, then ξ_{jk} is the only candidate variable which moves to the nonbasis and ξ_{ik} $(i=1,---,j-1)$ must not move to the nonbasis before ξ_{jk}.

The algorithm proposed is essentially constructed with two phases.

(a) First, without considering fixed charges, ordinary simplex calculations are carried out to obtain the initial feasible solution.

(b) Next, considering fixed charges, twice pivot calculations method are carried out to search for a better extreme point assuring the feasibility and monotone decreasing.

2. Preparations for the Algorithm

2.1 Definitions of the sets

Introducing slack variables y and z_k $(k=1,---,l)$, we can represent inequalities (2) as follows:

(5) $$\sum_{k=1}^{l} A_k x_k - y = b$$

(6) $$x_k + z_k = u \qquad (k=1,---,l)$$

(7) $x_k, y, z_k \geqq 0 \quad (k=1,---,l),$

where $y = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$, and $z_k = \begin{bmatrix} \zeta_{1k} \\ \vdots \\ \zeta_{n_k k} \end{bmatrix}$.

Let define the function G as follows:

(8) $G = \sum_{k=1}^{l} c_k' x_k.$

We will call the linear programming problem of minimizing (8) subject to (5), (6) and (7) *Problem A* and the fixed charge problem of minimizing (1) subject to (5), (6) and (7) *Problem B* hereafter.

Let X be the set of ξ_{jk} $(j=1,---,n_k;\ k=1,---,l)$, Y be the set of η_i $(i=1, ---,m)$ and Z be the set of ζ_{jk} $(j=1,---,n_k;\ k=1,---,l)$. At the any step of the simplex iteration, we define the sets of variables as follows:

$X_B^1 = \{\xi_{jk} \mid \xi_{jk} = 1,\ \xi_{jk} \in X\},$

$X_B^2 = \{\xi_{jk} \mid 0 < \xi_{jk} < 1,\ \xi_{jk} \in X\},$†

$X_N = X - (X_B^1 \cup X_B^2),$

Y_B = the set of basic variables of $\eta_i \in Y$,

$Y_N = Y - Y_B,$

† If there exists a basic variable which takes $\xi_{jk} = 0$ (that is, in the presence of degeneracy), we replace it by the nonbasic variable $\eta_i \in Y_N$. This pivot calculation is always attainable. Let proof this fact. Denote the coefficient matrix for the set of nonbasic variable Y_N as Y_N. Since the inverse matrix Λ^{-1} of the coefficient matrix Λ for the basic variable vector w_B defined as (11) is represented as (12) mentioned in chapter 3, the coefficient matrix Y_N^* of the current simplex tableau becomes as follows: ↘

$Z_B^1 = \{\zeta_{jk} \mid \zeta_{jk} = 1,\ \zeta_{jk} \in Z\}$,

Z_B^2 = the set of basic variables which take $0 \leqq \zeta_{jk} < 1,\ \zeta_{jk} \in Z$,

$Z_N = Z - (Z_B^1 \cup Z_B^2)$,

$W_B = X_B^1 \cup X_B^2 \cup Y_B \cup Z_B^1 \cup Z_B^2$,

and $W_N = X_N \cup Y_N \cup Z_N$.

2.2 Definitions of the decrease in the objective function

Let denote the variable which enters into the basis as $\omega_t^1 \in W_N$ and the variable which moves to the nonbasis as $\omega_s^1 \in W_B$. Then from the theory of linear programming, the variation ΔH_1 in the objective function of *Problem B* becomes as follows:

(9) $$\Delta H_1 = (\gamma_t^1 - \pi_t^1)\, \beta_s^{1*} / \sigma_{st}^{1*} + (\delta_t^1 - \delta_s^1),$$

where $(\gamma_t^1 - \pi_t^1)$ is the simplex criterion of the nonbasic variable ω_t^1, β_s^{1*} is the value of the basic variable ω_s^1, and σ_{st}^{1*} is the value of the pivot element. The asterisk represents the value of the current simplex tableau. δ_t^1 and δ_s^1 are

---

$$Y_N^* = \Lambda^{-1} Y_N = \begin{array}{c} \left[\begin{array}{c|c|c|c|c} P^{-1} & -P^{-1}Q & 0 & 0 & 0 \\ \hline 0 & E & 0 & 0 & 0 \\ \hline RP^{-1} & S-RP^{-1}Q & -E & 0 & 0 \\ \hline -P^{-1} & P^{-1}Q & 0 & E & 0 \\ \hline 0 & 0 & 0 & 0 & E \end{array}\right] \left[\begin{array}{c} -E \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \end{array}\right] \\ \begin{array}{ccccccc} m_1 & m_2 & m-m_1 & m_1 & n_0-m_1-m_2 & & m_1 \end{array} \end{array} = \begin{array}{cc} \left[\begin{array}{c} -P^{-1} \\ \hline 0 \\ \hline -RP^{-1} \\ \hline P^{-1} \\ \hline 0 \end{array}\right] & \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m-m_1 \\ \}\, m_1 \\ \}\, n_0-m_1-m_2 \end{array} \\ m_1 & \end{array}$$

Hence we know that there exists at least one nonzero element for $\eta_i \in Y_N$ corresponding to the basic variable which takes $\xi_{jk} = 0$.

fixed charges of ω_t^1 and ω_s^1, respectively. Let assume that, after we replace $\omega_s^1 \in W_B$ by $\omega_t^1 \in W_N$, we choose the variable which enters into the basis as $\omega_t^2 \in W_N - \{\omega_s^1\}$ and the variable which moves to the nonbasis as $\omega_s^2 \in W_B - \{\omega_t^1\}$. Then the variation ΔH_2 in the objective function becomes as follows:

$$(10) \qquad \Delta H_2 = \Delta H_1 + (\gamma_t^2 - \pi_t^2)\ \beta_s^{2*} \ / \ \sigma_{st}^{2*} + (\delta_t^2 - \delta_s^2),$$

where $(\gamma_t^2 - \pi_t^2)$, β_s^{2*} and σ_{st}^{2*} are defined as similar values as $(\gamma_t^1 - \pi_t^1)$, β_s^{1*} and σ_{st}^{1*} corresponding to ω_t^2 and ω_s^2. We can easily calculate ΔH_1 and ΔH_2 if ω_s^1, ω_t^1, ω_s^2 and ω_t^2 are determined. Then we define the decrease in the objective function of *Problem B* as follows:

Definition. We define that the objective function of *Problem B* decreases if it holds either

(a) $\Delta H_1 < 0$ or

(b) $\Delta H_1 \geqq 0$ and $\Delta H_2 < 0$.

3. Meaningful Pivit Calculations

When we choose the variable which enters into the basis as $\omega_t^1 \in W_N$ and the variable which moves to the nonbasis as $\omega_s^1 \in W_B$ for the first pivot calculation, we can formally consider fifteen cases as the combination of ω_s^1 and ω_t^1, that is,

1. $\omega_s^1 \in X_B^1$, $\omega_t^1 \in X_N$, 2. $\omega_s^1 \in X_B^2$, $\omega_t^1 \in X_N$, 3. $\omega_s^1 \in Y_B$, $\omega_t^1 \in X_N$,

4. $\omega_s^1 \in Z_B^1$, $\omega_t^1 \in X_N$, 5. $\omega_s^1 \in Z_B^2$, $\omega_t^1 \in X_N$, 6. $\omega_s^1 \in X_B^1$, $\omega_t^1 \in Y_N$,

7. $\omega_s^1 \in X_B^2$, $\omega_t^1 \in Y_N$, 8. $\omega_s^1 \in Y_B$, $\omega_t^1 \in Y_N$, 9. $\omega_s^1 \in Z_B^1$, $\omega_t^1 \in Y_N$,

10. $\omega_s^1 \in Z_B^2$, $\omega_t^1 \in Y_N$, 11. $\omega_s^1 \in X_B^1$, $\omega_t^1 \in Z_N$, 12. $\omega_s^1 \in X_B^2$, $\omega_t^1 \in Z_N$,

13. $\omega_s^1 \in Y_B$, $\omega_t^1 \in Z_N$, 14. $\omega_s^1 \in Z_B^1$, $\omega_t^1 \in Z_N$, 15. $\omega_s^1 \in Z_B^2$, $\omega_t^1 \in Z_N$.

Also we can formally consider fifteen cases as the combination of ω_s^2 and ω_t^2 for the second pivot calculation for each case of the first pivot calculation. But as soon seen, those combination mentioned above contain the cases which need not consider. The cases 1, 6, 9 and 14 never occur. Let proof these facts. Let denote the vector of variable ξ_{jk} which belongs to the set X_B^1 as x_B^1. Also we define the vectors x_B^2, y_B, z_B^1 and z_B^2 as the same manner as x_B^1. Let define the vector w_B as follows:

$$\text{(11)} \qquad w_B = m+n_0 \left(\begin{array}{c} x_B^2 \\ \hline x_B^1 \\ \hline y_B \\ \hline z_B^2 \\ \hline z_B^1 \end{array} \right) \begin{array}{l} m_1 \\ m_2 \\ m-m_1, \\ m_1 \\ n_0-m_1-m_2 \end{array} \text{where } n_0 = \sum_{k=1}^{l} n_k.$$

Then the coefficient matrix Λ for the vector w_B can be represented as follows:

$$\Lambda = m+n_0 \begin{array}{c} \overbrace{}^{m+n_0} \\ \left(\begin{array}{c|c|c|c|c} P & Q & 0 & 0 & 0 \\ \hline 0 & E & 0 & 0 & 0 \\ \hline R & S & -E & 0 & 0 \\ \hline E & 0 & 0 & E & 0 \\ \hline 0 & 0 & 0 & 0 & E \end{array} \right) \\ \begin{array}{ccccc} m_1 & m_2 & m-m_1 & m_1 & n_0-m_1-m_2 \end{array} \end{array} \begin{array}{l} m_1 \\ m_2 \\ m-m_1, \\ m_1 \\ n_0-m_1-m_2 \end{array}$$

where E and 0 represent the unit matrix and the zero matrix, respectively. Assuming that the squar matrix P is regular, we have the inverse matrix Λ^{-1} of Λ as follows:

$$(12)\qquad \Lambda^{-1} = m+n_0 \begin{array}{cl} \overbrace{\left(\begin{array}{c:c:c:c:c} P^{-1} & -P^{-1}Q & 0 & 0 & 0 \\ \hdashline 0 & E & 0 & 0 & 0 \\ \hdashline RP^{-1} & S-RP^{-1}Q & -E & 0 & 0 \\ \hdashline -P^{-1} & P^{-1}Q & 0 & E & 0 \\ \hdashline 0 & 0 & 0 & 0 & E \end{array}\right)}^{m+n_0} & \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m-m_1 \\ \}\, m_1 \\ \}\, n_0-m_1-m_2 \end{array} \\ \begin{array}{ccccc} m_1 & \quad m_2 & \quad m-m_1 & \quad m_1 & \quad n_0-m_1-m_2 \end{array} & \end{array}$$

From (12), we know that there exist no nonzero elements for $\eta_i \in Y_N$ corresponding to $\xi_{jk} \in X^1_B$. Also there exist no nonzero elements for $\eta_i \in Y_N$ corresponding to $\zeta_{jk} \in Z^1_B$ and for $\zeta_{jk} \in Z_N$ corresponding to $\zeta_{jk} \in Z^1_B$. Hence we cannot replace $\xi_{jk} \in X^1_B$ by $\eta_i \in Y_N$ (the case 6), $\zeta_{jk} \in Z^1_B$ by $\eta_i \in Y_N$ (the case 9) and $\zeta_{jk} \in Z^1_B$ by $\zeta_{jk} \in Z_N$ (the case 14). For the purpose of proving that the case 1 never occurs, let denote the column vector of coefficients of the variable $\xi_{jk} \in X_N$ as P_{jk}. Then P_{jk} is represented as follows:

$$P_{jk} = m+n_0 \left(\begin{array}{c} p \\ \hdashline 0 \\ \hdashline q \\ \hdashline 0 \\ \hdashline r \end{array}\right) \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m-m_1 \\ \}\, m_1 \\ \}\, m_0-m_1-m_2 \end{array}$$

As the column vector P^*_{jk} of the current simplex tableau becomes $\Lambda^{-1}P_{jk}$, P^*_{jk} is represented as follows:

$$(13)\qquad P^{*}_{jk} = \Lambda^{-1}P_{jk} = m + n_0 \left(\begin{array}{c} P^{-1}p \\ \hdashline 0 \\ \hdashline RP^{-1}p - q \\ \hdashline -P^{-1}p \\ \hdashline r \end{array}\right) \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \}\, m - m_1 \\ \}\, m_1 \\ \}\, n_0 - m_1 - m_2 \end{array}$$

From (13), we know that there exist no nonzero elements for $\xi_{jk} \varepsilon X_N$ corresponding to $\xi_{jk} \varepsilon X_B^1$. Hence we cannot replace $\xi_{jk} \varepsilon X_B^1$ by $\xi_{jk} \varepsilon X_N$. Thus we know that the cases 1, 6, 9 and 14 never occur.

And yet, for the case 7, 11 and 12, we know that it is enough to investigate the value ΔH_1. Therefore we may consider the combinations of remaining eight cases for the twice pivot calculations. But if the basic variable which moves to the nonbasis in the second iteration does not concern the fixed charge, it is meaningless for the purpose of decrease in the objective function. After the consideration of these facts, meaningful twice pivot calculations in our algorithm become as Table 1.

Table 1. Meaningful pivot calculations

the first pivot calculation			the second pivot calculation		
1.	$\omega_s \varepsilon X_B^2$,	$\omega_t \varepsilon X_N$	1.	$\omega_s \varepsilon X_B^2$,	$\omega_t \varepsilon X_N$
2.	$\omega_s \varepsilon Y_B$,	$\omega_t \varepsilon X_N$	2.	$\omega_s \varepsilon X_B^2$,	$\omega_t \varepsilon Y_N$
3.	$\omega_s \varepsilon Z_B^1$,	$\omega_t \varepsilon X_N$	3.	$\omega_s \varepsilon X_B^1$,	$\omega_t \varepsilon Z_N$
4.	$\omega_s \varepsilon Z_B^2$,	$\omega_t \varepsilon X_N$	4.	$\omega_s \varepsilon X_B^2$,	$\omega_t \varepsilon Z_N$
5.	$\omega_s \varepsilon Y_B$,	$\omega_t \varepsilon Y_N$			
6.	$\omega_s \varepsilon Z_B^2$,	$\omega_t \varepsilon Y_N$			
7.	$\omega_s \varepsilon Y_B$,	$\omega_t \varepsilon Z_N$			
8.	$\omega_s \varepsilon Z_B^1$,	$\omega_t \varepsilon Z_N$			

In Table 1, when we choose the variable $\omega_t \varepsilon X_N$ to enter the basis for the first or the second pivot calculation, it is apparent that we should only consider $\xi_{j^*+1k} \varepsilon X_N$ as the candidate variable of ω_t for each k $(k=1,---,l)$, where we assume that $\xi_{j^*k} \varepsilon (X_B^1 \cup X_B^2)$ $(k=1,---,l)$. Also, when we choose the variable $\omega_t \varepsilon Z_N$ to enter the basis, we should only consider $\zeta_{j^*-1k} \varepsilon Z_N$ as the candidate variable of ω_t for each k $(k=1,---,l)$, where we assume that $\zeta_{j^*k} \varepsilon (Z_B^1 \cup Z_B^2)(k=1,---,l)$. Moreover, if a basic variable $\xi_{j^*k^*} \varepsilon (X_B^1 \cup X_B^2)$ is chosen to be replaced by $\xi_{hk^*} \varepsilon X_N$ $(h=j^*+1,---,n_k)$, it is apparent that the objective function does not decrease. So we may delete such cases in the pivot calculations. Also we may delete to replace a basic variable $\zeta_{j^*k^*} \varepsilon (Z_B^1 \cup Z_B^2)$ by $\zeta_{hk^*} \varepsilon Z_N$ $(h=1,---,j^*-1)$.

4. Algorithm

In this section, we will propose the computational procedure to solve the fixed charge problem defined as *Problem B*. The fundamental idea is based on the simplex method of linear programming. Though this algorithm seems to resemble the heuristic method proposed by Steinberg [6] and Walker [7], it is slightly different from [6] and [7] in respect of selecting the pivot element by utilizing properties of the problem. We prepare two kinds of simplex tableau for the algorithm, that is, *Simplex Tableau 1 (ST1)* and *Simplex Tableau 2 (ST2)*. We use *ST1* for the pivot calculations when variables to enter the basis and move to the nonbasis are determined. On the other hand we use *ST2* only for the purpose of calculation of ΔH_2. The basic computational procedure is constructed from *Step 1* to *Step 14*. We use *ST1* from *Step 1* to *Step 6* and *Step 13* to *Step 14*, *ST2* from *Step 7* to *Step 12* in the algorithm.

Step 1. Solve *Problem A* by using ordinary simplex method.

Step 2. Set $j \leftarrow 1$, $F \leftarrow \infty$, and $\mu \leftarrow 1$.

Step 3. If $\omega_j \in W_B$, go to *Step 13*. Otherwise determine the basic variable $\omega_{i_1} \in W_B$ according to (14),

$$\theta_{i_1} = \min_{1 \leq i \leq m+n_0} \{\theta_i = \beta_i^{1*} / \sigma_{ij}^{1*}, \text{ for } \sigma_{ij}^{1*} > 0\} \tag{14}$$

where σ_{ij}^{1*} is the (i,j) element of current $ST1$.

Step 4. Calculate the value ΔH_1 by (15), that is,

$$\Delta H_1 = (\gamma_j^1 - \pi_j^1)\, \theta_{i_1} + (\delta_j^1 - \delta_{i_1}^1). \tag{15}$$

Step 5. For ΔH_1:

a) if $\Delta H_1 < 0$, then go to *Step 6*,

b) if $\Delta H_1 \geqq 0$, then go to *Step 7*.

Step 6. Compare ΔH_1 with F:

a) if $F > \Delta H_1$, set $F \leftarrow \Delta H_1$, $s_1 \leftarrow i_1$, $t_1 \leftarrow j$, $\mu \leftarrow 2$ and go to *Step 13*,

b) if $F \leqq \Delta H_1$, go to *Step 13* at once.

Ust $ST2$ from *Step 7* to *Step 12*.

Step 7. Set each element of $ST2$ as the same value as $ST1$.

Replace $\omega_{i_1} \in W_B$ by $\omega_j \in W_N$ and set $k \leftarrow 1$.

Step 8. If $\omega_k \in W_B$, go to *Step 12*. Otherwise determine the basic variable $\omega_{i_2} \in W_B$ according to (16).

$$\theta_{i_2} = \min_{1 \leq i \leq m+n_0} \{\theta_i = \beta_i^{2*} / \sigma_{ik}^{2*}, \text{ for } \sigma_{ik}^{2*} > 0\}, \tag{16}$$

where σ_{ik}^{2*} is the (i,k) element of current $ST2$.

Step 9. Calculate the value of ΔH_2 by (17):

$$\Delta H_2 = \Delta H_1 + (\gamma_k^2 - \pi_k^2)\, \theta_{i_2} + (\delta_k^2 - \delta_{i_2}^2). \tag{17}$$

Step 10. For ΔH_2:

a) if $\Delta H_2 < 0$, then go to *Step 11*,

b) if $\Delta H_2 \geqq 0$, then go to *Step 12.*

Step 11. Compare ΔH_2 with F:

a) if $F > \Delta H_2$, set $F \leftarrow \Delta H_2$, $s_1 \leftarrow i_1$, $t_1 \leftarrow j$, $s_2 \leftarrow i_2$, $t_2 \leftarrow k$, $\mu \leftarrow 3$ and go to *Step 12*,

b) if $F \leqq \Delta H_2$, go to *Step 12* at once.

Step 12. Set $k \leftarrow k+1$:

a) if $k \leqq m+2n_0$, then go to *Step 8*,

b) if $k > m+2n_0$, then go to *Step 13.*

Step 13. Set $j \leftarrow j+1$:

a) if $j \leqq m+2n_0$, then go to *Step 3*,

b) if $j > m+2n_0$, then go to *Step 14.*

Step 14. For μ:

a) if $\mu = 1$, the algorithm is terminated,

b) if $\mu = 2$, replace ω_{s_1} by ω_{t_1} by pivoting on term $\sigma^{1*}_{s_1 t_1}$ and return to *Step 2*,

c) if $\mu = 3$, replace ω_{s_1} by ω_{t_1} by pivoting on term $\sigma^{1*}_{s_1 t_1}$ for the first pivot calculation and then replace ω_{s_2} by ω_{t_2} by pivoting on term $\sigma^{2*}_{s_2 t_2}$ for the second pivot calculation. Return to *Step 2.*

If various methods are contrived under the consideration of the properties mentioned in chapter 3, we can improve the efficiency of the algorithm. Let $j^*_k(X)$ $(k=1,\text{---},l)$ be the maximum number of subscript of ξ_{jk} such that $\xi_{jk} \in (X^1_B \cup X^2_B)$ for each k and define $J^*(X) = \{j^*_1(X),\text{---},j^*_l(X)\}$. Also let $j^*_k(Z)$ $(k=1,\text{---},l)$ be the minimum number of subscript of ζ_{jk} such that $\zeta_{jk} \in (Z^1_B \cup Z^2_B)$ for each k and define $J^*(Z) = \{j^*_1(Z),\text{---},j^*_l(Z)\}$. Store the current $J^*(X)$ and $J^*(Z)$ after solving *Problem A* and at any step of the pivot calculation, that is, at *Step 1* and *Step 14.* Then we can contrive the algorithm as follows:

a) It is enough to investigate only $\xi_{j^*+1k} \in X_N$ for each k and only ζ_{j^*-1k}

$\varepsilon\ Z_N$ for each k at *Step 3* and *Step 8*.

b) When determining the basic variable which moves to the nonbasis by using eq. (14) in *Step 3*, we only consider such subscript that $i \in J^*(X)$ for $\omega_i \in (X_B^1 \cup X_B^2)$ and $i \in J^*(Z)$ for $\omega_i \in (Z_B^1 \cup Z_B^2)$. Also when determining the basic variable which moves to the nonbasis by using eq. (16) in *Step 8*, it is enough to investigate such variable that $\omega_i \in (X_B^1 \cup X_B^2)$ and $i \in J^*(X)$.

5. Numerical Experiments

5.1 Numerical example

To illustrate the algorithm mentioned in chapter 4, we show a simple numerical example. Let consider the problem of minimizing

$$F = [\tfrac{1}{5}\ \ \tfrac{3}{5}\ \ \tfrac{5}{5}\ \ \tfrac{7}{5}] \begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{31} \\ \xi_{41} \end{bmatrix} + [\tfrac{1}{6}\ \ \tfrac{3}{6}\ \ \tfrac{5}{6}] \begin{bmatrix} \xi_{12} \\ \xi_{22} \\ \xi_{32} \end{bmatrix} + [\tfrac{1}{3}\ \ \tfrac{3}{3}\ \ \tfrac{5}{3}] \begin{bmatrix} \xi_{13} \\ \xi_{23} \\ \xi_{33} \end{bmatrix}$$

$$+ [10\ 15\ 20\ 25] \begin{bmatrix} [\xi_{11}] \\ [\xi_{21}] \\ [\xi_{31}] \\ [\xi_{41}] \end{bmatrix} + [5\ 10\ 15] \begin{bmatrix} [\xi_{12}] \\ [\xi_{22}] \\ [\xi_{32}] \end{bmatrix} + [10\ 10\ 10] \begin{bmatrix} [\xi_{13}] \\ [\xi_{23}] \\ [\xi_{33}] \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{21} \\ \xi_{31} \\ \xi_{41} \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_{12} \\ \xi_{22} \\ \xi_{32} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} \xi_{13} \\ \xi_{23} \\ \xi_{33} \end{bmatrix} \geqq \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$$0 \leqq \xi_{jk} \leqq 1 \quad (j=1,\text{---},n_k;\ k=1,\text{---},3) \qquad \begin{cases} \text{if } \xi_{jk}>0, \text{ then } \xi_{ik}=1 \text{ for } i=1,\text{---},j-1 \\ \text{if } \xi_{jk}=0, \text{ then } \xi_{ik}=0 \text{ for } i=j+1,\text{---},n_k \end{cases} \quad (k=1,\text{---},3),$$

where $n_1 = 4,\ n_2 = n_3 = 3$ and $$[\xi_{jk}] = \begin{cases} 1 & \text{if } \xi_{jk} > 0 \\ 0 & \text{if } \xi_{jk} = 0 \end{cases} \quad (j=1,\text{---},n_k;\ k=1,\text{---},3).$$

The optimal state of *Problem A* is shown in Table 2 and Figure 3.

Table 2. Optimal tableau of *Problem A*¶

		δ_{jk}		10	15	20	25	5	10	15	10	10	10													
		γ_{jk}		$\frac{1}{5}$	$\frac{3}{5}$	$\frac{5}{5}$	$\frac{7}{5}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{3}{3}$	$\frac{5}{3}$													
d_B	c_B	ω_B	b	ξ_{11}	ξ_{21}	ξ_{31}	ξ_{41}	ξ_{12}	ξ_{22}	ξ_{32}	ξ_{13}	ξ_{23}	ξ_{33}	η_1	η_2	η_3	ζ_{11}	ζ_{21}	ζ_{31}	ζ_{41}	ζ_{12}	ζ_{22}	ζ_{32}	ζ_{13}	ζ_{23}	ζ_{33}
5	$\frac{1}{6}$	ξ_{12}	1					1													1					
10	$\frac{3}{6}$	ξ_{22}	$\frac{2}{15}$						1	1				$\frac{-11}{30}$	$\frac{-2}{3}$	$\frac{7}{30}$					-1					
10	$\frac{1}{3}$	ξ_{13}	1								1													1		
10	$\frac{1}{5}$	ξ_{11}	1	1													1									
15	$\frac{3}{5}$	ξ_{21}	1		1													1								
20	$\frac{5}{5}$	ξ_{31}	$\frac{7}{15}$			1	1							$\frac{-1}{30}$	$\frac{4}{15}$	$\frac{-13}{30}$	-1	-1								
		ζ_{41}	1				1													1						
10	$\frac{3}{3}$	ξ_{23}	$\frac{2}{3}$									1	1	$\frac{1}{6}$	$\frac{-1}{3}$	$\frac{1}{6}$								-1		
		ζ_{22}	$\frac{13}{15}$							-1				$\frac{11}{30}$	$\frac{1}{15}$	$\frac{-7}{30}$					1	1				
		ζ_{32}	1							1													1			
		ζ_{31}	$\frac{8}{15}$				-1							$\frac{1}{30}$	$\frac{-4}{15}$	$\frac{13}{30}$	1	1	1							
		ζ_{23}	$\frac{1}{3}$										-1	$\frac{-1}{6}$	$\frac{1}{3}$	$\frac{-1}{6}$								1	1	
		ζ_{33}	1										1													1
$\gamma_{jk}-\pi_{jk}$			2.5				$\frac{2}{5}$			$\frac{1}{3}$			$\frac{2}{3}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{4}{5}$	$\frac{2}{5}$			$\frac{1}{3}$			$\frac{2}{3}$		
ΔH_1			82.5				$\frac{389}{75}$			$\frac{227}{45}$			$\frac{4}{9}$	$\frac{13}{110}$	$\frac{1}{10}$	$\frac{347}{-35}$	$\frac{32}{75}$	$\frac{16}{75}$			$\frac{13}{45}$			$\frac{2}{9}$		

¶ We assume that each empty element in the tableau takes the value zero.

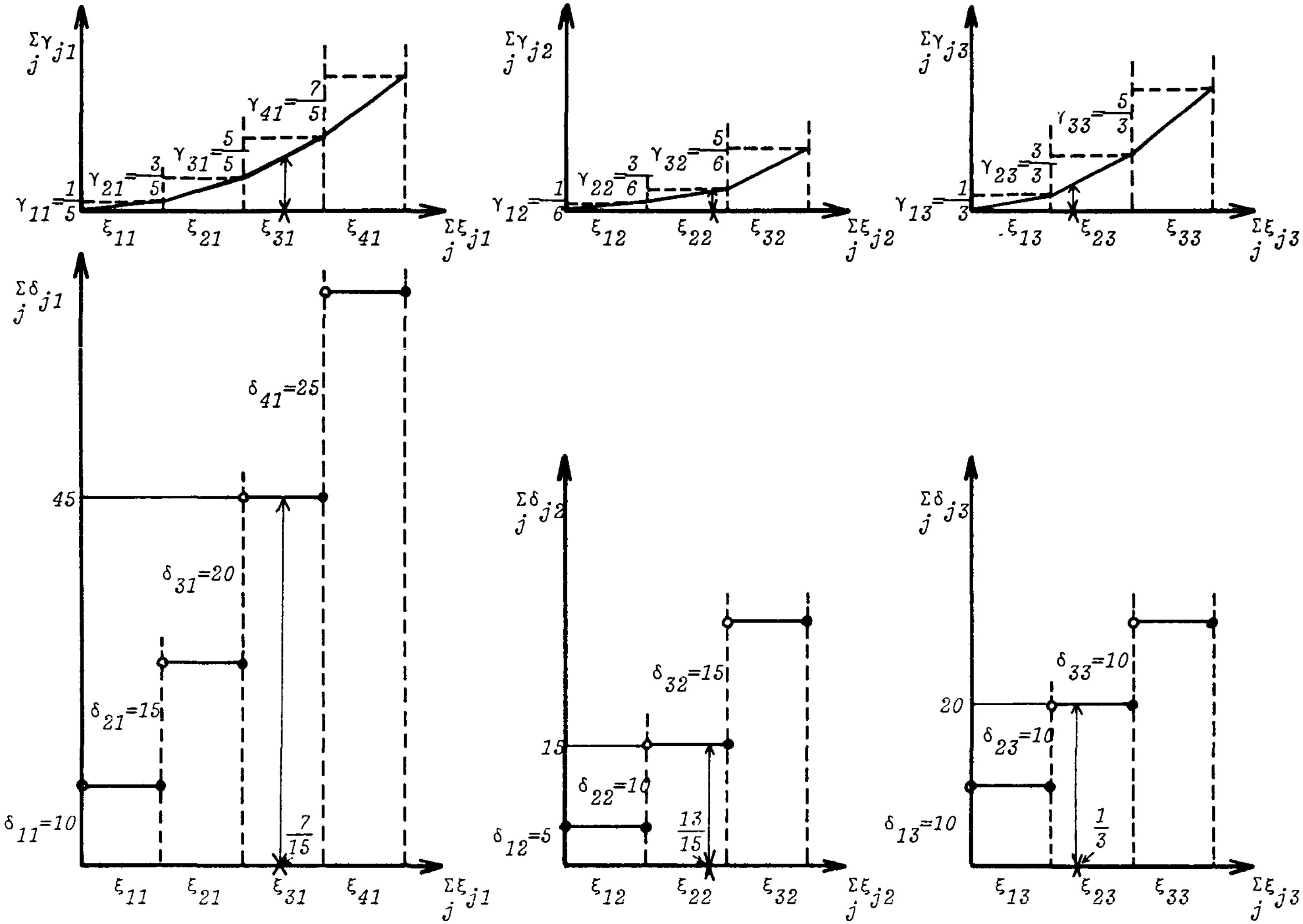

Fig. 3 Optimal state of *Problem A*

The value of the objective function is $F = 82.5$. According to the algorithm, we have the values of ΔH_1 and ΔH_2 as Table 3.

Table 3. Values of ΔH_1 and ΔH_2 for Table 2

the first pivot calculation		the second pivot calculation		values of ΔH_1 and ΔH_2	
ω_s^1	ω_t^1	ω_s^2	ω_t^2	ΔH_1	ΔH_2
ζ_{22}	η_1	ξ_{23}	ξ_{32}	$13/110$	$-95/22$
		ξ_{31}	η_2	$13/110$	$-97/10$
		ξ_{23}	η_3	$13/110$	$-97/10$
ζ_{23}	η_2	ξ_{31}	ξ_{33}	$1/10$	$-229/24$
		ξ_{31}	η_1	$1/10$	$-197/10$*
		ξ_{22}	η_3	$1/10$	$-97/10$
ξ_{22}	η_3			$-1045/105$	

* Minimum value of ΔH_1 and ΔH_2.

From Table 3, we know that we should replace $\zeta_{23} \in Z_B^2$ by $\eta_2 \in Y_N$ for the first pivot calculation and then $\xi_{31} \in X_B^2$ by $\eta_1 \in Y_N$ for the second pivot calculation. After these pivot calculations we have the state shown in Table 4 and Figure 4. The value of the objective function is $F = 62.8$. For Table 4, we have the values of ΔH_1 and ΔH_2 as Table 5. As there exists no ΔH_1 or ΔH_2 which takes the negative value, we know that Table 4 shows the terminal state of *Problem B*.

Table 4. Terminal tableau of *Problem B*¶

		δ_{jk}		10	15	20	25	5	10	15	10	10	10													
		γ_{jk}		$\frac{1}{5}$	$\frac{3}{5}$	$\frac{5}{5}$	$\frac{7}{5}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{3}{3}$	$\frac{5}{3}$													
d_B	c_B	ω_B	b	ξ_{11}	ξ_{21}	ξ_{31}	ξ_{41}	ξ_{12}	ξ_{22}	ξ_{32}	ξ_{13}	ξ_{23}	ξ_{33}	η_1	η_2	η_3	ζ_{11}	ζ_{21}	ζ_{31}	ζ_{41}	ζ_{12}	ζ_{22}	ζ_{32}	ζ_{13}	ζ_{23}	ζ_{33}
5	$\frac{1}{6}$	ξ_{12}	1					1													1					
10	$\frac{3}{6}$	ξ_{22}	1			4	4		1	1			3			-1	-4	-4			-1			-3	-3	
10	$\frac{1}{3}$	ξ_{13}	1								1													1		
10	$\frac{1}{5}$	ξ_{11}	1	1													1									
15	$\frac{3}{5}$	ξ_{21}	1		1													1								
		η_2	2			5	5						1		1	-2	-5	-5						-1	-1	
		ζ_{41}	1				1													1						
10	$\frac{3}{3}$	ξ_{23}	1									1													1	
		η_1	2			10	10						8	1		-3	-10	-10						-8	-8	
		ζ_{32}	1							1													1			
		ζ_{31}	1			1													1							
		ζ_{22}	0			-4	-4			-1			-3			1	4	4			1	1		3	3	
		ζ_{33}	1																							1
$\gamma_{jk}-\pi_{jk}$			2.8			-1	$\frac{-3}{5}$			$\frac{1}{3}$			$\frac{1}{6}$			$\frac{1}{2}$	$\frac{9}{5}$	$\frac{7}{5}$			$\frac{1}{3}$			$\frac{7}{6}$	$\frac{1}{2}$	
ΔH_1			62.8			$\frac{99}{5}$	$\frac{622}{25}$			$\frac{16}{3}$			$\frac{241}{24}$			0	0	0			0			0	0	

¶ We assume that each empty element in the tableau takes the value zero.

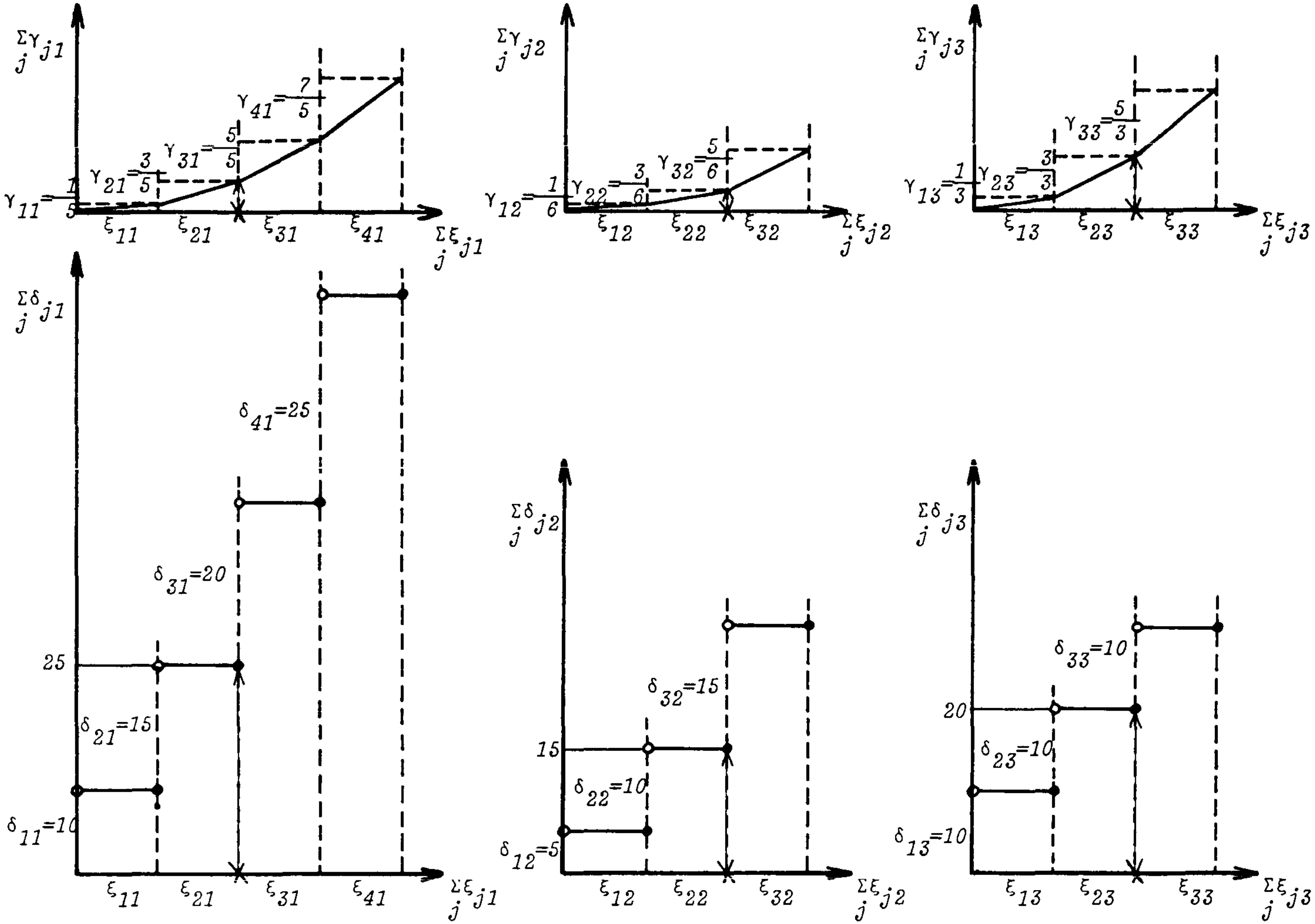

Fig. 4 Terminal state of *Problem B*

Table 5. Values of ΔH_1 and ΔH_2 for Table 4

the first pivot calculation		the second pivot calculation		values of ΔH_1 and ΔH_2	
ω_s^1	ω_t^1	ω_s^2	ω_t^2	ΔH_1	ΔH_2
η_1	ξ_{33}	ξ_{33}	ξ_{31}	8/81	1207/60
		ξ_{22}	η_3	8/81	5/4

5.2 Results of numerical experiments

In order to test the effectiveness of our algorithm mentioned in this paper, we prepare some numerical examples with the following properties:

(1) $l = 5$, $m = 5$, $n_k = 5$ $(k=1,---,l)$, hence $n_0 = 25$, $m + n_0 = 30$ (the number of inequalities) and $m + 2n_0 = 55$ (the number of variables including slack),

(2) $\alpha_{ij-1k} = \alpha_{ijk}$ $(i=1,---,m;\ j=2,---,n_k;\ k=1,---,l)$
and $0 \leqq \alpha_{i1k} \leqq 9$ $(i=1,---,m;\ k=1,---,l)$,

(3) β_{ik} are given about 0.6 times as many as $\sum_j \alpha_{ijk}$ for all i and k $(i=1,---,m;$ $k=1,---,l)$, and

(4) $\delta_{jk} \geqq \gamma_{jk}$ $(j=1,---,n_k;\ k=1,---,l)$.

Table 6 shows the input data we used. For each coefficients matrix and vector (A_k,b) of Table 6 (a), we examined all the cases of the cost and the fixed charge vector (c_k,d_k) of Table 6 (b), that is, we solved 5 x 4 = 20 cases. Table 7 shows the results of numerical experiments. In Table 7, *case 1-2*, for example, implies that *data no. 1* of Table 6 (a) and *data no. 2* of Table 6 (b) are combined.

Table 6. Input data for the numerical experiments

(a) Coefficients matrix and vector (A_k, b)

data no.	A_1	A_2	A_3	A_4	A_5	b
	2 2 2 2 2	3 3 3 3 3	2 2 2 2 2	2 2 2 2 2	0 0 0 0 0	30
	2 2 2 2 2	6 6 6 6 6	9 9 9 9 9	0 0 0 0 0	6 6 6 6 6	70
1	5 5 5 5 5	3 3 3 3 3	8 8 8 8 8	7 7 7 7 7	0 0 0 0 0	70
	5 5 5 5 5	7 7 7 7 7	0 0 0 0 0	0 0 0 0 0	5 5 5 5 5	50
	6 6 6 6 6	4 4 4 4 4	6 6 6 6 6	4 4 4 4 4	3 3 3 3 3	60
	6 6 6 6 6	7 7 7 7 7	2 2 2 2 2	5 5 5 5 5	8 8 8 8 8	80
	4 4 4 4 4	1 1 1 1 1	1 1 1 1 1	7 7 7 7 7	9 9 9 9 9	60
2	9 9 9 9 9	9 9 9 9 9	0 0 0 0 0	3 3 3 3 3	0 0 0 0 0	60
	3 3 3 3 3	3 3 3 3 3	0 0 0 0 0	3 3 3 3 3	5 5 5 5 5	40
	3 3 3 3 3	4 4 4 4 4	3 3 3 3 3	6 6 6 6 6	8 8 8 8 8	70
	2 2 2 2 2	8 8 8 8 8	2 2 2 2 2	7 7 7 7 7	3 3 3 3 3	70
	9 9 9 9 9	0 0 0 0 0	4 4 4 4 4	1 1 1 1 1	0 0 0 0 0	40
3	4 4 4 4 4	1 1 1 1 1	6 6 6 6 6	0 0 0 0 0	2 2 2 2 2	40
	8 8 8 8 8	0 0 0 0 0	9 9 9 9 9	7 7 7 7 7	3 3 3 3 3	80
	9 9 9 9 9	9 9 9 9 9	4 4 4 4 4	8 8 8 8 8	0 0 0 0 0	90
	7 7 7 7 7	9 9 9 9 9	2 2 2 2 2	3 3 3 3 3	2 2 2 2 2	70
	2 2 2 2 2	4 4 4 4 4	8 8 8 8 8	7 7 7 7 7	7 7 7 7 7	80
4	3 3 3 3 3	6 6 6 6 6	7 7 7 7 7	7 7 7 7 7	6 6 6 6 6	80
	6 6 6 6 6	1 1 1 1 1	7 7 7 7 7	1 1 1 1 1	9 9 9 9 9	70
	5 5 5 5 5	5 5 5 5 5	2 2 2 2 2	7 7 7 7 7	7 7 7 7 7	80
	0 1 2 3 4	8 8 8 8 8	0 0 0 0 0	6 6 6 6 6	8 8 8 8 8	70
	1 2 3 4 5	4 4 4 4 4	2 2 2 2 2	3 3 3 3 3	6 6 6 6 6	50
5	5 6 7 8 9	9 9 9 9 9	8 8 8 8 8	9 9 9 9 9	6 6 6 6 6	120
	3 4 5 6 7	5 5 5 5 5	0 0 0 0 0	2 2 2 2 2	5 5 5 5 5	50
	0 0 1 2 3	0 0 0 0 0	0 0 0 0 0	1 1 1 1 1	1 1 1 1 1	10

(b) Cost and fixed charge vector (c_k, d_k)

date no.	c_1	c_2	c_3	c_4	c_5	d_1	d_2	d_3	d_4	d_5
	25	20	60	80	60	100	150	200	250	300
	30	25	65	85	65	110	160	210	260	310
1	35	30	70	90	70	120	170	220	270	320
	40	35	75	95	75	130	180	230	280	330
	45	40	80	100	80	140	190	240	290	340
	60	80	60	20	25	100	150	200	250	300
	65	85	65	25	30	110	160	210	260	310
2	70	90	70	30	35	120	170	220	270	320
	75	95	75	35	40	130	180	230	280	330
	80	100	80	40	45	140	190	240	290	240
	25	20	60	80	60	300	250	200	150	100
	30	25	65	85	65	310	260	210	160	110
3	35	30	70	90	70	320	270	220	170	120
	40	35	75	95	75	330	280	230	180	130
	45	40	80	100	80	340	290	240	190	140
	60	80	60	20	25	300	250	200	150	100
	65	85	65	25	30	310	260	210	160	110
4	70	90	70	30	35	320	270	220	170	120
	75	95	75	35	40	330	280	230	180	130
	80	100	80	40	45	340	290	240	190	140

Table 7. Results of numerical experiments *

data	values of the objective function		number of pivot calculations after solving *Problem A*†		decrease in the objective function	time § (sec.)
	Problem A¶	*Problem B*				
case 1-1	*2924.44*	*2878.33*	*1*	*(0, 1)*	*46.11*	*20.7*
case 1-2	*4324.44*	*4095.95*	*3*	*(0, 3)*	*273.49*	*34.9*
case 1-3	*4413.96*	*4097.38*	*1*	*(0, 1)*	*316.58*	*21.0*
case 1-4	*4113.96*	*3997.38*	*1*	*(0, 1)*	*116.58*	*21.1*
case 2-1	*3633.33*	*3456.67*	*1*	*(1, 0)*	*176.66*	*20.3*
case 2-2	*3480.00*	*3480.00*	*0*	*(0, 0)*	*0.00*	*16.2*
case 2-3	*3633.33*	*3318.33*	*1*	*(1, 0)*	*315.00*	*21.2*
case 2-4	*3280.00*	*3186.87*	*3*	*(0, 3)*	*93.13*	*37.3*
case 3-1	*3538.67*	*3275.00*	*1*	*(0, 1)*	*263.67*	*23.5*
case 3-2	*4891.97*	*3743.00*	*7*	*(0, 7)*	*1148.97*	*57.8*
case 3-3	*4738.67*	*4215.00*	*5*	*(0, 5)*	*523.67*	*47.0*
case 3-4	*4391.97*	*4085.00*	*4*	*(1, 3)*	*306.97*	*43.5*
case 4-1	*4181.19*	*3930.00*	*2*	*(0, 2)*	*251.19*	*27.7*
case 4-2	*4620.96*	*4083.94*	*4*	*(2, 2)*	*537.02*	*40.5*
case 4-3	*4181.19*	*3739.04*	*4*	*(1, 3)*	*442.15*	*42.1*
case 4-4	*4220.96*	*3705.00*	*2*	*(0, 2)*	*515.96*	*27.1*
case 5-1	*3671.25*	*3422.50*	*1*	*(0, 1)*	*248.75*	*6.5*
case 5-2	*4708.75*	*3703.00*	*9*	*(1, 8)*	*1005.75*	*16.7*
case 5-3	*4571.25*	*3790.00*	*6*	*(0, 6)*	*781.25*	*13.5*
case 5-4	*3708.75*	*3635.00*	*1*	*(0, 1)*	*73.75*	*6.6*

* The computer used is the HITAC 8700 with OS/7 at the Computer Center of Hiroshima University except for the case *5*.

¶ The value of the objective function eq. (1) for the solution of *Problem A*.

† Total number of pivot calculations (the number of once pivot calculation (the case $\mu = 2$), the number of twice pivot calculations (the case $\mu = 3$)).

§ The computer used for the case *5* is the HITAC M-180 with VOS3.

6. Conclutions

We propose an approximate solution method for the problem defined in the introduction. As the algorithm mentioned in chapter 4 is based on the simplex procedure, we can easily treat our problem. If we are in the situation in which the more precise solution must be determined, we will prepare three kinds of simplex tableau for the algorithm and define the decrease in the objective function after three times of the pivot calculations. But it is apparent that the more the pivot calculations increase, the more the computational time and the computer memory required increase.

Acknowledgements

The author wishes to thank Professor Kenichi Aoki of Hiroshima University for his helpful comments, and to the referees for their kind and useful comments. This work was supported by the Zaidan Hojin Keiei Kagaku Shinko Zaidan.

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