Journal of the Operations Research Society of Japan
Vol. 23, No. 3, September 1980

A REPLACEMENT PROBLEM WITH TRADE-IN

Yukio Hatoyama
Tokyo Institute of Technology

(Received August 16, 1979; Revised January 25, 1980)

Abstract Consider the following replacement problem with trade-in. Suppose a system must operate for T time periods. There is an essential component of the system whose failure results in the failure of the system. There are n alternative types of components. The failure distribution for type i component is assumed to be exponential with rate λ_i. If type i component is replaced by type j component, replacement cost $C(i, j)$ is incurred. The dependency of the replacement cost on the failed component as well as a new component to replace the old one implies that the failed component has some trade-in value or salvage value. The problem is to determine an optimal policy for the purchase of a new type of component supposing type i component has just failed with t time periods remaining. Some properties of the optimal policy are discussed under the assumption that $C(i, j)$ has some cost structure such as a submodular function. In particular if $C(i, j)$ is additive, an explicit form of an optimal policy is derived. The model is then extended to multi-component coherent systems where it is shown that for most of the cases the results obtained for single-component system can be directly applicable.

1. Introduction

Machine replacement problems have been fully investigated, and much effort has been focussed on finding simple maintenance rules so that we can easily handle the maintenance systems (see for example [1],[3]). In these replacement models, it is usual to assume that old machines or components are discarded when they are replaced with new ones. However we often encounter the case where an old item can be traded-in on the purchase of a new item (and the replacement cost is the price of a new item minus the traded-in value of the replaced item). Suppose there are several alternative types of components. Then the selection of an optimal type of a new component may not be simple.

Consider the following problem. Suppose a system must operate for T time periods. There is a special component or a part of the system whose failure results in the failure of the system (for example, an automobile as a system, and a wheel tire or a battery as its component, or a record player of a stereo component system). Therefore, whenever a component fails to operate, it must be replaced with a new component at once. Now suppose there are n alternative types of components. The failure distribution for type i ($1 \le i \le n$) is assumed to be exponential with rate λ_i. If a type i component is replaced with a type j component, the replacement cost $C(i,j)$ is incurred when the replacement action is taken. This assumption is a slight generalization of Derman et. al. 's model [2], where the replacement cost depends only on j. We want to determine the best policy for the purchase of a new type of component supposing type i component has just failed with t ($0 \le t \le T$) time periods remaining.

Let $V(t,i)$ ($i \in A = \{1, \dots, n\}$, $0 \le t \le T$) be the additional expected cost incurred under the best policy when the system must operate for an additional amount of time t if type i component fails with t time to go. Then ($V(0,i) = 0$),

$$(1.1) \qquad V(t,i) = \min_{j \in A} \{C(i,j) + \int_0^t V(t-x,j)\lambda_j e^{-\lambda_j x} dx\} , \quad i \in A, \ 0 < t \le T.$$

In the following sections we determine the form of an optimal policy when the replacement cost $C(i,j)$ has some structure such as an additive function or a submodular function.

Thus far the system consists of one essential component. The last section treats the case where the system involves more than one component and has some coherent structure, and it is seen that in most cases the results obtained for the single component system directly applies to these more complicated systems.

2. Example

When a component is replaced with another component, several situations on the trade-in costs are to be considered depending on the characteristics of the components or the sales policies of their manufacturing and/or marketing companies. We explain some of the typical cases which seem to be practically important and easy to be handled.

Example 1. (additive cost function)

$$(2.1) \qquad C(i,j) = - f(i) + g(j) , \quad i, j \in A \subset R,$$

where $g(j)$ is the cost of purchasing a type j component, and $f(i)$ is the trade-in value of a type i component. In this additive cost case it is assumed that the trade-in value of a failed component is independent of the type of component to be purchased.

The above example is intuitively appealing because of its simplicity, and the case will be fully investigated later. In practice, however, the independence of the trade-in value of a failed component from the type of component to be purchased can be too restrictive. We often encounter the case where a salesman of a company buys his own products with higher trade-in values when he sells new products than those manufactured by other companies. It seems then, there exists a difference in the closeness or exchangeability between components. That is, failed components which are close to the component to be purchased can be traded-in with higher values, and those which are not close to it are traded-in with lower values. The following function is introduced to take the concept of closeness into consideration.

Consider a real valued function f on $S \times T$ where $S \subset R$ and $T \subset R$. Then f is called submodular[5] (strictly submodular) on $S \times T$ if

$$\Delta_{\sigma\tau} f(s,t) \equiv f(\sigma,\tau) - f(\sigma,t) - f(s,\tau) + f(s,t) \leq 0 \quad (< 0)$$
$$\text{for any } s,\ \sigma \in S,\ t,\ \tau \in T \text{ such that } (s-\sigma)(t-\tau) > 0.$$

f is called supermodular (strictly supermodular) on $S \times T$ if $-f$ is submodular (strictly submodular) on $S \times T$. An additive function as in Example 1 is both submodular and supermodular. Notice that submodularity of f is equivalent to

$$\frac{\partial^2 f}{\partial s \partial t} \leq 0 \quad \text{for } s \in S,\ t \in T,$$

if f is twice continuously differentiable on $S \times T$ and S, T are open intervals.

Example 2. (submodular cost function)

(2.2) $$C(i,j) = -f(i,j) + g(j)\ ,\ i,\ j \in A \subset R,$$

where $g(j)$ is as before and the trade-in value $f(i,j)$ of type i component depends on the type j component which buys it, and is assumed to be supermodular. The supermodularity of f implies the submodularity of C.

As an illustration, consider 3 types of components 1, 2, and 3, where type 1 and type 3 are not close and unable to be traded-in each other. The necessary and sufficient condition for f to be supermodular is

(2.3) $$f(i,i) + f(2,2) \geq f(i,2) + f(2,i) \qquad i = 1,\ 3,$$

(2.4) $$\min\{ f(2,1) + f(3,2),\ f(1,2) + f(2,3) \} \geq f(2,2).$$

(2.3) is usually satisfied since it is natural to think that a component can be traded-in with the highest value when it is purchased by the company who produces or sells it. In other words, it is natural to assume that for any $j \in A$, $f(i,i) \geq f(i,j)$. (2.4) holds when any trade-in value of any component is not less than one half of the highest trade-in value among all, which seems not too restrictive.

If the closeness between components of type i and j is a function only of $i-j$, we let

$$\phi_{i-j} = f(i,j),$$

and the necessary and sufficient condition for f to be supermodular is that ϕ is concave in the appropriate domain. The concavity of ϕ implies that as the distance between components to be traded approaches to zero, its trade-in value saturates, while its value decreases in acceleration as the distance between them gets apart.

The above examples implicitly assume that the trade-ins between components are acceptable (with positive trade-in values), with the exception of possibly a pair of components. In the actual cases, the sales strategies or whatever of a manufacturing company may often limit the types of used components which the salesmen can trade-in with their components. Some companies accept a type of component with a positive trade-in value, while others reject and throw away the same type of component when a new type of component is purchased. Next two examples deal with such a situation.

Example 3. (trade-in acceptable set)

$$C(i,j) = \begin{cases} -f(i) + g(j) & \text{for } j \in A(i), \\ g(j) & \text{otherwise,} \end{cases} \tag{2.5}$$

where f and g are as in Example 1, and $A(i) \subset A$ represents a set of types of components which accept type i component with a positive trade-in value $f(i)$. $A(i) = \{i\}$ implies the case where a type of component can be traded-in only if the same type of component is purchased, while $A(i) = A$ implies that it can be traded-in no matter what the purchased component may be.

As a particular case of the above example, consider the followings:

Example 4. (exclusive groups)

A market is divided into several exclusive groups A_k $(k = 1, \dots, K)$. If two types of components are in the same group, they are cooperative, but if they are not in the same group, they are competitive. That is, two types of components can be traded-in each other if and only if they belong to the same group. Mathematically, for $i \in A_m$, $j \in A_n$

$$C(i,j) = \begin{cases} -f(i) + g(j) & \text{if } m = n, \\ g(j) & \text{otherwise,} \end{cases} \tag{2.6}$$

where $\bigcup_{k=1}^{K} A_k = A$ and $A_m \cap A_n = \phi$ for $m \neq n$.

In the following sections each example will be taken into consideration, and some simplification on the algorithm for computing an optimal policy to each case will be discussed.

3. Optimal Policies and Computational Aspects

In this section some properties on the optimal decisions and the corresponding optimal costs of the functional equation (1.1) of the renewal decision model are discussed.

Proposition 1. If $C(i,j) \geq 0$ for all i, $j \in A$, then $V(t,i)$ is continuous and nondecreasing in $t > 0$ for all $i \in A$.

Proof: Nondecreasing property of V in t is obvious by definition and the nonnegativity of $C(i,j)$. Suppose type i component fails when t time units to go and type j component is optimal to replace it. Then for any $\varepsilon > 0$,

$$V(t,i) \leq V(t+\varepsilon,i) \leq C(i,j) + \int_0^{t+\varepsilon} V(t+\varepsilon-x,j)\lambda_j e^{-\lambda_j x} dx.$$

Now

$$C(i,j) + \int_0^{t+\varepsilon} V(t+\varepsilon-x,j)\lambda_j e^{-\lambda_j x} dx - V(t,i)$$

$$= \int_{-\varepsilon}^{t} V(t+\varepsilon-x'-\varepsilon,j)\lambda_j e^{-\lambda_j (x'+\varepsilon)} dx' - \int_0^t V(t-x,j)\lambda_j e^{-\lambda_j x} dx$$

$$= (e^{-\lambda_j \varepsilon} - 1)\int_0^t V(t-x,j)\lambda_j e^{-\lambda_j x} dx + \int_0^{\varepsilon} V(t-x+\varepsilon,j)\lambda_j e^{-\lambda_j x} dx.$$

The first term converges to zero when $t \to +0$. For the second term,

$$0 \leq \int_0^{\varepsilon} V(t-x+\varepsilon,j)\lambda_j e^{-\lambda_j x} dx \leq V(t+\varepsilon,j)\int_0^{\varepsilon} \lambda_j e^{-\lambda_j x} dx$$

$$= V(t+\varepsilon,j)(1 - e^{-\lambda_j \varepsilon}).$$

Hence the second term also goes to zero if $\varepsilon \to +0$, yielding that

$$\lim_{\varepsilon \to +0} V(t+\varepsilon,i) = V(t,i) \quad i \in A.$$

Similarly we can show that

$$\lim_{\varepsilon \to -0} V(t+\varepsilon,i) = V(t,i) \quad i \in A,$$

which gives the continuity of V in $t > 0$. □

Suppose type i component has just failed with t time units remaining, and that using type j component is uniquely optimal. Then $V(t,i)$ is differentiable at t and

$$\text{(3.1)}\quad \frac{d}{dt}V(t,i) = \frac{d}{dt}\{C(i,j) + \int_0^t V(t-x,j)\lambda_j e^{-\lambda_j x} dx\} = \lambda_j(V(t,j) - V(t,i) + C(i,j)).$$

If $V(t,j) - V(t,i)$ does not depend on t for any i, $j \in A$, the derivative is independent of t, and the curve of $V(t,i)$ becomes piecewise linear in $t > 0$ for each $i \in A$. In general, however, the piecewise linearity of V cannot be assured as is clear from (1.1). Suppose more than one types are optimal, say $j_1, \ldots, j_M$ are optimal to replace a failed type i component with t time units remaining. If j_m $(1 \leq m \leq M)$ is used at t, the right derivative of $V(t,i)$ equals the r.h.s. of (3.1) with j_m replacing j. Hence choosing type j_m^* component such that

$$j_m^* = \arg\min_{j_1 \leq j_m \leq j_M} \lambda_{j_m}(V(t,j_m) - V(t,i) + C(i,j_m))$$

is uniquely optimal for any $\tau \in (t,t+\varepsilon)$ for some positive ε.

Now return to the situation where type i^* (j^*, respectively) component optimally replaces type i (j, respectively) component upon failure with t time units remaining. Then,

$$V(t,i) = C(i,i^*) + \int_0^t V(t-x,i^*)\lambda_{i^*} e^{-\lambda_{i^*}x} dx \leq C(i,j^*) + \int_0^t V(t-x,j^*)\lambda_{j^*} e^{-\lambda_{j^*}x} dx,$$

and

$$V(t,j) = C(j,j^*) + \int_0^t V(t-x,j^*)\lambda_{j^*} e^{-\lambda_{j^*}x} dx \leq C(j,i^*) + \int_0^t V(t-x,i^*)\lambda_{i^*} e^{-\lambda_{i^*}x} dx,$$

which gives the following lemma.

Lemma 1. Assume that using type i^* (j^*, respectively) is optimal when type i (j, respectively) component fails with t time units remaining. Then the following inequality holds on the cost function:

$$\text{(3.2)}\quad C(i,i^*) + C(j,j^*) \leq C(i,j^*) + C(j,i^*).$$

Lemma 1 simplifies the algorithm to find an optimal policy since we only have to check those types of components j^*'s satisfying the inequality (3.2) as a candidate for the optimal type of component replacing type j component instead of examining all types of components in A if we have the additional information that type i^* component is optimal to replace a failed type i component. Lemma 1 leads to the following statement for a strictly submodular cost function C as is seen in Example 2.

Theorem 1. Assume that i^* and j^* satisfy the statement in Lemma 1. Furthermore, suppose that $C(i,j)$ is strictly submodular on $A \times A$. Then $i < j$ implies $i^* \leq j^*$.

Proof: Suppose $i < j$ and $i^* > j^*$. Then by the strict submodularity of C,

$$C(i,i^*) + C(j,j^*) > C(i,j^*) + C(j,i^*),$$

which contradicts the inequality (3.2). □

The knowledge of C being submodular further simplifies the algorithm to determine the optimal selection of a type of component to be used. If C is strictly submodular, after we calculate the optimal type of component to replace a failed type i component, say i^* to replace i, the search for the optimal type to replace a failed type j ($j \neq i$) component then can be restricted to those k's satisfying $k \geq i^*$ if $i < j$ and $k \leq i^*$ if $i > j$.

Fix a time period $T > 0$. Let $i \to j$ imply that type j component is optimal to replace a failed type i component with T periods remaining. Then for each $T > 0$, a directed graph showing the optimal selection of the type of component can be drawn, which will vary as T varies (see Fig. 1). If $i \to j$ $(i \neq j)$, then i is called transient, and if $i \to i$, i is called absorbing. A loop is a sequence of transient states $(i_0, i_1, \ldots, i_m)$ $(m \geq 2)$ such that

$$i_0 \to i_1 \to \cdots \to i_m \text{ and } i_0 = i_m.$$

Corollary 1. If the cost function C is strictly submodular on $A \times A$, there exists no loop in the graph representing the optimal selection of the type of components for each $T > 0$.

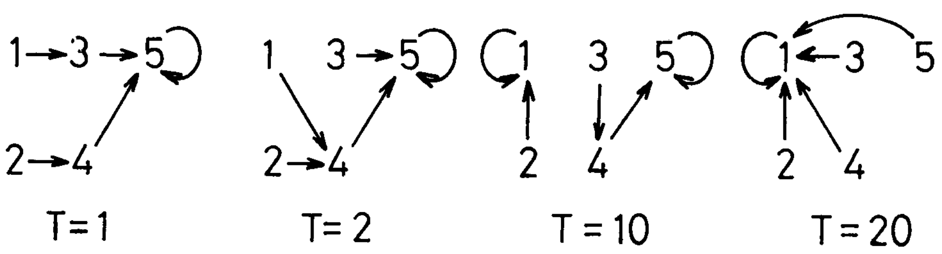

Figure 1. An Example of Time Varying Directed Graphs.

Proof: Suppose there exists a loop $(i_0, i_1, \dots, i_m)$. Then by using the submodularity of C,

$$i_0 < i_1 < \cdots < i_m \quad \text{or} \quad i_0 > i_1 > \cdots > i_m$$

by Theorem 1. In either case, $i_0 \neq i_m$, which contradicts the presumption that $(i_0, i_1, \dots, i_m)$ is a loop. □

From Corollary 1, the strict submodularity of C on $A \times A$ guarantees at least one absorbing state in the graph for each T. As λ_j is the failure rate of type j component and $C(j,j)$ is the replacement cost of type j component with the same type of component, $\lambda_j C(j,j)$ is the expected cost per period of staying in state j while j is an absorbing state. Hence, if type j^* component satisfying

$$\lambda_{j^*} C(j^*,j^*) = \min_{j \in A} \lambda_j C(j,j)$$

is unique, j^* will be the only absorbing state for a sufficiently large T under the condition that $C(i,j) < \infty$ for $i, j \in A$. This is equivalent to saying that type j^* component will be eventually used under the optimal policy for a sufficiently long period problem. This argument is, in general, not true when the strict submodularity of C is removed. Consider an example with only two types of components, 1 and 2. If $C(1,2) = C(2,1) = 0$ and $C(1,1) = C(2,2) = 1$, then it is clear that $1 \rightarrow 2$ and $2 \rightarrow 1$ for any T, and hence there is no absorbing state in this case.

Now let us take Examples 3 and 4 into consideration. When a component fails and must be replaced, not all the types of components necessarily accept that failed type of component with a positive trade-in value. $A(i)$ represents the set of types of components which accept type i component. Suppose type i^* (j^*, respectively) component optimally replaces type i (j, respectively) component. Then using Lemma 1 and examining all the combinations, we easily obtain the following relation between i^* and j^*.

Theorem 2. Suppose the cost structure is as is shown in (2.5). Then,

$$i^* \in A(i) \cap A(j) \quad \text{implies } j^* \in \begin{cases} A(i)^c \cup A(j) & \text{if } f(i) \geq f(j), \\ A(j) & \text{if } f(i) < f(j), \end{cases}$$

$$i^* \in A(i) \cap A(j)^c \quad \text{implies } j^* \in A,$$

$$i^* \in A(i)^c \cap A(j) \quad \text{implies } j^* \in A(i)^c \cap A(j),$$

$$i^* \in A(i)^c \cap A(j)^c \quad \text{implies } j^* \in \begin{cases} A(i)^c \cup A(j) & \text{if } f(i) \leq f(j), \\ A(i)^c & \text{if } f(i) > f(j). \end{cases}$$

Therefore if the cost structure is as is given in Example 3, the above theorem contributes to the simplification of the algorithm to determine the optimal selection of a type of component to be used. That is, if an additional information that $i \to i^*$ is obtained, the search for the optimal type of component to be used when any other type of component, say j ($j \neq i$), fails can be restricted to those types which j^* belongs. For example, suppose $i^* \in A(i) \cap A(j)$, namely suppose i^* is a type of component which accepts both types i and j components. Then j^* belongs to the types of components which accept type j component with a positive trade-in value if the trade-in value for type j component is higher than that for type i component. Otherwise j^* can be, in addition, a type of component which does not accept type i component. The most effective information on the search for j^* is that i^* is a type of component which accepts type j component but does not accept type i component. In that case j^* has the same property as i^* has, i.e., $j^* \in A(i)^c \cap A(j)$.

Example 4 is a special case of Example 3, and hence Theorem 2 can be transformed into the following statement by setting $A(i) = A_m$ if $i \in A_m$.

Corollary 2. Suppose the cost structure is as is shown in (2.6). Then if $i \in A_m$ and $j \in A_m$,

$$i^* \in A_m \text{ implies } j^* \in \begin{cases} A & \text{if } f(i) \geq f(j), \\ A_m & \text{if } f(i) < f(j), \end{cases}$$

$$i^* \in A_m^c \text{ implies } j^* \in \begin{cases} A & \text{if } f(i) \leq f(j), \\ A_m^c & \text{if } f(i) > f(j) \end{cases}$$

and if $i \in A_m$ and $j \in A_n$ ($m \neq n$),

$$i^* \in A_m \text{ implies } j^* \in A,$$

$$i^* \in A_n \text{ implies } j^* \in A_n,$$

$$i^* \in A_m^c \cap A_n^c \text{ implies } j^* \in \begin{cases} A & \text{if } f(i) \leq f(j), \\ A_m^c & \text{if } f(i) > f(j). \end{cases}$$

In the case where both types of components i and j are in the same cooperative group, j^* can be found in that group if i^* is also in that group and the trade-in value of a type j component is higher than that of a type i component. If i^* is not in that group and furthermore if the trade-in value of a type j component is lower than that of a type i component, then j^* cannot be in that same group. Consider the situation where types i and j components are competitive and not cooperative. It is reasonable that the knowledge that i^* and i are cooperative will not give any information on the selection of a type of component when a type j component fails to operate. However, if i^* and j

belong to the same group, then a type of component selected when a type j component fails to operate can be found in that group. In other words, if competitors select us, we should select ourselves.

Finally we will briefly discuss about the actual computation of the equation (1.1). The calculation of the optimal costs in (1.1) can be performed by the discrete approximations of the continuous time variable. Letting $\Delta t = T/K$ where K is the number of steps, we obtain recursively for all $i \in A$, the approximations:

$$(3.3) \qquad V(t_{k+1},i) = \min_{j\in A}\{ C(i,j) + \Delta t \sum_{l=0}^{k} V(t_{k-l},j)\lambda_j e^{-\lambda_j t_l} \}, \qquad k = 0, 1, \ldots , K-1,$$

where $t_k = k\Delta t$ and the initial values of V are

$$V(0,i) \equiv \lim_{t\to+0} V(t,i) = \min_{j\in A} C(i,j).$$

The accuracy of the approximations depends heavily on the length of Δt. Round-off and truncation error limit the choice of small values of Δt. In (3.3) minimization is taken over all j's in A, but Lemma 1 shows that the minimization over the set

$$S = \begin{cases} A, & i = 1, \\ \bigcap_{l<i} \{j \in A \mid C(i,j) - C(l,j) \le C(i,l^*) - C(l,l^*),\ l \to l^* \text{ at } t_k\}, & i > 1, \end{cases}$$

is sufficient. If C is strictly submodular, Theorem 1 can be utilized to further simplify the above set to

$$S = \begin{cases} A, & i = 1, \\ \bigcap_{l<i} \{j \in A \mid j \ge l^*,\ l \to l^* \text{ at } t_k\}, & i > 1. \end{cases}$$

Similar argument can be made for the cost structure (2.5) in Example 3 or (2.6) in Example 4. The former makes use of Theorem 2, while the latter makes use of Corollary 2.

The derivatives of V in (3.1) can also be used to reduce the number of steps on the calculation of the optimal cost. Suppose $i \to j$ with t remaining. Then if the cost of replacing i with k ($k \ne j$) at t is greater than the sum of the cost of replacing i with j at t and the increment of $V(t,i)$ in Δt, k is not optimal even at $t+\Delta t$. In other words, j is also optimal at $t+\Delta t$ if

$$\min_{k\ne j}\{C(i,k) + \int_0^t V(t-x,k)\lambda_k e^{-\lambda_k x} dx\} - V(t,i) > \Delta t \lambda_j (V(t,j) - V(t,i) + C(i,j)).$$

These techniques of reducing the number of computational steps have been shown effective in most cases by the digital computer simulation.

4. Additive Case

In the following, the cost function is assumed to be additive, i.e.,

$$C(i,j) = C_1(i) + C_2(j), \quad i, j \in A.$$

This seems to be a natural assumption to our replacement problem with trade-in since, as in Example 1, if the trade-in value of any type of a failed component does not depend on the type of component to be purchased and to replace the failed component, the cost for the replacement can be represented as the price $C_2(j)$ (= $g(j)$ in Example 1) of a type j component to be purchased minus the trade-in value $-C_1(i)$ (= $f(i)$ in Example 1) of a failed type i component.

4.1. Case where survived components are discarded

By formulating the replacement model as (1.1), we implicitly assume that survived components are just thrown away without any value. This would be reasonable if we consider that a trade-in of a used component is acceptable only when a new component is purchased. For the additive cost function C, (1.1) becomes, for $i \in A$ and $t > 0$,

$$(4.1) \qquad U(t,i) = \min_{j \in A} \{ C_2(j) + C_1(j)(1 - e^{-\lambda_j t}) + \int_0^t U(t-x,j)\lambda_j e^{-\lambda_j x} dx \},$$

where

$$U(t,i) \equiv V(t,i) - C_1(i), \quad i \in A, \ t \geq 0.$$

Notice that the right hand side of the equation (4.1) is independent of i, hence

$$(4.2) \qquad U(t,i) \equiv U(t) \quad \text{for any } i \in A, \ t > 0,$$

or equivalently,

Theorem 3. If the cost function C is additive, the optimal replacement policy is independent of the type of component to be replaced.

Therefore the functional equation can be simplified to

$$(4.3) \qquad U(t) = \min_{j \in A} \{ C^*(j,t) + \int_0^t U(t-x)\lambda_j e^{-\lambda_j x} dx \}, \quad t > 0,$$

where

$$(4.4) \qquad C^*(j,t) = C_2(j) + C_1(j)(1 - e^{-\lambda_j t}), \quad j \in A, \ t > 0.$$

For this additive case Proposition 1 implies that under the assumption:

Assumption: $\min_{j \in A} C_2(j) \geq - \min_{j \in A} C_1(j)$,

$U(t)$ is a nondecreasing, continuous function of t for $t > 0$. The above assumption states that the highest trade-in value among all types of used and failed components is less valuable than any type of new components. Usually, this assumption is satisfied. Under the same assumption, the following theorem is stated.

Theorem 4. If Assumption holds, $U(t)$ is a piecewise linear concave function of t for $t > 0$ with at most n pieces.

Proof: If type j component is uniquely optimal at $t > 0$, the derivative of $U(t)$ at t exists and is from (3.1),

$$\frac{d}{dt} U(t) = \frac{d}{dt} V(t,i) = \lambda_j C(j,j). \tag{4.5}$$

As $dU(t)/dt$ is constant within the intervals where one type of component is optimal, $U(t)$ is linear within the interval. (4.5) also shows that among those types of components that are optimal at t, that type j minimizing $\lambda_j C(j,j)$ is uniquely optimal over some interval $(t, t+\varepsilon)$, $\varepsilon > 0$, and that type j maximizing $\lambda_j C(j,j)$ is uniquely optimal over some interval $(t-\varepsilon', t)$, $\varepsilon' > 0$. At each change of optimal type as t increases, a type with a smaller $\lambda_j C(j,j)$ becomes optimal, and hence there can be at most n-1 changes in the interval $(0,+\infty)$. The concavity of $U(t)$ is assured since $dU(t)/dt$ is nonincreasing. □

A type of component which attains the smallest C_2 is selected when the time is sufficiently small since $C^*(j,t)$ approaches to $C_2(j)$ as $t \to 0$. Let i_1 be that type of component. Then it is easily seen by the concavity of the cost function $U(t)$ that any type j component where $\lambda_j C(j,j)$ is larger than $\lambda_{i_1} C(i_1,i_1)$ will never be used in an optimal policy and hence can be deleted from our consideration. The next proposition also contributes to reduce the number of types of components to be considered.

Proposition 2. Suppose $C_1(i) \leq 0$. Then if $\lambda_j \geq \lambda_i$, $C_2(j) \geq C_2(i)$, and $\lambda_j C_1(j) \geq \lambda_i C_1(i)$ hold (not all are =), then type j component is not used in an optimal policy. If $C_1(i) > 0$, the last inequality can be replaced by $C_1(j) \geq C_1(i)$.

Proof: Let π_i be a policy which uses type i component at t and then follows optimally, and let U_{π_i} be the corresponding cost U. Now

$$U_{\pi_j}(t) - U_{\pi_i}(t) = (C_2(j)-C_2(i)) + (C_1(j)(1-e^{-\lambda_j t}) - C_1(i)(1-e^{-\lambda_i t}))$$
$$+ \int_0^t U(t-x)\lambda_j e^{-\lambda_j x} dx - \int_0^t U(t-x)\lambda_i e^{-\lambda_i x} dx .$$

The first expression on the right is nonnegative by the assumption, and so is the last expression since U is nonincreasing in x and $\lambda_j \geq \lambda_i$. Let $h(t)$ be the second expression. Then,

$$h'(t) = \lambda_j C_1(j) e^{-\lambda_j t} - \lambda_i C_1(i) e^{-\lambda_i t}.$$

$h(t)$ is trivially nonnegative for $t > 0$ if $C_1(j) \geq 0$ and $C_1(i) \leq 0$. If both $C_1(i)$ and $C_1(j)$ are nonpositive, $h'(t) \geq 0$ for all t such that

$$t \geq t^* = \frac{1}{\lambda_j - \lambda_i} \{\log(-\lambda_j C_1(j)) - \log(-\lambda_i C_1(i))\} .$$

As $t^* \leq 0$ and $h(0) = 0$, we have that $h(t) \geq 0$ for all $t \geq 0$. If $C_1(i) > 0$, the fact that $h(0) = 0$ and $\lim_{t\to\infty} h(t) = C_1(j) - C_1(i) \geq 0$ give the desired property since the sign change of $h'(t)$ is from positive to negative. Careful examination shows that $U_{\pi_j} > U_{\pi_i}$ if at least one of the inequalities is strict.□

After the deletion of unnecessary types of components, we can assume without loss of generality that the types of components are arranged so that

$$\lambda_1 C(1,1) > \lambda_2 C(2,2) > \cdots > \lambda_n C(n,n). \tag{4.6}$$

Then the optimal policy uses type 1 component when the time remaining is sufficiently small, then switches to type 2 component at some time t_1 $(0 \leq t_1 \leq \infty)$ and uses type 2 component until time increases to t_2 $(t_1 \leq t_2 \leq \infty)$, when type 3 component replaces type 2 component, and so on. The interval (t_{i-1}, t_i) for use of type i component may be empty. Actually such switching points can be obtained and the tedious calculations show that the solution is as below:

$$V(t,i) = C_1(i) + C_2(1) + \sum_{j=1}^{k-1} (\lambda_j C(j,j) - \lambda_{j+1} C(j+1,j+1)) t_j + \lambda_k C(k,k) t \tag{4.7}$$
$$\text{for } t_{k-1} < t \leq t_k , \quad i, k \in A,$$

$$t_k = \frac{1}{\lambda_{k+1}} \log \frac{H(k)}{\lambda_k C(k,k) - \lambda_{k+1} C(k+1,k+1)} \tag{4.8}$$

where $t_0 \equiv 0$ and $t_n \equiv +\infty$, and

$$H(k) = \lambda_1 C(1,1) - \lambda_{k+1} C_2(1) - \lambda_{k+1} C_1(k+1) + \sum_{j=1}^{k-1} (\lambda_{j+1} C(j+1,j+1) - \lambda_j C(j,j)) e^{\lambda_{k+1} t_j}.$$

On the derivation of the above expressions, $\lambda_k C(k,k)$'s are assumed to be decreasing, and furthermore t_k's are assumed increasing. If any type of component is used at some interval in the optimal policy, t_k becomes strictly increasing, and conversely if t_k's obtained from (4.8) are strictly increasing, none of the types of components are extraneous. However, it should not be in general assumed before calculating t_k's that t_k's obtained from (4.8) are increasing. Actually this condition can be relaxed. Suppose at k-th recursion, the increasing property of t_k's is violated, i.e., $t_k \le t_{k-1}$. Then we can safely conclude that type k component may never be used in the optimal policy and hence can be excluded from our consideration. Type k+1 component now replaces type k component, and t_{k-1} is reevaluated by (4.8). The repitition of this procedure will lead to an optimal policy with at most $2n$-3 computations of t_k's. (Notice that the violations of the increasing property of t_k's will occur at most n-2 times since at each violation a type of component is excluded from consideration and there are n types of components.)

4.2. Case where survived components are salvaged

We briefly discuss about the case where survived components are salvaged with a salvage value which is naturally assumed to be equal to its trade-in value. The following functional equation replaces (1.1) now ($V(0,i) \equiv C_1(i)$):

$$(4.9) \qquad V(t,i) = \min_{j \in A} \{ C(i,j) + \int_0^t V(t-x,j) \lambda_j e^{-\lambda_j x} dx + C_1(j) \int_t^\infty \lambda_j e^{-\lambda_j x} dx \},$$

for $i \in A$, $0 < t \le T$.

The same argument as before reduces the problem to

$$(4.10) \qquad U(t) = \min_{j \in A} \{ C(j,j) + \int_0^t U(t-x) \lambda_j e^{-\lambda_j x} dx \},$$

which exactly coincides with Derman et. al.'s model [2] when C_j replaces $C(j,j)$. Hence the derivation of the solution there and the result of Smith [4] are directly applicable, and the solution is as below:

$$(4.11) \qquad V(t,i) = C_1(i) + C(1,1) + \sum_{j=1}^{k-1} (\lambda_j C(j,j) - \lambda_{j+1} C(j+1,j+1)) t_j + \lambda_k C(k,k) t$$

for $t_{k-1} < t \le t_k$, $i, k \in A$, where t_k is as is shown in (4.8) and

$$H(k) = C(1,1)(\lambda_1 - \lambda_{k+1}) + \sum_{j=1}^{k-1}(\lambda_{j+1}C(j+1,j+1) - \lambda_j C(j,j))\ e^{\lambda_{k+1}t_j} .$$

The forms of V and H are slightly different from those where survived components are discarded. Also notice that type i component in (4.11) does not necessarily correspond to type i component in (4.7) since the way of excluding those types of components which should never be used in an optimal policy here is through Proposition 2 and the succeeding remark in Derman et. al. [2], which are different from our Proposition 2.

5. Extension to Series-Parallel Structures

Thus far it was assumed that only one component was essential to the operation of the system. In the real world, on the contrary, a system usually consists of many components, and some of them are essential to the system operation. In the present section we treat the cases where there is more than one essential component. It is shown that many of the structures which are of practical importance can be reduced to one component system, which implies that the results obtained in the previous sections are directly applicable to these more complicated systems.

5.1. Series structure

Frequently the failure of any component of a system implies the failure of the whole system; that is, the system has a series structure regardless of its actual mechanical or electrical structure as far as the system reliability is concerned (for example in a car the tires, battery, and engine are all connected in series from the point of view of its reliability).

In general, suppose a system consists of R components in series, and component r has n_r ($n_r \geq 1$) different types. Assume that the failure distribution of type m ($1 \leq m \leq n_r$) of component r is exponential with rate λ_m. Cost $C_r(m,m')$ is incurred when type m of component r is replaced by type m' of the same component. Under the assumption that each component fails independently, a policy minimizing the expected replacement cost for the whole system in a finite horizon case is given as below:

Suppose type m_r of component r fails with t time remaining, when component j uses type m_j ($j = 1, \ldots, R$). Formulate a subproblem r as a replacement problem where only one component r exists in the system, and let $V_r(t,m_r)$ be the additional expected cost incurred under the best policy in the period

$(0,t]$ if type m_r of component r fails with t time units remaining. Then to replace type m_r with m_r' is optimal for the subproblem r where m_r' realizes the right side of the following functional equation

$$V_r(t,m_r) = \min_{1\le m_r'\le n_r} \{ C_r(m_r,m_r') + \int_0^t V_r(t-x,m_r')\lambda_{m_r'} e^{-\lambda_{m_r'}x} dx\}.$$

Now the requirement that the series system must survive for T time periods implies that each subsystem r must operate for T periods. Furthermore each component is assumed to fail independently, and the notion of preventive maintenance has no meaning in this model because of its cost structure. Hence a policy which optimizes every subproblem r is optimal for the whole system, and the expected minimal cost in the period $(0,t]$ for the whole system is given by

$$V_r(t,m_r) + \sum_{j\ne r}\int_0^t V_j(t-x,m_j)\lambda_{m_j} e^{-\lambda_{m_j}x} dx,$$

and this value can be obtained from the solutions of the subproblems j's.

5.2. Parallel structure

If a system consists of parallel components, the failure of a component does not necessarily imply the failure of the system, and hence each component is not "essential" to the system in this sense. However, our previous results can be easily applied to this parallel structure as is shown in the following discussion. First notice that it will never be optimal to replace a failed type of component unless the system fails.

Suppose the system fails with t time remaining, when component j uses type m_j ($j = 1, \dots, R$), and let $V(t, m_1, \dots, m_R)$ be its optimal additional expected cost. Then the functional equation becomes

$$V(t,m_1, \dots ,m_R) = \min_{1\le r\le R} \min_{1\le m_r'\le n_r} \{ C_r(m_r,m_r') + \int_0^t V(t-x,m_1, \dots ,m_r', \dots ,m_R)\lambda_{m_r'} e^{-\lambda_{m_r'}x} dx \},$$

and it is optimal for the whole system to use a new type m_r' of component r which attains the minimum of the right side of the equation (in place of the failed type m_r of component r).

Because of the simple structure on the cost function, an optimal policy has the form that doing nothing until the system failure, when a component is selected and is replaced with a new one. The above class of policies will rarely be optimal if the time horizon is infinite and a suitable additional cost structure is assumed such as the replacement of more than one components at the same time is less expensive than the sum of the corresponding individual replacements. Even for the finite horizon case, this sort of cost structure will sometimes make the type of optimal policies on the maintenance of the parallel system more practically attractive. The modification of the model in functional equation form will be easy but the formulation itself will become complicated since the action space to be concerned then will be much enlarged.

In particular, if the trade-in is not accepted, or if any failed type of component is discarded as a scrap, the minimal expected cost V does not depend on m_r's and hence it can be written simply as $V(t)$. On the other extreme and special case, if the trade-in among different components is accepted and furthermore if we place the condition that the type of component which fails last and therefore brings the system to fail should be traded-in for a new component which may differ from the traded-in component, then the optimal expected cost is a function only of the remaining time t and the lastly failed type of component m_r. Moreover, if the replacement cost $C(m_r, m'_r)$ is additive, the optimal policy is given as a function of only t as has been shown in the previous discussion.

5.3. Series-parallel structure and others

Consider as the combination of the above two cases a series-parallel structure where parallel subsystems are connected in series (see Fig. 2). As subsystems are connected in series and components are independent, an optimal policy for each subsystem calculated by ignoring other subsystems optimizes

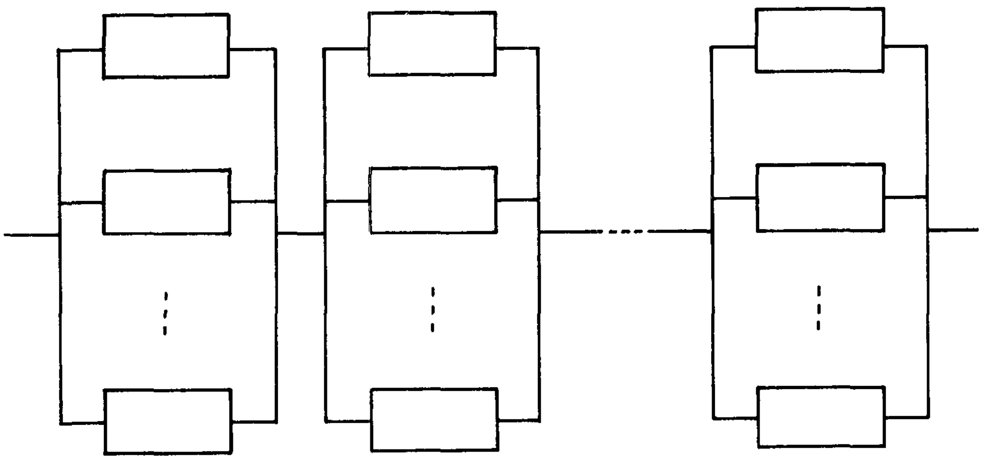

Figure 2. Series-Parallel Structure

the whole system. Now the problem is to obtain an optimal policy for each subsystem, and this can be easily accomplished by the argument in section 5.2 since each subsystem is a parallel system.

For most of other coherent systems, it will be possible to formulate the models in functional equation forms and to compute their numerical solutions, but it would seem difficult to find some interesting qualitative remarks on them. Further research will be expected.

Acknowledgments

I am indebted to the referees of this paper for insightful comments and helpful suggestions.

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Yukio HATOYAMA: Department of Management Science and Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo, 152, Japan.