Journal of the Operations Research Society of Japan
Vol. 23, No. 3, September 1980

STATISTICAL ANALYSIS OF RELIABILITY DATA IN RAMDOMLY CENSORED LIFE TESTING

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(Received November 9, 1978)

Abstract For independent nonnegative continuous random variables $X_1, \ldots, X_k$, $Z = min(X_1, \ldots, X_k)$ and $\delta_i (i=1, \ldots, k)$, where $\delta_i = 1$ if $Z = X_i$ and $\delta_i = 0$ otherwise, are statistically independent random variables if and only if the distribution of X_i is written as $F_i(x) = 1 - exp(-p_i Q(x))$ $(i=1, \ldots, k)$. From this characterization theorem for the random variable X_i, an unbiased estimate for failure rate of distribution function is presented using reliability data in random life testing. A saving of time in a life testing experiment by allowing random censoring is also discussed.

1. Introduction

In reliability analysis, it is practically important to know the failure characteristics of the components which compose the system, either as a "series system" (where the system fails if any component fails), or a "parallel system" (where the system fails only if all components fail). Usually they may be estimated properly from the component test data and, therefore, a great number of papers have been devoted to the matter.

In some cases, however, the series system data are available for estimating the probability of failure for each of the components. Here, the system data may be consist of two kinds of sets: one is the time-to-failure of the system and the other the cause of the system failure, which implies that the system failure is occured by the failure of the one of k independent, but different components.

To illustrate the situation, we consider an integrated circuit (IC) in the area of a semi-conductor. When the IC is connected with an external terminal, the surface of the alminum electrode is contacted with the external gold line by the method of thermal compression. In this life test, the gold line is pulled for the purpose of measuring the strength of contactness (denoted by X_1), and then the value of X_1 is observed if $X_1 \leq$ the strength of the gold line (denoted by X_2): otherwise, just the information that $X_1 > X_2$ is provided. After all we observe $Z = min(X_1, X_2)$ and the cause of the breakdown, which means that our observation is based upon either X_1 or X_2.

In this article, estimation of the failure characteristics of the component under random censorship will be considered. Random censorship means that the observable value is restricted to $(Z = min(X_1, \cdots, X_k), Z = X_i)$ = (age of failure, cause of failure) where $X_1, \cdots, X_k$ are independent nonnegative continuous random variables. The model arises from many practical situations such as medical follow-up studies, competing risks except life testing stated as above.

Notations and assumptions are stated in Section 2. In section 3 a characterization of distribution function is constructed: the characterization theorem is related to the theorem due to Berman[2]. Section 4 contains applications of the characterization theorem. An unbiased estimate for failure rate of distribution function is presented and some comparisons of efficiency of this estimate with the minimum variance unbiased estimate under Type II censored data are made. These comparisons show that randomly censored life testing gives much more information per unit time than Type II censoring when underlying distribution is Weibull with the shape parameter $m \leq 1$. Section 4 is devoted to related works.

2. Notations and Assumptions

It is assumed that a system is composed of k independent, but non-identical components in a series. If $X_i (i = 1, \cdots, k)$ is a nonnegative continuous random variable indicating the life of the ith component, the life of the series system is represented by the following way: that is,

$$Z = min(X_1, \cdots, X_k).$$

For $i = 1, \cdots, k$, denote

$$F_i(x) = P(X_i \leq x) \quad .$$

Conveniently we put

$$R_i(x) = 1 - F_i(x) \qquad (i = 1, \cdots, k)$$

and define a new measure $Q_i(x)$ by the relation

$$R_i(x) = exp\{-Q_i(x)\} \qquad (i = 1, \cdots, k) \quad .$$

It is obvious that there exist one-to-one correspondence between $Q_i(x)$ and $R_i(x)$. Then

$$F(x) = P(Z \leqq x) = 1 - exp\{-\sum_{i=1}^{k} Q_i(x)\} = 1 - exp\{-Q(x)\}$$

$$R(x) = 1 - F(x) = exp\{-Q(x)\}$$

where $Q(x) = \sum_{i=1}^{k} Q_i(x)$.

Suppose that observations are to be taken from the distribution of Z with the restriction that the value of X_i can be observed if and only if $X_i \leqq Z$; otherwise just information that $X_i > Z$ is provided. From a sample of observations with this sort of random censoring, it is required that an inference be made about the parameters of each of the random variables $X_1, \cdots, X_k$.

The details of observations are represented in the following way. We assume that the series systems have been subjected to life test until the time of the rth failure with Type II censoring at r out of n. For the system, we observe

$$z_1 < z_2 < \cdots < z_r, \qquad (r > 2)$$

where z_j is the jth smallest failure time of the r failures of the system. To apply a binomial sampling plan to this situation, define the random variable

$$\delta_i = \begin{cases} 1 & , \text{ if } X_i \leqq Z \\ 0 & , \text{ if } X_i > Z \end{cases}$$

for $i = 1, \cdots, k$, and the joint distribution function of δ_i and Z

$$G_i(x) = P(\delta_i = 1, \ Z \leqq x) \quad .$$

Clearly, the random variable δ_i has a binomial distribution with the parameter

$$p_i = P(\delta_i = 1) = P(X_i \leqq Z) = \int_0^\infty \prod_{j \neq i} \{1 - F_j(t)\} dF_i(t) \quad .$$

For a given, positive integer r, it is assumed that the failures of the ith component are observed r_i times: that is,

$$r_i = \sum_{j=1}^{r} \delta_{ij} \quad , \qquad r = \sum_{i=1}^{k} r_i$$

where

$$\delta_{ij} = \begin{cases} 1 , & \text{if } X_{ij} \leqq z_j \\ 0 , & \text{if } X_{ij} > z_j \end{cases} \quad .$$

Remark 1. In the life table, let X_1 denote the true survival time for an individual and X_2 the period of observation, or follow-up, for the individual.

That is, the X_1 is censored on the right by the X_2 since one observes only

$$Z = min(X_1, X_2) \quad and \quad \delta$$

where δ indicates whether X_1 is censored $(\delta=0)$ or not $(\delta=1)$.

Remark 2. Considering a competing risks study in which it is presumed that there are k risks of death competing for the life of an individual. Upon death, the age and cause of death are recorded $(Z=min(X_1,\cdots,X_k),\ Z=X_i)$. Based on these data one wishes to examine and predict the mortality pattern under the hypothetical conditions when certain risks of death are elminated.

3. Fundamental Theorem

We will be concerned with the case where $\delta_i (i=1,\cdots,k)$ and Z are independent random variables. Although the set $\{\delta_i\}$ and Z are not independent in general, Theorem below shows that $\{\delta_i\}$ and Z are independently distributed if and only if $F_i(x)$ is written as $F_i(x)=1\text{-}exp\{-p_iQ(x)\}$ for $i=1,\cdots,k$.

Lemma due to Berman[2] is needed for the proof of Theorem.

Lemma. (Berman[2]) The set of functions $\{F_i(x)\}$ is given by

$$F_i(x) = 1\text{-}exp\{-\int_0^x (1-\sum_{j=1}^k G_j(t))^{-1}dG_i(t)\} \quad , \quad (i=1,\cdots,k)$$

using the set $\{G_i(x)\}$.

If $\{\delta_i\}$ and Z are independent, then $G_i(x)$ is written as

$$(3.1) \quad G_i(x) = P(\delta_i=1)\cdot P(Z\leqq x) = p_i\{1\text{-}exp\{-Q(x)\}\}$$

for $i=1,\cdots,k$. Applying Equation (3.1) and Lemma the following theorem can be constructed.

Theorem. A necessary and sufficient condition for $\{\delta_i\}$ and Z to be independent is that $F_i(x)$ is given by $F_i(x)=1\text{-}exp\{-p_iQ(x)\}$ for $i=1,\cdots,k$.

Proof: To see that the condition is necessary, we assume $\{\delta_i\}$ and Z to be independent, that is, $G_i(x)=p_iF(x)$ for $i=1,\cdots,k$. Then we have, from lemma,

$$F_i(x) = 1\text{-}exp\{-\int_0^x (1-\sum_{j=1}^k G_j(t))^{-1}dG_i(t)\}$$

$$= 1\text{-}exp\{-\int_0^x \frac{p_i dF(t)}{1-\sum_{j=1}^k p_jF(t)}\}$$

$$= 1-exp\{-p_i\int_0^x \frac{dF(t)}{R(t)}\} \quad .$$

By applying $Q(x) = \int_0^x dF(t)/R(t)$, we have

$$F_i(x) = 1-exp\{-p_i Q(x)\} \quad .$$

Now consider the sufficiency. Assuming that $F_i(x) = 1-exp\{-p_i Q(x)\}$ for $i = 1,\cdots,k$, then we have

$$\begin{aligned} dG_i(x) &= P(\delta_i = 1,\ Z \varepsilon\, dx) \\ &= P(X_i \varepsilon dx,\ \min_{j\neq i}\{X_j\} > x) \\ &= dF_i(x)\cdot \prod_{j\neq i}\{1-F_j(x)\} \\ &= p_i Q'(x) exp\{-Q(x)\}dx \\ &= p_i dF(x) \\ &= P(\delta_i = 1)\ P(Z \varepsilon\, dx) \end{aligned}$$

and

$$P(\delta_i = 0,\ Z\varepsilon dx) = P(\delta_i = 0)\ P(Z\varepsilon dx)$$

in a similar way. Hence the proof is completed.

Remark. The assumption that $\{\delta_i\}$ and Z are independent means that there is no loss of information included in the data even though the age and cause of failure are recorded individually. Theorem above, therefore, enables the reduction of efforts required to record and analyse the data when $F_i(x)$ is written as $F_i(x) = 1-exp\{-p_i Q(x)\}$ for $i = 1,\cdots,k$.

4. Applications to Life Testing under Random Censorship

4.1 Unbiased estimate of failure rate

In this section it is assumed that $Q_i(x)$ is written as $Q_i(x) = \lambda_i H(x)$ and $H(x)$ is known. Then, $Q(x) = \lambda H(x)$ where $\lambda = \sum_{i=1}^{k} \lambda_i$, and $p_i = \lambda_i/\lambda$. Practically important cases are the exponential and the Weibull distribution, which satisfy $H(x) = x$ and $H(x) = x^m$ respectively.

We consider estimation of λ_i under random censorship. The statistic u is defined as

$$u = \sum_{i=1}^{r} H(z_i) + (n-r)H(z_r) \quad ,$$

which has the sampling distribution such as

$$f(t) = (\lambda^r t^{r-1}/\Gamma(r))exp(-\lambda t) \quad .$$

From this result, it is found that an unbiased estimate for λ is given by

$$\hat{\lambda} = (r-1)/u \quad ,$$

and this variance equals

$$\mathrm{Var}(\hat{\lambda}) = \lambda^2/(r-2) \quad .$$

On the other hand, it is easily shown that the random variable $(r_1,\cdots,r_k)$ has a multinomial distribution with the parameter $(p_1,\cdots,p_k) = (\lambda_1/\lambda,\cdots,\lambda_k/\lambda)$, and an unbiased estimate for λ_i/λ is

$$\widehat{(\lambda_i/\lambda)} = r_i/r \quad .$$

Since Theorem shows that the random variables r_i and u are independently distributed, an unbiased estimate for λ_i is given by

$$\hat{\lambda}_i = (r_i/r)\hat{\lambda} = (r_i/r)\cdot((r-1)/u) \quad .$$

Under random censorship the unbiased estimate for λ_i is gained combining the statistics u and r_i for $i=1,\cdots,k$.

In the next, let us compare the estimate $\hat{\lambda}_i$ with the minimum variance unbiased estimate $\tilde{\lambda}_i$ for λ_i from the ith component test data which is given by

$$\tilde{\lambda}_i = (r-1)/(\sum_{j=1}^{r} H(x_{ij}) + (n-r)H(x_{ir})) \quad ,$$

where $x_{ij} < x_{i,j+1}$, $j=1,\cdots,r$, is the observable failure time for the jth prototype of the ith component tested, with life testing of the ith component terminated at the observed time x_{ir} of the rth failure. Since the variance of $\tilde{\lambda}_i$ equals

$$\mathrm{Var}(\tilde{\lambda}_i) = \lambda_i^2/(r-2) \quad ,$$

the variance of $\hat{\lambda}_i$ can be written as the following:

$$\mathrm{Var}(\hat{\lambda}_i) = \{1 + (1-1/r)\phi_i\}\mathrm{Var}(\tilde{\lambda}_i)$$

where $\phi_i = (\lambda-\lambda_i)/\lambda_i$. The derivation is in Appendix. From this, it is seen, as is now supposed, to be

$$\mathrm{Var}(\hat{\lambda}_i) > \mathrm{Var}(\tilde{\lambda}_i) \quad .$$

Efficiency is a measure intended to provide a convenient standard of comparison for estimates. This is done for two estimates to be compared by dividing the variance of $\hat{\lambda}_i$ into the variance of $\tilde{\lambda}_i$. That is, *Eff1* is defined as the following:

$$Eff1 = \mathrm{Var}(\tilde{\lambda}_i)/\mathrm{Var}(\hat{\lambda}_i)$$
$$= 1/\{1+(1-1/r)\phi_i\} \quad .$$

Table 4.1 shows the efficiency values so obtained, for the case $r=3,5,10,20,\infty$, as regards the parameter $\phi_i = 0.2,\ 0.4,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0$. These values show that the efficiency is above 50 percent for all the values of r when ϕ_i is less than or equal to 1. However, if ϕ_i is greater than 1, the efficiency is below 50 percent.

The estimate $\hat{\lambda}_i$ is not so efficient as $\tilde{\lambda}_i$ when ϕ_i is greater than 1, since r_i the number of the observed failures for the ith component decreases with increasing ϕ_i. However, the time required for randomly censored life testing

Table 4.1 Efficiency of the Estimate $\hat{\lambda}$ for $\tilde{\lambda}$

ϕ_i	r	$Eff1(\%)$	ϕ_i	r	$Eff1(\%)$
0.2	3	88.2	1.5	3	50.0
	5	86.2		5	45.5
	10	84.7		10	42.6
	20	84.0		20	41.2
	∞	83.3		∞	40.0
0.4	3	78.9	2.0	3	42.9
	5	75.8		5	38.5
	10	73.5		10	35.7
	20	72.5		20	34.5
	∞	71.4		∞	33.3
0.5	3	75.0	3.0	3	33.3
	5	71.4		5	29.4
	10	69.0		10	27.0
	20	67.8		20	26.0
	∞	66.7		∞	25.0
1.0	3	60.0			
	5	55.6			
	10	52.6			
	20	51.3			
	∞	50.0			

is less than that needed for the ith component life testing, because $min(X_1,\cdots,X_k) \leqq X_i$. The efficiency of $\hat{\lambda}_i$ will be investigated in the next section considering both the variance of the estimate and the time required for life testing.

Remark 1. (Another derivation of $\hat{\lambda}_i$) It can be shown that the maximum likelihood estimate $\hat{\hat{\lambda}}_i$ of λ_i is given by $\hat{\hat{\lambda}}_i = r_i/u$ and this estimate is not unbiased $(E(\hat{\hat{\lambda}}_i) = r\lambda_i/(r-1))$. The estimate $\hat{\lambda}_i$ is also derived changing $\hat{\hat{\lambda}}_i$ into an unbiased estimate noting that $\{\delta_i\}$ and Z are independent.

Remark 2. It may be unrealistic assuming that $H(x) = x^m$, that is, the time-to-failure of each of the components in the system is Weibull distributed with the same shape parameter m. However, there are many cases in which the unknown parameter λ_i must be infered conducting life test of the component when m is known. In this situation, Randomly censored life testing is recommended. The details are stated in the next section.

4.2 Time saving in random censorship

In order to plan an experiment in which individual components are observed until failure we need to consider not only sample size but also test time. If we insist on observing all individuals until failure we may need to wait unacceptably. If, however, we are willing to accept a randomly censored life testing we may save a large proportion of the time until the last failure.

Let $X_{1j} (j=1,\cdots,n)$ be the survival time for the jth prototype of the component having a distribution function $F_1(x) = 1-exp\{-\lambda_1 H(x)\}$. The period of observation for the jth prototype will typically be limited by an ammount X_{2j}. Formally speaking, the X_{1j} is censored on the right by the X_{2j} since one observes only

$$Z_j = min(X_{1j}, X_{2j}) \quad and \quad \delta_{1j} = \begin{cases} 1 , & if\ X_{1j} \leqq X_{2j} \\ 0 , & if\ X_{1j} > X_{2j} \end{cases}$$

where δ_{1j} indicates whether X_{1j} is censored $(\delta_{1j} = 0)$ or not $(\delta_{1j} = 1)$. Under the random censorship model the censoring variable X_{2j} is also assumed to be a random sample, drawn independently of the X_{1j}, from a distribution $F_2(x) = 1-exp\{-\lambda_2 H(x)\}$. For the Z's, it is assumed that we observe r ordered values with Type II censoring at r out of n: that is,

$$z_1 < z_2 < \cdots < z_r \quad .$$

The ratio $c = E(z_r)/E(x_{1r})$ is the proportion of expected time we must wait if the X_1's are not randomly censored. The ratio c is evaluated as the following way:

$$c = \frac{E(z_r)}{E(x_{1r})}$$

$$= \frac{n\binom{n-1}{r-1}\int_0^\infty x\lambda H'(x)e^{-\lambda H(x)}(1-e^{-\lambda H(x)})^{r-1}(e^{-\lambda H(x)})^{n-r}dx}{n\binom{n-1}{r-1}\int_0^\infty x\lambda_1 H'(x)e^{-\lambda_1 H(x)}(1-e^{-\lambda_1 H(x)})^{r-1}(e^{-\lambda_1 H(x)})^{n-r}dx}$$

$$= \frac{\int_0^\infty H^{-1}(x/\lambda)e^{-x}(1-e^{-x})^{r-1}(e^{-x})^{n-r}dx}{\int_0^\infty H^{-1}(x/\lambda_1)e^{-x}(1-e^{-x})^{r-1}(e^{-x})^{n-r}dx}$$

where $\lambda = \lambda_1 + \lambda_2$.

Thus, if $H(x) = x^m$, we obtain

$$c = (\lambda_1/\lambda)^{1/m} = (1+\phi_1)^{-1/m}$$

since $H^{-1}(x) = x^{1/m}$. The quantity $(\lambda_1/\lambda) < 1$ shows that the time saving increase with decreasing m.

Furthermore, the comparison of the variance of test time says that the variance of that for random censoring can be reduced, that is,

$$d = \mathrm{Var}(z_r)/\mathrm{Var}(x_{1r}) < 1 \quad .$$

If $H(x) = x^m$, d is given by

$$d = (1+\phi_1)^{-2/m} \quad .$$

The values of c and d for $\phi_1 = 0.2, 0.4, 0.5, 1.0, 1.5, 2.0, 3.0$ and for $m = 0.5, 0.8, 1.0, 1.5, 2.0$ are presented in Table 4.2. Table 4.2 shows that c and d decrease for fixed m as the value of ϕ_1 increases. Especially randomly censored life testing can reduce the variance of test time comparing Type II censoring very much.

The result above shows that, for the purpose of comparing the random censoring with Type II censoring, it is necessary to take both the variance of the estimate for λ_1 and the mean test time into consideration, espesially in the area of reliability theory. Considering both them, the efficiency for comparison of the two life testing methods may be defined as

$$Eff2 = \frac{\mathrm{Var}^{-1}(\hat{\lambda}_1)}{E(z_r)} \Big/ \frac{\mathrm{Var}^{-1}(\tilde{\lambda}_1)}{E(x_{1r})} \quad ,$$

which denotes the ratio of the information per unit time obtained for random censoring and that for Type II censoring.

If $H(x) = x^m$, that is, X_1's and X_2's are Weibull distributed with the same shape parameter m, then $Eff2$ is given by

$$Eff2 = (1+\phi_1)^{1/m}/\{1+(1-1/r)\phi_1\} \quad .$$

Table 4.3 gives the numerical results of the calculations of $Eff2$ for $\phi_1 = 0.2$, 0.4, 0.5, 1.0, 1.5, 2.0, 3.0, for $r = 3$, 5, 10, 20, ∞, and for $m = 0.5$, 0.8, 1.0, 1.5, 2.0. The numerical results show that Random censoring gives much more information per unit time than Type II censoring when $m \leqq 1$ and the relative efficiency increases with decreasing m and r. Hence it may be said that Random censoring can be used in place of Type II censoring when the time required for life testing is limited.

Table 4.2 Ratios of the Mean and the Variance of the Relative Waiting Time for Weibull Distribution with the Shape Parameter m and Various Choices of ϕ_1 and m

ϕ_1		m=0.5	m=0.8	m=1.0	m=1.5	m=2.0
0.2	c	0.694	0.796	0.833	0.866	0.913
	d	0.482	0.634	0.694	0.784	0.833
0.4	c	0.510	0.657	0.714	0.799	0.845
	d	0.260	0.431	0.510	0.639	0.714
0.5	c	0.444	0.602	0.667	0.763	0.817
	d	0.198	0.363	0.444	0.582	0.667
1.0	c	0.250	0.420	0.500	0.630	0.707
	d	0.063	0.177	0.250	0.397	0.500
1.5	c	0.160	0.318	0.400	0.543	0.632
	d	0.026	0.101	0.160	0.295	0.400
2.0	c	0.111	0.253	0.333	0.481	0.577
	d	0.012	0.0645	0.111	0.231	0.333
3.0	c	0.063	0.177	0.250	0.397	0.500
	d	0.004	0.031	0.063	0.157	0.250

Finally, it is important that the choice of the parameter λ_2 must be done taking both the variance of the estimate and the time needed for life testing into consideration.

Remark. The result that $Eff2 \geqq 1$ when $H(x) = x^m$ and $m \leqq 1$ can be shown in the following way. Let $m \leqq 1$, then $(1+\phi_1)^{1/m} \geqq 1+\phi_1$. Therefore,

Table 4.3 Efficiency of the Estimate $\hat{\lambda}$ for $\tilde{\lambda}$ for Weibull Distribution Considering the Test Time

(%)

ϕ_1	r	m=0.5	m=0.8	m=1.0	m=1.5	m=2.0
0.2	3	127.1	110.8	105.9	99.6	96.7
	5	124.1	108.3	103.4	97.3	94.4
	10	122.0	106.4	101.7	95.7	92.8
	20	121.0	105.5	100.8	94.9	92.1
	∞	120.0	104.7	100.0	94.1	91.3
0.4	3	172.9	120.2	110.5	98.8	93.4
	5	169.0	115.4	106.1	94.8	89.6
	10	166.1	112.0	102.9	92.1	87.0
	20	164.7	110.4	101.4	90.7	85.7
	∞	163.3	108.8	100.0	89.4	84.5
0.5	3	168.8	124.5	112.5	98.3	91.9
	5	160.7	118.6	107.1	93.6	87.5
	10	155.1	114.5	103.4	90.4	84.5
	20	152.5	112.5	101.7	88.8	83.0
	∞	150.0	110.7	100.0	87.4	81.6
1.0	3	240.0	142.7	120.0	95.2	84.8
	5	222.2	132.1	111.1	88.2	78.6
	10	210.5	125.2	105.3	83.5	74.4
	20	205.1	122.0	102.6	81.4	72.5
	∞	200.0	118.9	100.0	79.4	70.7
1.5	3	312.5	157.2	125.0	92.1	79.1
	5	284.1	142.9	113.6	83.7	71.9
	10	266.0	133.8	106.4	78.4	67.3
	20	257.7	129.6	103.1	76.0	65.2
	∞	250.0	125.7	100.0	73.7	63.2
2.0	3	385.7	169.2	128.6	89.1	74.2
	5	346.2	151.9	115.4	80.0	66.6
	10	321.4	141.0	107.1	74.3	61.9
	20	310.3	136.1	103.4	71.7	59.7
	∞	300.0	131.6	100.0	69.3	57.7
3.0	3	533.3	188.6	133.3	84.0	66.6
	5	470.6	166.4	117.6	74.1	58.8
	10	432.4	152.9	108.1	68.1	56.1
	20	415.6	146.9	103.9	65.5	51.9
	∞	400.0	141.4	100.0	63.0	50.0

$$Eff2 = \frac{(1+\phi_1)^{1/m}}{1+(1-1/r)\phi_1} \geqq \frac{1+\phi_1}{1+(1-1/r)\phi_1} \geqq 1 \quad .$$

5. Related Works

Under random censorship Kaplan and Meier[5] give the product-limit (PL) estimate and reduced-sample (RS) estimate for the proportion $P(t)$ of items in the population whose lifetimes would exceed t, without making any assumption about the form of the function $P(t)$. PL estimate is the distribution, unrestricted as to form, which maximizes the likelihood of the observations. Breslow and Crowley[3] give a necessary and sufficient condition for the consistency of the standard (actuarial) life table estimate of $P(t)$ and asymptotic normality of this estimate, using the model of random censorship. Abe[1] gives a non-parametric estimate for the life time distribution from observations of an aggregate of renewal processes. Since the life time X_0 of each unit is determined by

$$X_0 = min(X_1, X_2)$$

where X_1 and X_2 are mutually independent random variables with continuous distribution functions, Abe's model is considered as one of the random censorship models.

Elveback[4] discusses a simple frequency estimate assuming that the survivorship function is approximated by the polygonal function and shows that the method proposed is appropriate and highly efficient for large scale follow-up studies. Yang[7] deals with estimation of life expectancy used in survival analysis and competing risk study under the condition that the data are randomly censored by k independent censoring variables and shows that the estimate converges weakly to a Gaussian process.

Muenz and Green[6] studies time savings in Type II censored life testing. Their measure of time savings is $t_{(r)}/t_{(n)}$ which is the proportion of time we must wait if failures $r+1$ through to n are not observed. They give numerical results and outline the application of this approach to the evaluation of early stopping procedures.

Acknowledgement. The author wishes to thank Professor Hajime Makabe, Tokyo Institute of Technology, for his guidance and encouragement.

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Appendix. Derivation of the Variance of $\hat{\lambda}_i$.

Noting that the random variables r_i and u are independent, the variance of $\hat{\lambda}_i$ is computed as the fllowing way:

$$\mathrm{Var}(\hat{\lambda}_i) = \mathrm{E}\{(r_i/r)^2\}\mathrm{E}\{((r-1)/u)^2\} - \lambda_i^2$$

$$= \frac{r\frac{\lambda_i}{\lambda}(1-\lambda_i/\lambda) + r^2(\lambda_i/\lambda)^2}{r^2}\{1/(r-2)+1\}\lambda^2 - \lambda_i^2$$

$$= (r-1)\{1+((\lambda-\lambda_i)/\lambda_i)(1/r)\}\lambda_i^2/(r-2) - \lambda_i^2$$

$$= (r-1)(1+\phi_i/r)\mathrm{Var}(\tilde{\lambda}_i) - \lambda_i^2$$

$$= \{1+(1-1/r)\phi_i\}\mathrm{Var}(\tilde{\lambda}_i) \quad .$$

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