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GENERALIZED GOOFSPIEL

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Abstract Suppose there are two persons each with limited resources to gain the value of n targets. The targets appear one by one in sequential order, and the value of each target is a random variable X. One of the players has two kinds of resources (high and low) and the other has only one type (middle). Whenever the target arrives with the realized value x, both player either choose one of their cards or pass. If both player choose the card, the one who discards the high wins an amount equal to x. The card chosen in this stage is never used again. This case is a variant of the game Goofspiel discussed by Ross [2]. We formulate the problem as a two-person zero-sum sequential game and derive a system of recursive equations with some boundary conditions which is solvable in sequence and determines the optimal strategy. A deterministic case is solved completely.

1. Introduction

We shall consider a two-person zero-sum sequential n-stage game, which is a variant of the card game Goofspiel discussed by Ross [2].

The game called Goofspiel is played by two players, using a normal deck of cards, as follows. The 13 clubs are first taken out of the deck and of the remaining 39 cards the 13 hearts are given to player I, the 13 diamonds to player II, and the 13 spades are placed face down in the center. The spades are shuffled and one is turned face up. At this point, the two players choose one of their cards and then simultaneously discard it. The one who discards the higher cards wins from the other an amount equal to the value of the upturned spade. If both plyers discard the same card, then neither wins. The three cards are then thrown away, a new spade upturned and the game continues. After 13 plays, there are no remaining cards and the game ends. In his paper [2], if the game is considered under the assumption that player II discards his cards in a completely random manner, the best thing for player I to do is to always match the upturned spade, i.e., if the upturned card is an ace then player I should play his ace, etc. However without this assumption the optimal strategies and the value of the game are very complicated. So we consider the following variant of this game.

Suppose $G_n(k,l,m)$ be the two person zero-sum sequential n-stage game played as follows. There are two players (player I and player II) to gain the value of n targets. (In the above case, they are the 13 spades face down in the center.) Player I has l cards of type L (low) and k cards of type H (high), and player II has m cards of type M (middle), where $k+l \leqq n$ and $m \leqq n$. The n targets arrive in sequential order. (The one spade is turned face up.) In the follwing we consider the (n,k,l,m) as a state variable. The j-th target is a nonnegative random variable X_j which takes on the value $x_j (j=1,\dots,n)$. We assume that X's are independent and identically distributed random variables with a known cumulative distribution function $F(x)$. On arrival of a target with a realized value x, (the time when one spade is turned face up) the two players, simultaneously and independently, either choose one of their cards or pass. If each player doesn't pass, the one who chooses the higher card wins from the other an amount equal to x. If the game is $G_n(k,l,m)$, player I plays H and player II plays, then player I wins from player II an amount equal to x and the game goes to $G_{n-1}(k-1,l,m-1)$. The other cases can be described similarly. The immediate payoff to player I at this stage is descrided as follows.

		II: play M	II: pass
I	play L	$-x$	0
	play H	x	0
	pass	0	0

Even if one of the players passes, when the card is chosen in this stage, it is abandoned and never used again. For instance when player I plays H and player II passes, player I doesn't get a payoff and the game goes to $G_{n-1}(k-1, l,m)$, i.e., it is disadvantageous for player I to play H. When both players pass, the immidiate payoff is 0 and the game at the next stage is $G_{n-1}(k,l,m)$. If the game is $G_n(k,l,m)$ and $k+l=n$, player I is unable to pass and if $m=n$, so is player II, i.e., these players are restricted on the decisions. This game has perfect recall and we consider the behavioral strategy. If one of the players has no card, this game ends. From the above assumption, after n stages, there is no one with any cards in his hand and the game ends until the n-th stage.

For relatively small values of the targets, if $k=0$, player I would discard and player II wouldn't discard, and if $l=0$, they would behave conversely. For the targets with intermediate values, both players choose a card. For the higher values, both players use the mixed strategy. For example we explain the case where $l=0$. In this case if both players discard, player I always wins from the other. So player II spends his cards for the relatively small values of the targets which are not valuable for player I to play. Player I always wins from player II whenever the realized value of the target is in the intermediate interval which is thought to be reasonable for player I to win. Player II uses the mixed strategy for defending the attack of player I for the higher value of the target. The other cases are considered similarly. If $k+l=m=n$, our game is a variant of Goofspiel, where the deck consists of three types of cards.

There are two interesting papers related to our model. Sakaguchi [3] treats a special case of this paper where $l=0$. We use a similar method of that paper. A sequential allocation problem, discussed in Derman, Lieberman and Ross [1], includes the case of this paper where $l=0$ and $m=n$.

In Section 2 we shall derive a fundamental recursive equation by dynamic programming and observe some preliminary lemmas. In Section 3 we shall observe the boundary conditions of the recursive equations and obtain the main result of this paper. A simple example in explained. In the last section the game is completely solved in the case of deterministic value of the targets.

2. Preliminary Lemmas

Let $G_n(k,l,m)$ be the game described in the previous section. (n,k,l,m) denote the state of the game where player I has l cards of type L and k cards of type H, and player II has m cards of type M and they have n stages to go. If the target appears with the realized value x, the normalized form of the game $G_n(k,l,m)$ has the matrix

$$\begin{array}{ccc} & & \text{II} \\ & & \begin{array}{cc} \text{play M} & \text{pass} \end{array} \\ \text{I} & \begin{array}{c} \text{play L} \\ \text{play H} \\ \text{pass} \end{array} & \begin{pmatrix} -x+G_{n-1}(k,l-1,m-1) & G_{n-1}(k,l-1,m) \\ x+G_{n-1}(k-1,l,m-1) & G_{n-1}(k-1,l,m) \\ G_{n-1}(k,l,m-1) & G_{n-1}(k,l,m) \end{pmatrix} \end{array}$$

The meaning of $x+G_{n-1}(k-1,l,m-1)$ is that when player I plays H and player II plays M, the immidiate payoff to player I is x and the state of the game goes to $(n-1,k-1,l,m-1)$ where the game is $G_{n-1}(k-1,l,m-1)$ and the value of this game can be obtained independent of the previous stage. From the definition of the game $G_n(k,l,m)$, this normalized form will be degenerated, i.e., when $m = n$, player II must play at every stages and this form is a column vector, etc.

Denote the value of the game $G_n(k,l,m)$ by $v_n(k,l,m)$. Because $G_n(k,l,m)$ is a two person zero-sum sequential n-stage game and has perfect recall, we can consider the behavior strategy. When the both players use their first pure strategy and the realized value of the target is x, the immidiate payoff to player I is $-x$ and the game goes to $G_{n-1}(k,l-1,m-1)$ whose value is obtained and denoted as $v_{n-1}(k,l-1,m-1)$. The other cases are considered similarly. So $v_n(k,l,m)$ satisfies the following recursive equation by a dynamic programming formulation of the problem.

$$(1)\quad v_n(k,l,m) = \int_0^\infty val \begin{pmatrix} -x+v_{n-1}(k,l-1,m-1) & v_{n-1}(k,l-1,m) \\ x+v_{n-1}(k-1,l,m-1) & v_{n-1}(k-1,l,m) \\ v_{n-1}(k,l,m-1) & v_{n-1}(k,l,m) \end{pmatrix} dF(x)$$

The initial condition of (1) is $v_0(0,0,0)=0$. The notation val A for the matrix A is used for the value of the matrix game A.

We shall discuss some properties of $v_n(k,l,m)$ from now on. First of all we observe the value for $n=1$ which is the one stage game $G_1(k,l,m)$. From the definition $v_1(0,1,0)=v_1(1,0,0)=v_1(0,0,1)=0$, and

$$v_1(1,0,1) = \int_0^\infty x dF(x) = \mu ,$$

$$v_1(0,1,1) = -\int_0^\infty x dF(x) = -\mu ,$$

where we assume $0 < \mu \equiv E(X) < \infty$. We remark that, for $n=1$, $v_1(k,l,m)$ satisfies the following lemmas.

The proof of the following lemmas is easy by induction on n.

Lemma 1. For any k, l, m and n, we have

$$(2)\qquad v_n(k,l+1,m) \leqq v_n(k,l,m) \leqq v_n(k+1,l,m).$$

Lemma 2. For any k, l, m and n, we have

$$(3a)\qquad v_n(k,0,m) \geqq v_n(k,0,m-1)$$

$$(3b)\qquad v_n(0,l,m) \leqq v_n(0,l,m-1)$$

The following lemma is trivial by the definition of the game and the fact that $X \geqq 0$.

Lemma 3. For any k, l, m and n, we have
$v_n(k,0,m) \geqq 0$ and $v_n(0,l,m) \leqq 0$.

We note here that these lemmas are intuitively reasonable. The properties of $v_n(k,l,m)$ mentioned above are used in the next section to derive the optimal strategy.

3. Boundary Conditions of $v_n(k,l,m)$

First we have to find the boundary conditions of $v_n(k,l,m)$ for the recursive equation (1). We define the well known function in opportunity analysis,

$$T_F(z) = \int_z^\infty (x-z)dF(x)$$

and this function is well defined because of $\mu < \infty$. Then $S_F(z) = z+T_F(z)$ is convex and increasing.

Let $\{g_{n,i}\}_{1\leqq i\leqq n}$ be a trianguler array defined by the following recursive relations.

$$g_{n,1}=S_F(g_{n-1,1}) \qquad (n \geqq 2,\ g_{1,1}=\mu\ )$$

$$(4) \qquad g_{n,i}=S_F(g_{n-1,i}) - \textstyle\sum_{j=1}^{i-1}(g_{n,j}-g_{n-1,j})$$

$$(2 \leqq i \leqq n-1,\ g_{n,n}= n\mu- \textstyle\sum_{j=1}^{n-1}g_{n,j})$$

Similarly we define a trianguler array $\{h_{n,i}\}_{1\leqq i\leqq n}$,

$$h_{n,1}= \mu-T_F(h_{n-1,1}) \qquad (n \geqq 2,\ h_{1,1}=\mu\ )$$

$$(5) \qquad h_{n,i}= \mu-T_F(h_{n-1,i})- \textstyle\sum_{j=1}^{i-1}(h_{n,j}-h_{n-1,j})$$

$$(2 \leqq i \leqq n-1,\ h_{n,n}=n\mu- \textstyle\sum_{j=1}^{n-1}h_{n,j})$$

Then we get the following lemma. Since the proof of (6) in Lemma 4 is similar to that of Proposition 1 discussed in Sakaguchi [3], we skech the outline of the proof.

Lemma 4. We have

$$(6)_n \qquad v_n(k,0,n)= -v_n(n,k,0)= \textstyle\sum_{i=1}^{k}g_{n,i}.$$

$$(7)_n \qquad -v_n(0,l,n)=v_n(l,n,0)= \textstyle\sum_{i=1}^{l}h_{n,i}.$$

Proof: We use the induction on n.

$$v_n(k,0,n)= \int_0^\infty max\{x+v_{n-1}(k-1,0,n-1),\ v_{n-1}(k,0,n-1)\}dF(x)$$

$$=v_{n-1}(k,0,n-1)+T_F(v_{n-1}(k-1,0,n-1)-v_{n-1}(k-1,0,n-1)).$$

Substitute the equation $(6)_{n-1}$ on the right-hand side of the equation then we get $(6)_n$. For the other cases the proof is the same for $v_n(k,0,n)$. This completes the proof.

In these cases, we get the optimal strategy easily. So we explain the optimal strategy for player I in $G_n(k,0,n)$ only.

$$\text{if } x\left\{\begin{matrix} < \\ \geqq \end{matrix}\right\} g_{n-1,k}, \text{ then } \left\{\begin{matrix} \text{pass that stage} \\ \text{play a card H} \end{matrix}\right.$$

The other cases can be derived similarly.

Concerning the game $G_n(n-l,l,n)$, define a triangular array $\{p_{n,i}\}_{0\leqq i\leqq n}$.

$$p_{n,0}=\frac{n}{2}\mu$$

$$p_{n,i}=-\mu+T_F(-p_{n-1,i})+\frac{\mu}{2}-\sum_{j=0}^{i-1}(p_{n,j}-p_{n-1,j}) \qquad (2\leqq i\leqq n-1) \tag{8}$$

$$p_{n,n}=-\frac{n}{2}\mu-\sum_{j=1}^{n-1}p_{n,j}.$$

Now by lemma 1 and 4, we get that even $g_{n,i}$ and $h_{n,i}$ are positive but $p_{n,i}$ is not always positive.

Related to the boundary conditions, Lemma 4 and the next two lemmas are considered as a dynamic programming problem where player I is the only decision maker.

Lemma 5.

$$(9)_n \qquad v_n(n-l,l,n)=2\sum_{j=0}^{l}p_{n,j}$$

and the optimal strategy for player I is as follows,

$$\text{if } x\left\{\begin{matrix} > \\ \leqq \end{matrix}\right\}(-h_{n-1,l})\wedge 0, \quad \text{then } \left\{\begin{matrix} \text{play a card H} \\ \text{play a card L} \end{matrix}\right.$$

where $\alpha\wedge\beta=\min\{\alpha,\beta\}$.

Proof: We use induction on n. For $n=1$, $v_1(0,1,1)=-2p_{1,0}=-\mu$, $v_1(1,0,1)=2((\frac{\mu}{2}-p_{1,0})+p_{1,0})=\mu$. We have

$$v_n(n-l,l,n)=\int_0^\infty \max\{-x+v_{n-1}(n-l,l-1,n-1),\ x+v_{n-1}(n-l-1,l,n-1)\}dF(x)$$

$$=-\mu+v_{n-1}(n-l-1,l,n-1)+2S_F(-p_{n-1,l}).$$

By the inductive hypothesis, substitute $(9)_{n-1}$ for the right-hand side of the last equation gettng $(9)_n$. For $l=0$ and $l=n$, we get the equation $(9)_n$ by a simple calculation. We get the proof.

Concerning the game $G_n(k,l,n)$ we define a following function by

$$C_F(z,\alpha,\beta)=-(\alpha+z)F((\alpha+z)\wedge\frac{\alpha-\beta}{2})+\beta(F(\frac{\alpha-\beta}{2})-1)$$

$$-\int_0^{(\alpha+z)\wedge\frac{\alpha-\beta}{2}}xdF(x)+\int_{\frac{\alpha-\beta}{2}}^{\infty}xdF(x).$$

This function is well defined because of $\mu<\infty$. Let $\{f_{n,l,k}\}_{0\leqq k,\, l\leqq n,\, k+l\leqq n}$ be a sequence as follows.

$$f_{n,l,k}=f_{n-1,l,k}+C_F(f_{n-1,l,k},\ \sum_{i=0}^{l-1}f_{n-1,i,k},\ \sum_{j=0}^{k-1}f_{n-1,l,j})$$

$$+\sum_{i=1}^{l}\sum_{\substack{j=1\\(i,j)\neq(l,k)}}^{k}(f_{n,i,j}-f_{n-1,i,j}),\qquad (0<k+l<n,\ n\geqq 2)$$

$$f_{n,l,n-l}=v_n(n-l,l,n)-\sum_{i=0}^{l}\sum_{\substack{j=0\\(i,j)\neq(l,n-l)}}^{n-l}f_{n,i,j},$$

$$f_{n,n,0}=-n\mu-\sum_{i=0}^{n-1}f_{n,i,0},$$

$$f_{n,0,n}=n\mu-\sum_{j=0}^{n-1}f_{n,0,j},$$

$$f_{n,0,k}=h_{n,k},\quad f_{n,l,0}=-g_{n,l},$$

$$f_{n,0,0}=0,\quad f_{1,0,1}=\mu,\quad f_{1,1,0}=-\mu,$$

where $\sum_{i=0}^{l}\sum_{\substack{j=0\\(i,j)\neq(l,k)}}^{k}f_{n,i,j}=\sum_{i=0}^{l}\sum_{j=0}^{k}f_{n,i,j}-f_{n,l,k}$.

Lemma 6.

$$(10)_n\qquad v_n(k,l,n)=\sum_{i=0}^{l}\sum_{j=0}^{k}f_{n,i,j}.$$

The optimal strategy for player I is

$$\text{if}\begin{cases} 0\leqq x<(\alpha+z)\wedge\frac{\alpha-\beta}{2} & \text{then play a card L}\\ (\alpha+z)\wedge\frac{\alpha-\beta}{2}\leqq x<\frac{\alpha-\beta}{2} & \text{then pass this stage}\\ \frac{\alpha-\beta}{2}\leqq x<\infty & \text{then play a card H}\end{cases}$$

where $z=f_{n-1,l,k}$, $\alpha=\sum_{i=0}^{l-1}f_{n-1,i,k}$, $\beta=\sum_{j=0}^{k-1}f_{n-1,l,j}$.

Proof: We employ the proof by induction on n. For $n=1$, it is obvious from the definition. We get, for $1<k,l<n$,

$$v_n(k,l,n)=\int_0^\infty max\{-x+v_{n-1}(k,l-1,n-1),\ x+v_{n-1}(k-1,l,n-1),v_{n-1}(k,l,n-1)\}dF(x)$$

$$=v_{n-1}(k,l,n-1)+C_F(z,\alpha,\beta).$$

Substitute $(10)_{n-1}$ on the right hand side of the last equation, then we get $(10)_n$. The other cases are derived by a simple examination.

In the next place we consider the case where $kl=0$ and $0<m<n$. Define the function by

$$A_F(z,\alpha,\beta)=\alpha F(\alpha)-(\alpha+\beta+z)F(\alpha\vee\beta)$$
$$+\int_{\alpha}^{\alpha\vee\beta} x\,dF(x)-(\alpha+z)(\beta+z)\int_{\alpha\vee\beta}^{\infty}\frac{1}{x+z}dF(x),$$

where $\beta > -z$ and $\alpha\vee\beta=\max\{\alpha,\beta\}$. This function is well defined because of $\mu < \infty$, and is convex in z.

Let's define a sequence $\{g_{n,k,m}\}_{1\leqq k,m\leqq n}$ as follows:

$$g_{n,k,m}=g_{n-1,k,m}+A_F\Big(g_{n-1,k,m},\ \sum_{j=1}^{m-1}g_{n-1,k,j},\ \sum_{i=1}^{k-1}g_{n-1,j,m}\Big)$$
$$-\sum_{i=1}^{k}\sum_{\substack{j=1\\(i,j)\neq(k,m)}}^{m}(g_{n,i,j}-g_{n-1,i,j}),$$
$$(\ n\geqq 2,\ 1<k,m<n-1\ )$$

$$g_{n,k,n}=v_n(k,0,n)-\sum_{i=1}^{k}\sum_{\substack{j=1\\(i,j)\neq(k,n)}}^{n}g_{n,i,j},$$

$$g_{n,n,m}=v_n(n,0,m)-\sum_{i=1}^{n}\sum_{\substack{j=1\\(i,j)\neq(n,m)}}^{m}g_{n,i,j},\qquad g_{n,n,n}=n\mu-\sum_{i=1}^{n}\sum_{\substack{j=1\\(i,j)\neq(n,n)}}^{n}g_{n,i,j},$$

$$g_{1,1,1}=\mu,\ g_{1,0,1}=0,\ g_{1,1,0}=0.$$

Proposition 1. In the game $G_n(k,0,m)$,

$(11)_n$ $$v_n(k,0,m)=\sum_{i=1}^{k}\sum_{j=1}^{m}g_{n,i,j},$$

and the optimal strategy for each player is as follows,

Condition	For I	For II
$0\leqq x<\alpha$	pass	play M
$\alpha\leqq x<\alpha\vee\beta$	play H	play M
$\alpha\vee\beta\leqq x<\infty$	Use the mixed strategy $\left(0,\frac{\beta+z}{x+z},\frac{x-\beta}{x+z}\right)$	Use the mixed strategy $\left(\frac{\alpha+z}{x+z},\frac{x-\alpha}{x+z}\right)$

where $z=g_{n-1,k,m}$, $\alpha=\sum_{i=1}^{m-1}g_{n-1,k,i}$, $\beta=\sum_{j=1}^{k-1}g_{n-1,j,m}$.

Proof: The proof by induction on n is employed. For $n=1$, the statement is trivial from the fact that $v_1(1,0,1)=g_{1,1,1}=\mu$.

We note by Lemmas 1 and 2 that

$$\alpha=v_{n-1}(k,0,m-1)-v_{n-1}(k-1,0,m-1)\geqq 0$$
$$\beta=v_{n-1}(k-1,0,m)-v_{n-1}(k-1,0,m-1)\geqq 0$$
$$z=v_{n-1}(k,0,m)-v_{n-1}(k,0,m-1)-v_{n-1}(k-1,0,m)+v_{n-1}(k-1,0,m-1).$$

We get, for $k\neq n$ and $m\neq n$,

$$v_n(k,0,m)= \int_0^\infty val \begin{pmatrix} x+v_{n-1}(k-1,0,m-1) & v_{n-1}(k-1,0,m) \\ v_{n-1}(k,0,m-1) & v_{n-1}(k,0,m) \end{pmatrix} dF(x)$$

$$= \int_0^\alpha v_{n-1}(k,0,m-1)dF(x)+ \int_\alpha^{\alpha\vee\beta} (x+v_{n-1}(k-1,0,m-1))dF(x)$$

$$+ \int_{\alpha\vee\beta}^\infty (v_{n-1}(k,0,m)- \frac{(\alpha+z)(\beta+z)}{x+z})dF(x)$$

$$=v_{n-1}(k,0,m) + A_F(z,\alpha,\beta)$$

This recursive equation has the solution $(11)_n$. In fact, substituting $(11)_{n-1}$ into the right-hand side of the equation, we get

$$\sum_{i=1}^k \sum_{j=1}^m g_{n-1,i,j}+A_F(g_{n-1,k,m}, \sum_{i=1}^{m-1} g_{n-1,k,i}, \sum_{j=1}^{k-1} g_{n-1,j,m})$$

$$= \sum_{\substack{i=1\\(i,j)\neq(k,m)}}^k \sum_{j=1}^m g_{n,i.j}- \sum_{\substack{i=1\\(i,j)\neq(k,m)}}^k \sum_{j=1}^m (g_{n,i,j}-g_{n-1,i,j})+g_{n-1,k,m}$$

$$+A_F(g_{n-1,k,m}, \sum_{i=1}^{m-1} g_{n-1,k,i}, \sum_{j=1}^{k-1} g_{n-1,j,m})$$

$$= \sum_{i=1}^k \sum_{j=1}^m g_{n,i,j}.$$

For the other cases, the equation $(11)_n$ is seen to be valid from a simple examination. This completes the proof.

Concerning the game $G_n(0,l,m)$ we get

Proposition 2.

$(12)_n$ $$v_n(0,l,m)= - \sum_{i=1}^l \sum_{j=1}^m g_{n,j,i},$$

and the optimal strategy for each player is as follows,

Condition	For I	For II
$0\leqq x<\beta$	play L	pass
$\beta\leqq x<\alpha\vee\beta$	play L	play M
$\alpha\vee\beta\leqq x<\infty$	Use the mixed strategy $(\frac{\beta+z}{x+z}, 0, \frac{x-\beta}{x+z})$	Use the mixed strategy $(\frac{\alpha+z}{x+z}, \frac{x-\alpha}{x+z})$

where $z=g_{n-1,m,l}$, $\alpha= \sum_{j=1}^{m-1} g_{n-1,j,l}$, $\beta= \sum_{i=1}^{l-1} g_{n-1,m,i}$.

Proof: We use the similar method as in Proposition 1. We note,

$$\alpha=v_{n-1}(0,l-1,m-1)-v_{n-1}(0,l,m-1) \geqq 0$$
$$\beta=v_{n-1}(0,l-1,m-1)-v_{n-1}(0,l-1,m) \geqq 0$$
$$z=v_{n-1}(0,l,m-1)-v_{n-1}(0,l-1,m-1)+v_{n-1}(0,l-1,m)-v_{n-1}(0,l,m),$$

and, if $l \neq n$ and $m \neq n$,

$$v_n(0,l,m)=\int_0^\infty val\begin{pmatrix} -x+v_{n-1}(0,l-1,m-1) & v_{n-1}(0,l-1,m) \\ v_{n-1}(0,l,m-1) & v_{n-1}(0,l,m)\end{pmatrix} dF(x)$$

$$=v_{n-1}(0,l,m)-A_F(z,\beta,\alpha).$$

Substituting $(12)_{n-1}$ into the right-hand side of the last equation, we get $(12)_n$. The other cases are gained from simple examinations.

The last case is the game $G_n(n-l,l,m)$ $(l \leqq n)$. Define the function by,

$$B_F(z,\alpha,\beta)=\beta F(-\beta)+(-\beta-(\alpha+z)/2)F(\tfrac{\alpha}{2}\vee(-\beta))+\frac{\alpha+z}{2}$$
$$-\int_{-\beta}^{\frac{\alpha}{2}\vee(-\beta)} x dF(x)+z(z+\alpha)\int_{\frac{\alpha}{2}\vee(-\beta)}^{\infty}\frac{1}{2x+2\beta+z}dF(x)$$

where $\frac{\alpha}{2}\vee(-\beta) > -\beta-\frac{z}{2}$. Let $\{h_{n,l,m}\}_{0\leqq l\leqq n,\,1\leqq m\leqq n}$ be a following sequence,

$$h_{n,l,m}=h_{n-1,l,m}+B_F(h_{n-1,l,m},\ \textstyle\sum_{j=1}^{m-1}h_{n-1,l,j},\ \sum_{i=0}^{l-1}h_{n-1,i,m})$$
$$-\sum_{\substack{i=0\\(i,j)\neq(l,m)}}^{l}\sum_{j=1}^{m}(h_{n,i,j}-h_{n-1,i,j}),\quad (n\geqq 2,\ 1\leqq l,m\leqq n)$$

$$h_{n,n,m}=v_n(0,n,m)-\sum_{\substack{i=0\\(i,j)\neq(n,m)}}^{n}\sum_{j=1}^{m}h_{n,i,j},$$

$$h_{n,l,n}=v_n(n-l,l,n)-\sum_{\substack{i=0\\(i,j)\neq(l,n)}}^{l}\sum_{j=1}^{n}h_{n,i,j},\qquad (1\leqq l<n)$$

$$h_{n,0,m}=v_n(n,0,m)-\sum_{j=1}^{m-1}h_{n,0,j},$$

$$h_{1,0,1}=\mu.$$

Proposition 3.

$(13)_n$ $\quad v_n(n-l,l,m)=\sum_{i=0}^{l}\sum_{j=1}^{m}h_{n,i,j}.$

and the optimal strategy for each player is as follows.

Condition	For I	For II
$0\leqq x<(-\beta)\vee 0$	play L	pass
$(-\beta)\vee 0\leqq x<(\frac{\alpha}{2})\vee(-\beta)\vee 0$	play L	play M
$(\frac{\alpha}{2})\vee(-\beta)\vee 0\leqq x<\infty$	Use the mixed strategy $(\frac{x+\beta+z}{2x+2\beta+z},\frac{x+\beta}{2x+2\beta+z},0)$	Use the mixed strategy $(\frac{\alpha+z}{2x+2\beta+z},\frac{2x+2\beta-\alpha}{2x+2\beta+z})$

where $z=h_{n-1,l,m}$, $\alpha=-\sum_{i=0}^{l-1}h_{n-1,i,m}$, $\beta=\sum_{j=1}^{m-1}h_{n-1,l,j}$.

Proof: Since the method of the proof is similar to that of Proposition 1., we only skech the outline of the proof.

From the definition, we have the following representation.

$$\alpha=v_{n-1}(n-l,l-1,m-1)-v_{n-1}(n-l-1,l,m-1)$$
$$\beta=v_{n-1}(n-l,l-1,m)-v_{n-1}(n-l,l-1,m-1)$$
$$z=v_{n-1}(n-l,l-1,m-1)-v_{n-1}(n-l,l-1,m)-v_{n-1}(n-l-1,l,m-1)+v_{n-1}(n-l-1,l,m)$$

and, if $0 < l < n$ and $1 < m < n$,

$$v_n(n-l,l,m)=\int_0^\infty val\begin{pmatrix}-x+v_{n-1}(n-l,l-1,m-1) & v_{n-1}(n-l,l-1,m)\\ x+v_{n-1}(n-l-1,l,m-1) & v_{n-1}(n-l-1,l,m)\end{pmatrix}dF(x)$$

$$=v_{n-1}(n-l-1,l,m)+B_F(z,\alpha,\beta).$$

For getting the relation $(13)_n$, substitute $(13)_{n-1}$ into the right-hand side of the last equation. From a similar manner we obtain the other cases. Thus the proof is complete.

Now we get the following proposition. From this proposition we can obtain the optimal strategy and the value of this game.

Proposition 4. The value of the game, $v_n(k,l,m)$, satisfies the recursive equation (1) with the boundary condition (6), (7), (9), (10), (11), (12), (13) and $v_n(0,0,m)=v_n(k,l,0)=0$. When the target has the value x, the optimal strategy for each player is that of the matrix game in the integrand of (1).

Here we poit out the following fact from the calculation of the recursive equation (1). The optimal strategy of each player is; from the lower value of the target to the high value, the optimal strategy pair changes from (play L, pass), (play L, play M), (pass, play M) to (play H, play M), and for the higher value of the realized value x, both players use the mixed strategy. It may be considered that for the relatively small value of a target player II wins or passes, for the intermediate value player I wins or passes, and for the higher value both player use the mixed strategy. The similar condition is considered for the lemmas and propositions of this section. Let's consider the case of Proposition 1. In this case, if both player discard, player I always wins from player II. So player II spends his cards for the small values of the targets which are not valuable for player I to win from player II and he passes for these values of the targets. For the higher value of the targets, both plyers use the mixed strategy in order to defend the attack of player I for getting these values. When the value is of the intermediate interval, both players play (player I always wins from player II) and these values are thought to be reasonable for both players. The other cases are considered similarly.

Remark that when we consider that X's are independent but not identically distributed, the result of lemmas and propositions of Section 2. and 3. go through in a similar manner.

Example. We consider that $F(x)=x$ $(x\epsilon[0,1])$. We find, for $n=1$,

$$v_1(0,1,1)=\int_0^1 xdx=\frac{1}{2}\ (=g_{1,1}),\quad v_1(1,0,1)=-\int_0^1 xdx=-\frac{1}{2},$$

and, for $n=2$,

$$v_2(0,1,1)=g_{2,1,1}=g_{1,1,1}+A_F(g_{1,1,1},0,0)=\int_0^1 val\begin{pmatrix} x & 0 \\ 0 & \frac{1}{2}\end{pmatrix} dx=\frac{2-log3}{4}\equiv a,$$
$$(g_{1,1,1}=\frac{1}{2})$$

$$v_2(1,0,1)=-a,$$
$$v_2(1,0,2)=\int_0^1 max\{-x,-\frac{1}{2}\}dx=-\frac{1}{2}+T_F(h_{1,1})=-\frac{3}{8},$$
$$v_2(1,0,2)=S_F(g_{1,1})=\frac{5}{8},$$
$$v_2(1,1,2)=p_{2,1}+p_{2,0}=\int_0^1 max\{x-\frac{1}{2},-x+\frac{1}{2}\}dx=-\frac{1}{4},$$
$$v_2(1,1,1)=\int_0^1 val\begin{pmatrix} -x & \frac{1}{2} \\ x & -\frac{1}{2}\end{pmatrix}dx=0,\quad (h_{1,0,1}+h_{1,1,1}=\mu-\mu=0)$$

and, for $n=3$,

$$v_3(1,1,2)=\int_0^1 val\begin{pmatrix} -x+a & \frac{5}{8} \\ x-a & -\frac{3}{8} \\ 0 & \frac{1}{4}\end{pmatrix} dx=\int_0^a(-x+a)dx+\frac{1}{4}\int_a^1\frac{x-a}{8x-8a+5}dx$$
$$=\frac{a^2}{2}+\frac{1}{32}(1-a)+\frac{5}{4}log\frac{5}{13-8a},$$

and the optimal strategy for each player is,

Condition	For I	For II
$0\leqq x<a$	play L	play M
$a\leqq x\leqq 1$	Use the mixed strategy $(0,\ \frac{a}{8x-8a+5},\frac{8x-8a+3}{8x+8a+5})$	Use the mixed strategy $(\frac{5}{8x-8a+5},\frac{8x-8a}{8x-8a+5})$

4. Deteministic Case

In this section, we consider the case of deterministic value, i.e., $X=1$, with probability 1. The fundamental recursive equation is given, from (1), by

$$(14)\qquad v_n(k,l,m)=val\begin{pmatrix} -1+v_{n-1}(k,l-1,m-1) & v_{n-1}(k,l-1,m) \\ 1+v_{n-1}(k-1,l,m-1) & v_{n-1}(k-1,l,m) \\ v_{n-1}(k,l,m-1) & v_{n-1}(k,l,m)\end{pmatrix}$$

with the boundary conditions that $v_n(k,l,0)=0$ and $v_n(0,0,m)=0$.

Proposition 5. The solution of the recursive equation (14) is

$$v_n(k,l,m)=\frac{m(k-l)}{n}.$$

The optimal strategy for each player is;

for player I, $x^*=tx_1+(1-t)x_2$, $(t\in[0,1])$

where
$$x_1=\left(\frac{n+k-l}{2n},\ \frac{n+l-k}{2n},\ 0\right),$$
$$x_2=\begin{cases}\left(0,\ \frac{k-l}{n},\ \frac{n+l-k}{n}\right) & \text{if } k \geqq l,\\ \left(\frac{l-k}{n},\ 0,\ \frac{n-l+k}{n}\right) & \text{if } k < l,\end{cases}$$

and for player II, $y^*=\left(\frac{m}{n},\ \frac{n-m}{n}\right)$.

Proof: A proof by induction on n is employed. For $n=1$, the statements are obvious. Assume the validity for $n-1$. From the inductive hypothesis we get, for $1 < k,l,m < n$,

$$v_n(k,l,m)=val\begin{pmatrix} -1+\frac{(m-1)(k-l+1)}{n-1} & \frac{m(k-l+1)}{n-1}\\ 1+\frac{(m-1)(k-l-1)}{n-1} & \frac{m(k-l-1)}{n-1}\\ \frac{(m-1)(k-l)}{n-1} & \frac{m(k-l)}{n-1}\end{pmatrix} \tag{15}$$

and note that,

$$\frac{m(k-l+1)}{n-1} \geqq -1+\frac{(m-1)(k-l+1)}{n-1},$$
$$\frac{m(k-l-1)}{n-1} \leqq 1+\frac{(m-1)(k-l-1)}{n-1},$$

and the one-half of the sum of the first and the second row vectors of the matrix in (15) is the third row vector. We can solve this matrix game easily. So we get the value $v_n(k,l,m)=\frac{m(k-l)}{n}$, and the optimal strategy for each player descrived in this proposition. (For the zero-sum game, if the value is known then the optimal strategy for each player is obtained easily.) This completes the proof.

In order to obtain the value of the game, player II must allocate his available card such that his probability of playing is equal to the ratio of the numbers of his cards in hand to the total number of the remaining period. Player I must use his optimal strategy which only depends the number of his cards in hand and the total number of the remaining period, and which doesn't depend on the number of the cards in the opponent's hand. Both players need not consider the opponent's hand, and only consider their own hand and the remaining period.

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