

# A GROUP THEORETIC DUAL PROBLEM WITHOUT DUALITY GAPS FOR BOUNDED INTEGER PROGRAMS

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*Abstract* We present a procedure for constructing a group theoretic dual problem with no duality gap to a given bounded integer programming problem. An optimal solution of this dual problem is easily determined and an optimal solution of the integer programming problem can be obtained by solving only one group optimization problem.

## Introduction

Since Held and Karp [11,12] reported a very promising result about their algorithm for the travelling salesman problem, their approach, which is now interpreted as a dual method, has been extended to general integer programmings (see [8]). Especially Fisher and Shapiro [5] combined the dual method with the relaxation of linear constraints into a group congruence (see also [6]). They showed that their dual problem provides a better bound than the ordinary linear programming relaxation. However it can be seen that there generally exist duality gaps. Bell [1,2], and Bell and Shapiro [3] have proved that duality gaps can be closed up by introducing increasingly stronger group congruence conditions in finite Abelian groups.

In the subsequent sections of this paper we shall propose a procedure to construct a dual problem for bounded integer programmings, and show that duality gaps are eliminated and an optimal solution is easily obtained in this dual problem.

## 1. Lagrangean dual problem

We consider the following integer programming problem:

$$(1-1) \quad \begin{aligned} z^* = & \text{minimize } cx, \\ & \text{subject to } x \in X = \{ x \mid Ax \leq b, 0 \leq x \leq M, x \in Z^n \}, \end{aligned}$$

where  $A = (a_{ij}) \in Z^{m \times n}$ ,  $b = (b_1, b_2, \dots, b_m)^t \in Z^m$ ,  $M = (M_1, M_2, \dots, M_n)^t \in Z^n$  and  $c = (c_1, c_2, \dots, c_n) \in R^n$ . Without loss of generality we assume that the region  $\{ x \mid Ax \leq b, 0 \leq x \}$  is bounded (e.g., add  $\sum_{j=1}^n x_j \leq \sum_{j=1}^n M_j$ ). By relaxing the inequality constraints  $Ax \leq b$  in  $X$ , we define the set

$$\bar{X} = \{ x \mid 0 \leq x \leq M, x \in Z^n \},$$

which we shall also write as  $\bar{X} = \{ x^s \mid s \in S \}$ . Putting nonnegative multipliers  $u = (u_1, u_2, \dots, u_m)$  on the constraints  $Ax \leq b$ , we define

$$L(u, s) = -ub + (c + uA) x^s$$

and call

$$L(u) = \min \{ L(u, s) \mid s \in S \}$$

the Lagrangean function for the problem (1-1). It is known and easily shown that  $L(u) \leq z^*$  for any nonnegative multiplier vector  $u$ . Thus the problem to find the best of these lower bounds is formulated as

$$(1-2) \quad \begin{aligned} L^* = & \text{maximize } L(u), \\ & \text{subject to } u \geq 0. \end{aligned}$$

This problem, a maximization problem of the piecewise linear concave function  $L(u)$ , is called the Lagrangean dual problem of (1-1) [8]. Although several algorithms have been proposed for this problem [5,6,10], the optimal multiplier vector is not easy to obtain and there is no guarantee that  $L^* = z^*$ , i.e., duality gaps generally exist.

For a positive integer vector  $d = (d_1, d_2, \dots, d_m)^t$  define the sets

$$Y_d = \left\{ (x, y) \left| \begin{array}{l} y + Ax \leq b, y + Ax \equiv b \pmod{d}, \\ 0 \leq x \leq M, 0 \leq y < d, x \in Z^n, y \in Z^m \end{array} \right. \right\},$$

$$X_d = \{ x \mid (x, y) \in Y_d \text{ for some } y \},$$

then we obtain Lemma 1, where  $a \equiv b \pmod{d}$  means that  $a_i \equiv b_i \pmod{d_i}$  for  $i = 1, 2, \dots, m$ . Though the proof of the following lemma can be found in [1], we shall give the proof for consistency.

Lemma 1.

$X = X_d$  for any positive integer vector  $d$ .

proof. It is sufficient to show that  $X \subset X_d$ . For an arbitrary vector  $x$  in  $X$ ,  $\bar{y} = b - Ax$  is a nonnegative integer vector. Then there uniquely exists a pair of vectors  $\bar{y}$  and  $y$  such that  $\bar{y} = \Delta \bar{y} + y$ ,  $0 \leq \bar{y}$ ,  $0 \leq y < d$ , where

$$\Delta = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{pmatrix}.$$

Hence  $b - (y + Ax) = \Delta \bar{y} \geq 0$  and  $\equiv 0 \pmod{d}$ , i.e.,  $x$  is in  $X_d$ . □

Relaxing again the inequality conditions in  $Y_d$  and  $X_d$  defines

$$\bar{Y}_d = \left\{ (x,y) \mid \begin{array}{l} y + Ax \equiv b \pmod{d}, \\ 0 \leq x \leq M, 0 \leq y < d, x \in Z^n, y \in Z^m \end{array} \right\},$$

$$\bar{X}_d = \{ x \mid (x,y) \in \bar{Y}_d \text{ for some } y \}.$$

It is seen that  $\bar{X} = \bar{X}_d$  for any positive integer vector  $d$  in the same way as in the proof of Lemma 1 and for any  $x \in \bar{X}_d$  the vector  $y$  such that  $(x,y) \in \bar{Y}_d$  is uniquely determined. Hence we also denote  $\bar{Y}_d = \{ (x^s, y^s) \mid s \in S \}$ . Therefore the problem (1-1) may be written equivalently as

$$(1-3) \quad \begin{array}{l} z^* = \text{minimize } cx^s, \\ \text{subject to } y^s + Ax^s \leq b, s \in S. \end{array}$$

For any nonnegative multiplier vector  $u$ , define

$$\begin{array}{l} L_d(u,s) = -ub + (c + uA) x^s + uy^s, \\ L_d(u) = \min \{ L_d(u,s) \mid s \in S \}, \end{array}$$

then the problem (1-3) is dualized as

$$(1-4) \quad \begin{array}{l} L_d^* = \text{maximize } L_d(u), \\ \text{subject to } u \geq 0. \end{array}$$

It should be noted that if  $d = (1,1,\dots,1)^t$  then  $L_d(u,s) = L(u,s)$  for any  $s \in S$ .

In the next section we shall show that there exists a finitely large  $d$  for which an optimal multiplier vector  $u^*$  is easily determined and it is sufficient to solve only one group optimization problem to obtain an optimal solution of the problem (1-1).

2. Resolution of the duality gap

Let  $D = (D_1, D_2, \dots, D_m)^t$  be defined such that for  $i = 1, 2, \dots, m$

$$(2-1) \quad D_i = \max \{ b_i - A_i x^s \mid s \in S \},$$

where  $A_i$  is the  $i$ -th row of the constraint matrix  $A$ . Note that  $D_i \geq 0$  if (1-1) has a feasible solution.

Lemma 2.

Let  $d > D$  be a positive integer vector. Then for any  $(x^s, y^s) \in \bar{Y}_d$ ,

- (i)  $y_i^s + A_i x^s - b_i = 0$  if  $A_i x^s \leq b_i$ ,
- (ii)  $y_i^s + A_i x^s - b_i \geq d_i$  otherwise.

proof. (i) Suppose that  $A_i x^s \leq b_i$ , then by (2-1) and  $0 \leq y_i^s < d_i$   
 $-d_i < y_i^s + A_i x^s - b_i < d_i$ .

Hence it follows from  $y_i^s + A_i x^s - b_i \equiv 0 \pmod{d_i}$  that  $y_i^s + A_i x^s - b_i = 0$ .

(ii) For  $x^s$  such that  $A_i x^s > b_i$ ,  $0 < y_i^s + A_i x^s - b_i \equiv 0 \pmod{d_i}$ . Then for some positive integer  $k$ ,  $y_i^s + A_i x^s - b_i = k \cdot d_i$  which is not less than  $d_i$ .  $\square$

Lemma 2 reveals that if  $x^s$  is feasible in (1-1) then the value of  $L_d(u, s)$  is independent of the amount of  $u$ , while if  $x^s$  is infeasible, i.e.,  $A_i x^s > b_i$  for some  $i$ , then  $L_d(u, s)$  ascends in the  $i$ -th coordinate direction (see Fig.1).

Define

$$\underline{z} = \min \{ cx^s \mid s \in S \},$$

and  $\bar{z}$  as an upper bound on  $z^*$ .

Theorem 3.

Let  $U_i = (\bar{z} - \underline{z}) / d_i$  for an arbitrarily fixed  $d > D$ . If (1-1) has a feasible solution, then (i) and (ii) hold for any  $u^* > U = (U_1, U_2, \dots, U_m)$ .

- (i)  $L_d(u^*) = L_d^* = z^*$ ,
- (ii)  $x^s$  is an optimal solution of (1-1) for any  $s \in S_d(u^*) = \{ s \mid s \in S, L_d(u^*) = L_d(u^*, s) \}$ .

proof. For an infeasible solution  $x^s$  of (1-1), define the nonempty set

$$I = \{ i \mid A_i x^s > b_i \},$$

then by Lemma 2, we have

$$\begin{aligned}
(2-2) \quad L_d(u^*, s) &= cx^s + u^* ( y^s + Ax^s - b ) \\
&= cx^s + \sum_{i \in I} u_i^* ( y_i^s + A_i x^s - b_i ) \\
&\geq cx^s + \sum_{i \in I} u_i^* \cdot d_i && \text{(by Lemma 2 and } u^* > 0 \text{)} \\
&> cx^s + \sum_{i \in I} ( \bar{z} - \underline{z} ) && \text{(by the choice of } u^* \text{)} \\
&\geq \bar{z} + ( cx^s - \underline{z} ) \\
&\geq \bar{z} && \text{(by the definition of } \underline{z} \text{)}
\end{aligned}$$

Let  $s^*$  be any index in  $S_d(u^*)$ . Since  $y^s + Ax^s - b = 0$  for any feasible solution  $x^s$  of (1-1), we have

$$(2-3) \quad L_d(u^*, s^*) \leq L_d(u^*, s) = cx^s$$

for any feasible solution  $x^s$  of (1-1), which implies that

$$(2-4) \quad L_d(u^*, s^*) \leq \bar{z}.$$

Then by (2-2) and (2-4) we have that  $x^{s^*}$  is feasible in (1-1), thus

$$(2-5) \quad L_d(u^*, s^*) = cx^{s^*}.$$

Comparing (2-3) and (2-5) gives the desired result (ii).

(i) follows directly from the weak duality  $L_d^* \leq z^*$  and  $z^* = cx^{s^*} = L_d(u^*)$ .  $\square$

Note that if  $x^{s^*}$  is infeasible in (1-1), then (1-1) has no feasible solution. Also note that  $\underline{z} = \sum_{j \in J_0} c_j M_j$  and  $D_i = b_i - \sum_{j \in J_i} a_{ij} M_j$ ,  $i = 1, 2, \dots, m$ , where  $J_0 = \{ j \mid c_j < 0 \}$ ,  $J_i = \{ j \mid a_{ij} < 0 \}$  and the summation is assumed to be zero if  $J_i$  is empty.

It follows from Theorem 3 that an optimal solution of (1-1) is obtained by solving the group optimization problem with the additional bounded-variable constraints:

$$\begin{aligned}
(2-6) \quad &\text{minimize } -u^*b + (c + u^*A)x + u^*y, \\
&\text{subject to } y + Ax \equiv b \pmod{d}, \\
&0 \leq x \leq M, 0 \leq y < d, x \in Z^n, y \in Z^m,
\end{aligned}$$

where the upper bound constraints on  $y$  might be deleted since  $u^* > 0$  and  $y_i - d_i \equiv y_i \pmod{d_i}$ .

For group optimization problems several algorithms have been proposed [4, 7, 10], and some of those are applicable to the bounded-variable case [9]. Although most of the algorithms for group optimization problems require the non-negativity of the objective function coefficients, it is not necessary in (2-6) since the bounded-variable constraints serve to make the number of possible

solutions finite. It is preferable, however, to make the objective function coefficients nonnegative from the computational point of view.

Lemma 4.

If the bounded region  $\{ x \mid Ax \leq b, 0 \leq x \}$  is nonempty, then there exists a multiplier vector  $u^*$  satisfying both  $c + u^*A \geq 0$  and  $u^* \geq V$  for a given  $V$ .

proof. Suppose on the contrary no such  $u^*$  exists, then by Gale's alternative theorem there exist nonnegative vectors  $z$  and  $y$  such that  $Az + y = 0$  and  $cz - Vy = -1$ . Then  $z \neq 0$  for otherwise  $cz - Vy = 0$ , which is a contradiction. Therefore we obtain that  $z \geq 0$  and  $Az = -y \leq 0$  which contradicts to the assumption that the region is bounded and nonempty.  $\square$

It is known that the problem (2-6) can be considered as a shortest path problem on a network having  $\prod_{i=1}^m d_i$  number of nodes. The edges of the network are classified into  $n+1$  groups and it is required that the number of edges of the same group contained in the path must not exceed the bounded constant value. Hence it is desired from the computational point of view in solving (2-6) that each component  $d_i$  is as small as possible. But if  $d_i$  is smaller than the value presented in Lemma 2, we must solve a maximization problem (1-4) of the piecewise linear concave function  $L_d(u)$  to find an optimal multiplier vector. Furthermore there is no guarantee that the duality gap is closed up. Thus further investigation and computational experiments are required in order to find the way of deciding "optimal" value of  $d$ .

### 3. Illustrative example

Consider the following knapsack problem:

$$\begin{aligned} & \text{minimize} && -(12x_1 + 7x_2 + 6x_3), \\ & \text{subject to} && 3x_1 + 2x_2 + 2x_3 \leq 4, \\ & && x_j \text{ is 0 or 1 for } j = 1, 2, 3 \end{aligned}$$

with an optimal solution  $x = (0, 1, 1)$  and the optimal value  $-13$ . By the definition we have  $d = 5$  and  $\underline{z} = -25$ . since  $x = (0, 0, 0)$  is a trivial feasible solution,  $\bar{z} = 0$  is an upper bound on  $z^*$ . Then  $U = 25/5 = 5$  and  $u^* = 6$  is sufficient to make the objective function coefficients nonnegative. The group optimization problem is as follows:

$$\begin{aligned}
 &\text{minimize } -24 + 6x_1 + 5x_2 + 6x_3 + 6y, \\
 &\text{subject to } 3x_1 + 2x_2 + 2x_3 + y \equiv 4 \pmod{5}, \\
 &\quad x_j \text{ is 0 or 1 for } j = 1, 2, 3, \\
 &\quad 0 \leq y \text{ integer.}
 \end{aligned}$$

To illustrate how  $d$  and  $u^*$  work,  $L_d(u, s)$  are depicted for all possible solutions  $x^s$  and for  $d = 1, 3, 5$  in Fig. 2 where darkened lines represent  $L_d(u)$ .

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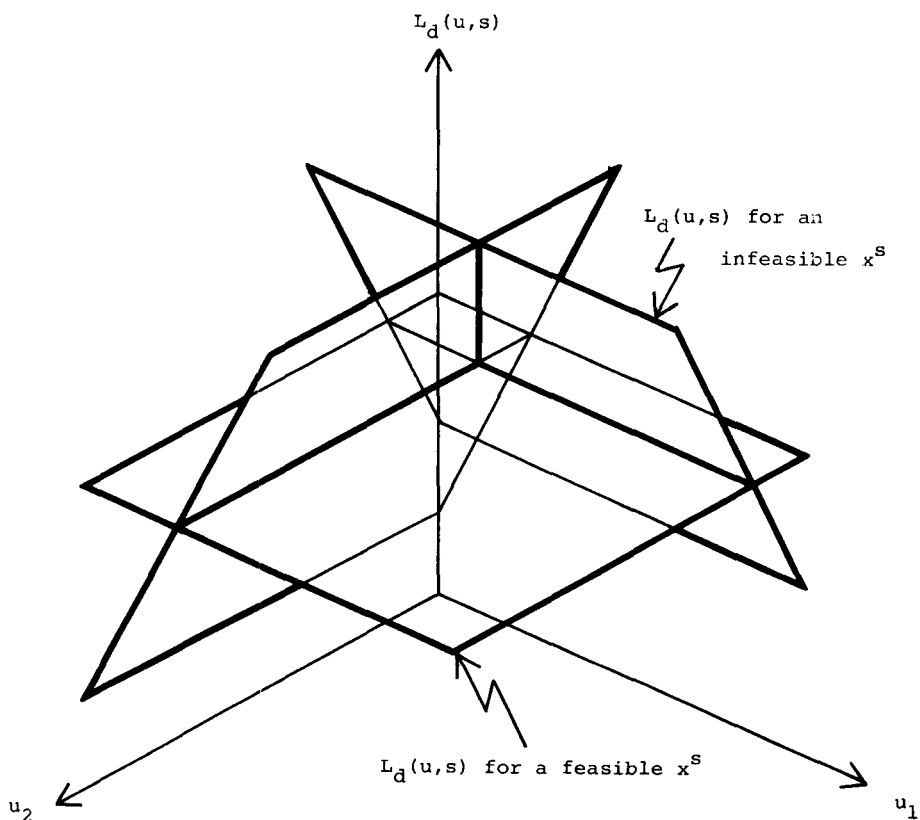


Fig. 1

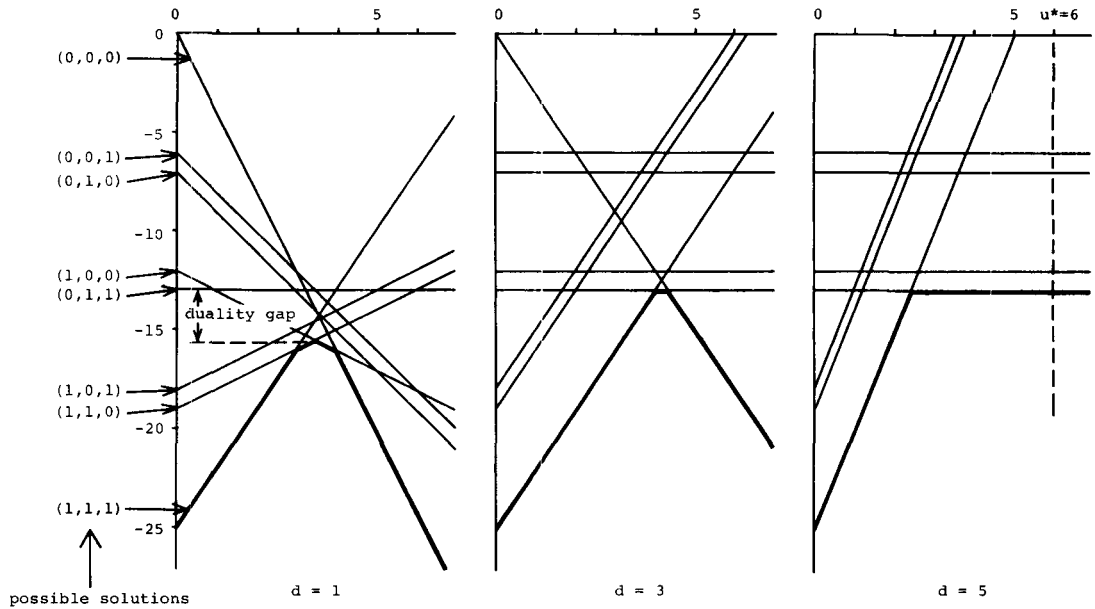


Fig. 2

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