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MAINTAINABILITY ANALYSIS OF A SYSTEM WITH PARTIAL REDUNDANCY

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Abstract This paper deals with a maintainability analysis of a system which consists of three subsystems. Two subsystems in the system are the same and constitute a warm standby redundant system. This warm standby redundant system and the residual subsystem are connected in series. In this paper, two repair disciplines are considered. One is a priority repair discipline and the other is a non priority repair discipline. The Laplace transform of the point-wise availability and the steady state availability for the system are derived under each repair discipline, and comparisions of the two results are made. Then the effect of the priority repair discipline on an increment of the system availability is discussed.

1. Introduction

An availability of a repairable system can be increased by allocating redundant systems. In many cases, it is enough to allocate the redundancy to a part of the system. For example, many on-line computer systems have two computers since the host computer system is the most important subsystem. This type of system is considered to be the system with partial redundancy.

This paper considers the system composed of three subsystems. Two subsystems in the system are the same and constitute a warm standby redundant system. This warm standby redundant system and the residual subsystem are connected in series. The availability of the system varies according to the repair discipline. Two repair disciplines are considered here. One is a preemptive-resume repair discipline, in which low priorities are assigned to the repairs of the warm standby redundant system and high priorities are assigned to the repairs of the residual subsystem. The other is a first come first served(FCFS) repair discipline. In the preemptive-resume repair discipline, the repair delay time is considered. The repair delay time is the time for the preemption of the repair. The Laplace transform of the point-wise availability and the steady state availability for the system are derived under each repair discipline, and comparisions of the two results are made. Then the effect of the repair delay time on an increment of the system availability under the preemptive-resume repair discipline is examined.

The system with partial redundancy was analyzed by Takamatsu, et al.[2] and Sahiar, et al.[3], and there are many other papers which deal with the generalizations of [3]. Takamatsu, et al. assumed that the repair discipline was a preemptive-repeat repair discipline. But a repair rate function is generally an increasing function. In this case, an availability of the system under a preemptive-resume repair discipline is supposed to be higher than the one under a preemptive-repeat repair discipline. In fact, Kodama, et al.[4] showed the validity of the surmise when the system is a two units warm standby redundant system. Sahiar, et al. analyzed the system under a preemptive-resume repair discipline and a preemptive-repeat repair discipline on the assumption that the system failures were caused by the failures of the warm standby redundant system, and the failures of the residual subsystem only debased system efficiency.

2. Models

In this paper, two models are defined according to the repair discipline.

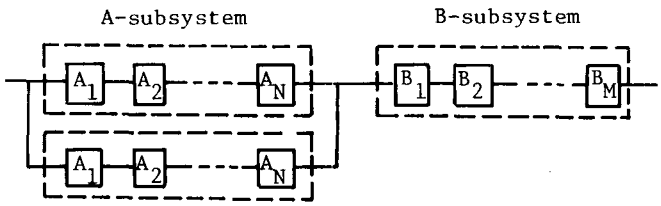


Fig.1. The structure of the system.

2.1. Assumptions of Model I

(I-1) The system consists of two A-subsystems and B-subsystem(Fig.1). Two A-subsystems constitute the warm standby redundant system. This warm standby redundant system and B-subsystem are connected in series. The switchover time of a failed A-subsystem and a warm standby A-subsystem can be neglected.

(I-2) A-subsystem consists of N different units(A_i-unit; $i=1,2,\ldots,N$) in series. B-subsystem consists of M different units(B_j-unit; $j=1,2,\ldots,M$) in series.

(I-3) When a subsystem is in a failed state, the residual good units in the subsystem do not fail. When the system is in a failed state, the residual good units in the system do not fail.

(I-4) At time t=0, all the units in the system are good.

(I-5) There is only one repair facility. The repaired units are like new and are set on the system as soon as the repair completes.

(I-6) The failures of the units in the system are stochastically independent. The failure rate of an active x-unit is $\lambda_x(x=A_1,\ldots,A_N,B_1,\ldots,B_M)$. The failure rate of a warm standby x-unit is $\alpha\lambda_x(x=A_1,\ldots,A_N;\ 0\leqq\alpha\leqq 1)$.

(I-7) The repair time distribution function of x-unit has the probability density function $g_x(t)=\mu_x(t)\exp[-\int_0^t\mu_x(u)du]$, where $\mu_x(t)$ is the repair rate function of x-unit.

(I-8) The repair discipline is a preemptive-resume repair discipline[1]. The repairs of failed units in B-subsystem are assigned high priorities and the repairs of failed units in A-subsystem are assigned low priorities. Therefore, if the B_j-unit fails when an A_i-unit is being repaired, the repair of the A_i-unit is interrupted and the repair of the B_j-unit begins. As soon as the repair of the B_j-unit completes, the repair of the A_i-unit is resumed from where it was left off.

(I-9) When a repair of the B_j-unit preempts a repair of an A_i-unit, the repair time of the B_j-unit is prolonged and its probability density function is $g_{B_jD}(t)=g_{B_j}*g_D(t)$, where $g_D(t)=\mu_D(t)\exp[-\int_0^t\mu_D(u)du]$ and it is considered as the probability density function of the repair delay time caused by preemption. "*" denotes the convolution over [0,t]. If the repair delay time can be neglected, $g_D(t)=\delta(t)$(delta function) and therefore, $g_{B_jD}(t)$ $=g_{B_j}(t)$.

2.2. Assumptions of Model II

(II-1) Assumptions (I-1)-(I-7) are applied to Model II.

(II-2) The repair discipline is the FCFS repair discipline.

2.3. Notations

x,y: elapsed repair times of failed units in A-subsystem and B-subsystem, respectively

$\lambda_A=\sum_{i=1}^{N}\lambda_{A_i}$

$\lambda_B=\sum_{j=1}^{M}\lambda_{B_j}$

$\lambda'_x=(1+\alpha)\lambda_x \quad (x=A_1,\ldots,A_N,A)$

$g_A(t)=\sum_{i=1}^{N}(\lambda_{A_i}/\lambda_A)g_{A_i}(t)$

$g_B(t)=\sum_{j=1}^{M}(\lambda_{B_j}/\lambda_B)g_{B_j}(t)$

$g_{BD}(t)=g_B*g_D(t)$

$R_x(t)=1-\int_0^t g_x(\xi)d\xi \quad (x=A,B,BD,A_1,\ldots,A_N,B_1,\ldots,B_M)$

$\bar{\mu}_x=\int_0^\infty t\cdot g_x(t)dt \quad (x=A,B,D,BD)$

$A_x(t)$: system availability for Model x at time t $(x=I,II)$

$f^*(s)=\int_0^\infty \exp(-st)f(t)dt$, where f(t) is a general function

$A_x\equiv A_x(\infty)$

$P_\phi(t)$: probability that all the units in the system are good at time t

$P_{A_i}(t,x)\Delta x$: probability that only one A_i-unit fails and is under repair, and its elapsed repair time lies in $[x,x+\Delta x]$ at time t

$P_{B_j}(t,y)\Delta y$: probability that only B_j-unit fails and is under repair, and its elapsed repair time lies in $[y,y+\Delta y]$ at time t

$P_{A_i,B_j}(t,x)\Delta x$: probability that only one A_i-unit and B_j-unit fail and A_i-unit is under repair, and the elapsed repair time of A_i-unit lies in $[x,x+\Delta x]$ at time t

$P_{B_j,A_i}(t,x,y)\Delta x\Delta y$: probability that only one A_i-unit and B_j-unit fail and B_j-unit is under repair, and the elapsed repair times of A_i-unit and B_j-unit lie in $[x,x+\Delta x]$ and $[y,y+\Delta y]$, respectively, at time t

$P_{A_i,A_k}(t,x)\Delta x$: probability that only A_i-unit and A_k-unit(if $i=k$, two A_i-units) fail and A_i-unit is under repair, and its elapsed repair time lies in $[x,x+\Delta x]$

3. System Availability

From the assumptions, the following simultaneous integro-differential equations are derived for Model I and Model II.

(1) $(d/dt+\lambda_A'+\lambda_B)P_\phi(t)=\sum_{\ell=1}^{M}\int_0^\infty P_{B_\ell}(t,y)\mu_{B_\ell}(y)dy+\sum_{\ell=1}^{N}\int_0^\infty P_{A_\ell}(t,x)\mu_{A_\ell}(x)dx$ (Model I and Model II)

(2) $(\partial/\partial t+\partial/\partial x+\lambda_A+\lambda_B+\mu_{A_i}(x))P_{A_i}(t,x)=\sum_{\ell=1}^{M}\int_0^\infty P_{B_\ell,A_i}(t,x,y)\mu_{B_\ell D}(y)dy$ (Model I)

$=0$ (Model II)

(3) $(\partial/\partial t+\partial/\partial y+\mu_{B_j}(y))P_{B_j}(t,y)=0$ (Model I and Model II)

(4) $(\partial/\partial t+\partial/\partial y+\mu_{B_jD}(y))P_{B_j,A_i}(t,x,y)=0$ (Model I)

(4') $(\partial/\partial t+\partial/\partial x+\mu_{A_i}(x))P_{A_i,B_j}(t,x)=\lambda_{B_j}P_{A_i}(t,x)$ (Model II)

(5) $(\partial/\partial t+\partial/\partial x+\mu_{A_i}(x))P_{A_i,A_k}(t,x)=\lambda_{A_k}P_{A_i}(t,x)$ (Model I and Model II)

The followings are the boundary conditions and the initial condition.

(6) $P_{A_i}(t,0)=\lambda_{A_i}'P_\phi(t)+\sum_{\ell=1}^{N}\int_0^\infty P_{A_\ell,A_i}(t,x)\mu_{A_\ell}(x)dx$ (Model I and Model II)

(7) $P_{B_j}(t,0)=\lambda_{B_j}P_\phi(t)$ (Model I)

$=\lambda_{B_j}P_\phi(t)+\sum_{\ell=1}^{N}\int_0^\infty P_{A_\ell,B_j}(t,x)\mu_{A_\ell}(x)dx$ (Model II)

(8) $P_{B_j,A_i}(t,x,0)=\lambda_{B_j}P_{A_i}(t,x)$ (Model I)

(8') $P_{A_i,B_j}(t,0)=0$ (Model II)

(9) $P_{A_i,A_k}(t,0)=0$ (Model I and Model II)

(10) $P_{\phi}(0)=1$ (Model I and Model II)

In Equations (1)-(10), $i,k=1,2,\ldots,N$ and $j=1,2,\ldots,M$.

From Equations (1)-(10), Laplace transforms of point-wise availabilities for Model I and Model II are obtained as follows. For the detail of the calculations, see Appendix.

(11) $A^*_I(s)=a(s)/(s\cdot b(s))$,

(12) $A^*_{II}(s)=a'(s)/(s\cdot b'(s))$,

where

$$a(s)=\lambda_A(1+(\lambda'_A-s')R^*_A(s'))+s\{\lambda_A R^*_A(s)+\lambda_B R^*_{BD}(s)(1+\lambda'_A R^*_A(s'))\},$$

$$b(s)=\lambda_A(1-s'R^*_A(s'))(1+\lambda'_A R^*_A(s)+\lambda_B R^*_B(s))+\lambda'_A s'R^*_A(s')(\lambda_A R^*_A(s)+\lambda_B R^*_{BD}(s))$$
$$+s(1+\lambda_B R^*_B(s))(\lambda_A R^*_A(s)+\lambda_B R^*_{BD}(s)),$$

$$a'(s)=(\lambda_A+\lambda_B)(1+(\lambda'_A-\lambda_A)R^*_A(s''))+\lambda_A(R^*_A(s)-R^*_A(s''))s,$$

$$b'(s)=(\lambda_A+\lambda_B)\{1+\lambda'_A R^*_A(s)+\lambda_B R^*_B(s)+(-\lambda_A+(\lambda'_A-\lambda_A)\lambda_B R^*_B(s))R^*_A(s'')\}$$
$$+s(R^*_A(s)-R^*_A(s''))(\lambda_A-\lambda_B(\lambda'_A-\lambda_A)R^*_B(s)),$$

$$s'=s(1+\lambda_B R^*_{BD}(s))+\lambda_A,$$

and

$$s''=s+\lambda_A+\lambda_B.$$

From Equations (11) and (12), the steady state availabilities are obtained, using the final value theorem, as follows.

(13) $A_I=\{1+(\lambda'_A-\lambda_A)R^*_A(\lambda_A)\}/\{(1-\lambda_A R^*_A(\lambda_A))(1+\lambda'_A\bar{\mu}_A+\lambda_B\bar{\mu}_B)+\lambda'_A R^*_A(\lambda_A)(\lambda_A\bar{\mu}_A+\lambda_B\bar{\mu}_B+\lambda_B\bar{\mu}_D)\}$,

(14) $A_{II}=\{1+(\lambda'_A-\lambda_A)R^*_A(\lambda_A+\lambda_B)\}/\{1+\lambda'_A\bar{\mu}_A+\lambda_B\bar{\mu}_B+(-\lambda_A+(\lambda'_A-\lambda_A)\lambda_B\bar{\mu}_B)R^*_A(\lambda_A+\lambda_B)\}$.

The results are the same to the ones when A-subsystem and B-subsystem are single unit systems whose failure rates and repair density functions are λ_A,λ_B and $g_A(t),g_B(t)$, respectively.

4. Numerical Examples and Some Considerations

From Equations (13) and (14), the following results are obtained easily.

[Property 1]

Put $\tau^*=(1+\alpha\lambda_A\bar{\mu}_A)(R^*_A(\lambda_A)-R^*_A(\lambda_A+\lambda_B))/\{\lambda_B(1+\alpha\lambda_A R^*_A(\lambda_A+\lambda_B))R^*_A(\lambda_A)\}$.

Then, (1-i) $A_I\geq A_{II}$ i.f.f $\bar{\mu}_D\leq\tau^*$,

(1-ii) τ^* is an increasing function in α.

τ^* can be considered as one of measures which estimates the effect of the preemptive-resume repair discipline on the system availability, since as τ^* increases, the domain of $A_I \geq A_{II}$ extends.

Here we consider the gamma distribution and the Weibull distribution as the repair time distributions of A-subsystem. When $g_x(t)$ is the probability density function of the gamma distribution with the shape parameter m and the scale parameter β^{-1}, $g_x^*(s)$, $R_x^*(s)$ and $\bar{\mu}_x$ are as follows.

(15) $$g_x^*(s)=\frac{\beta^m}{(s+\beta)^m},$$

(16) $$R_x^*(s)=\frac{(s+\beta)^m-\beta^m}{s(s+\beta)^m},$$

(17) $$\bar{\mu}_x=\frac{m}{\beta}.$$

When $g_x(t)$ is the probability density function of the Weibull distribution with the shape parameter 2 and the scale parameter β^{-1}, $g_x^*(s)$, $R_x^*(s)$ and $\bar{\mu}_x$ are as follows.

(18) $$g_x^*(s)=1-(\sqrt{\pi}/2\sqrt{\beta})\exp(s^2/4\beta)\mathrm{erfc}(s/2\sqrt{\beta})s,$$

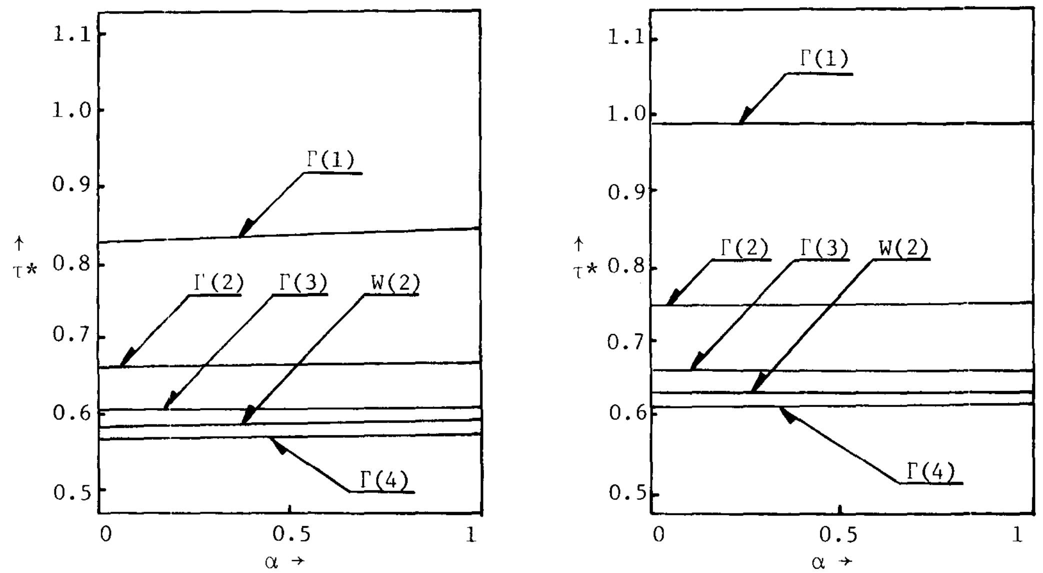
(19) $$R_x^*(s)=(\sqrt{\pi}/2\sqrt{\beta})\exp(s^2/4\beta)\mathrm{erfc}(s/2\sqrt{\beta}),$$

(20) $$\bar{\mu}_x=\sqrt{\pi}/2\sqrt{\beta},$$

where $\mathrm{erfc}(t)=1-(2/\sqrt{\pi})\int_0^t \exp(-x^2)dx$ (complementary error function).

Fig.2(a) and Fig.2(b) show the relation between τ^* and α. Fig.2(a) shows the case of $\lambda_A\cdot\bar{\mu}_A=0.1$, and Fig.2(b) shows the case of $\lambda_A\cdot\bar{\mu}_A=0.01$. When $\lambda_A\cdot\bar{\mu}_A$ is less than 0.01, the results are similar to that of Fig.2(b) except that τ^* increases as $\lambda_A\cdot\bar{\mu}_A$ decreases. From Fig.2(a) and Fig.2(b) and the fact just mentioned above, two insights are obtained. One is that τ^* is a decreasing function in $\bar{\mu}_A$ when the other parameters are fixed. The other is that τ^* is an increasing function in the variance of $g_A(t)$. In fact, the first one is easily seen from Fig.2(a) and Fig.2(b), and the second one is verified by the fact that the variance of $\Gamma(i)$(the gamma distribution with the shape parameter i and the mean value 1), $V_\Gamma(i)$, and the variance of W(2)(the Weibull distribution with the shape parameter 2 and the mean value 1), $V_W(2)$, are ordered as follows.

$$V_\Gamma(1)>V_\Gamma(2)>V_\Gamma(3)>V_W(2)>V_\Gamma(4)$$



(a) $\lambda_A=\lambda_B=0.1$, $\bar{\mu}_A=1$ (b) $\lambda_A=\lambda_B=0.01$, $\bar{\mu}_A=1$

Fig.2. The relations between τ^* and α, where $\Gamma(i)[W(i)]$ denotes that $g_A(t)$ is the probability density function of the gamma[Weibull] distribution function with a shape parameter i.

It is difficult to prove the validities of these conjectures rigorously since $R_A^*(s)$ includes $\bar{\mu}_A$ and the variance implicitly. But approximately they are true. Indeed, when s is sufficiently small,

(21) $R_A^*(s) \simeq \bar{\mu}_A - m_A s/2$,

where m_A is the second moment of $g_A(t)$. Equation (21) is obtained by neglecting the second order terms of the Maclaurin expansion of $R_A^*(s)$. Substituting Equation (21) to τ^*, we obtain

(22) $\tau^* \simeq \tilde{\tau}^* \equiv 2m_A(1+\alpha\lambda_A\bar{\mu}_A)/\{(2+\alpha\lambda_A(2\bar{\mu}_A - m_A(\lambda_A+\lambda_B)))(2\bar{\mu}_A - m_A\lambda_A)\}$.

Equation (22) is rewritten as follows.

(22') $\tilde{\tau}^* = m_A\{(1+\alpha\lambda_A\bar{\mu}_A)/(1-\alpha\lambda_A m_A(\lambda_A+\lambda_B)/2+\alpha\lambda_A\bar{\mu}_A)\}\{1/(2\bar{\mu}_A - m_A\lambda_A)\}$

If $m_A<2\bar{\mu}_A/(\lambda_A+\lambda_B)$, both $(1+\alpha\lambda_A\bar{\mu}_A)/\{1-\alpha\lambda_A\bar{\mu}_A(\lambda_A+\lambda_B)/2+\alpha\lambda_A\bar{\mu}_A\}$ and $1/(2\bar{\mu}_A - m_A\lambda_A)$ are positive decreasing functions in $\bar{\mu}_A$ and positive increasing functions in m_A.

Then $\tilde{\tau}^*$ is a decreasing function in $\bar{\mu}_A$ and an increasing function in m_A. The variance of $g_A(t)$ is $m_A-\bar{\mu}_A^2$, therefore, $\tilde{\tau}^*$ is an increasing function in the variance of $g_A(t)$.

The assumption, $m_A<2\bar{\mu}_A/(\lambda_A+\lambda_B)$, is fairly reasonable from the reason described below. The assumption is equivalent to

$$(23)\qquad \frac{m_A-\bar{\mu}_A^2}{\bar{\mu}_A^2} < \frac{2}{(\lambda_A+\lambda_B)\bar{\mu}_A} -1.$$

Generally, a repair time distribution has an increasing hazard rate, and in this case, the coefficient of variation of the distribution is less than or equal to 1[5]. The left hand side of Inequality (23) is the square of the coefficient of variation of $g_A(t)$. The right hand side of Inequality (23) is larger than 1 if $\bar{\mu}_A(\lambda_A+\lambda_B)<1$. Then in almost every practical case, Inequality (23) holds. Consequently, we obtain the following result.

[property 2]

If $m_A<2\bar{\mu}_A/(\lambda_A+\lambda_B)$, then

(2-i) $\tilde{\tau}^*$ is a decreasing function in $\bar{\mu}_A$,

(2-ii) $\tilde{\tau}^*$ is an increasing function in the variance of $g_A(t)$.

5. Conclusion

We analyzed the system with partial redundancy and compares the two results, one is the result on the assumption that the repairs of failed units in B-subsystem have precedence to the repairs of failed units in A-subsystems, and the other is the results on the assumption that the repair discipline is FCFS. We obtained the following results.

The effectiveness of the priority repair discipline increases in the following three cases:

(i) when redundancy of the warm standby redundant system decreases,

(ii) when the mean repair time of A-subsystem decreases,

(iii) when the variance of the repair times of A-subsystem increases.

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Appendix Derivations of (11) and (13)

Laplace transforms of the equations for Model I, Eqs. (1)-(9), are obtained as follows.

(A1) $$(s+\lambda_A'+\lambda_B)P_\phi^*(s)=\sum_{\ell=1}^{M}\int_0^\infty P_{B_\ell}^*(s,y)\mu_{B_\ell}(y)dy+\sum_{\ell=1}^{N}\int_0^\infty P_{A_\ell}^*(s,x)\mu_{A_\ell}(x)dx+1$$

(A2) $$(s+\partial/\partial x+\lambda_A+\lambda_B+\mu_{A_i}(x))P_{A_i}^*(s,x)=\sum_{\ell=1}^{M}\int_0^\infty P_{B_\ell A_i}^*(s,x,y)\mu_{B_\ell D}(y)dy$$

(A3) $$(s+\partial/\partial y+\mu_{B_j}(y))P_{B_j}^*(s,y)=0$$

(A4) $$(s+\partial/\partial y+\mu_{B_j D}(y))P_{B_j A_i}^*(s,x,y)=0$$

(A5) $$(s+\partial/\partial x+\mu_{A_i}(x))P_{A_i A_k}^*(s,x)=\lambda_{A_k}P_{A_i}^*(s,x)$$

(A6) $$P_{A_i}^*(s,0)=\lambda_{A_i}'P_\phi^*(s)+\sum_{\ell=1}^{N}\int_0^\infty P_{A_\ell A_i}^*(s,x)\mu_{A_\ell}(x)dx$$

(A7) $$P_{B_j}^*(s,0)=\lambda_{B_j}P_\phi^*(s)$$

(A8) $$P_{B_j A_i}^*(s,x,0)=\lambda_{B_j}P_{A_i}^*(s,x)$$

(A9) $$P_{A_i A_k}^*(s,0)=0$$

From (A4) and (A8), we obtain

(A10) $$P_{B_\ell A_i}^*(s,x,y)=\lambda_{B_\ell}P_{A_i}^*(s,x)\exp[-\int_0^y(s+\mu_{B_\ell D}(u))du].$$

Substituting (A10) into (A2) and solving (A2), we get

(A11) $P^*_{A_i}(s,x)=P^*_{A_i}(s,0)\exp[-\int_0^x(s'+\mu_{A_i}(u))du]$,

where $s'=s(1+\lambda_B R^*_{BD}(s))+\lambda_A$.

From (A5), (A9) and (A11), we have

(A12) $P^*_{A_iA_k}(s,x)=\lambda_{A_k}P^*_{A_i}(s,0)\{\exp[-\int_0^x(s+\mu_{A_i}(u))du]-\exp[-\int_0^x(s'+\mu_{A_i}(u))du]\}/(s'-s)$.

Substituting (A12) into (A6), we get

(A13) $P^*_{A_i}(s,0)=P^*_\phi(s)[\lambda'_{A_i}+\lambda'_A\lambda_{A_i}(g^*_A(s)-g^*_A(s'))/\{s'-s-\lambda_A(g^*_A(s)-g^*_A(s'))\}]$.

From (A11), (A13) and (A3), (A7), we obtain

(A14) $\sum_{\ell=1}^{N}\int_0^\infty P^*_{A_\ell}(s,x)\mu_{A_\ell}(x)dx=\lambda'_A g^*_A(s')P^*_\phi(s)+\lambda'_A\lambda_A(g^*_A(s)-g^*_A(s'))g^*_A(s')P^*_\phi(s)/\{s'-s-\lambda_A(g^*_A(s)-g^*_A(s'))\}$,

(A15) $\sum_{\ell=1}^{M}\int_0^\infty P^*_{B_\ell}(s,y)\mu_{B_\ell}(y)dy=\lambda_B g^*_B(s)P^*_\phi(s)$.

Substituting (A14) and (A15) into (A1), we get

(A16) $P^*_\phi(s)=[s(1+\lambda_B R^*_B(s))+\lambda'_A R^*_A(s')s'+\lambda'_A\lambda_A(sR^*_A(s)-s'R^*_A(s'))g^*_A(s')/\{\lambda_A g^*_A(s')+s(\lambda_A R^*_A(s)+\lambda_B R^*_{BD}(s))\}]^{-1}$

From (A11) and (A13), we have

(A17) $P^*_A(s)\equiv\sum_{\ell=1}^{N}\int_0^\infty P^*_{A_\ell}(s,x)dx$

$=\lambda'_A R^*_A(s')(\lambda_A+\lambda_B R^*_{BD}(s)s)P^*_\phi(s)/\{\lambda_A g^*_A(s')+s(\lambda_A R^*_A(s)+\lambda_B R^*_{BD}(s))\}$.

Finally we obtain

(A18) $A^*_I(s)=P^*_\phi(s)+P^*_A(s)$

$=a(s)/(s\cdot b(s))$,

where a(s) and b(s) are given in Eq.(11).

Using the final value theorem, we get

$A_I=\lim_{s\to 0} sA_I(s)=a(0)/b(0)$.

Noticing that $R^*_A(0)=\bar{\mu}_A$, $R^*_B(0)=\bar{\mu}_B$ and $R^*_{BD}(0)=\bar{\mu}_B+\bar{\mu}_D$, we obtain Eq.(11).

Derivations of Eqs. (12) and (14) are similar to ones for Eqs.(11) and (13).

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