

## MAINTAINABILITY ANALYSIS CONSIDERING THE DISCARD OF FAILED UNITS

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(Received November 17, 1977; Final December 21, 1979)

*Abstract* The maintainability analysis considering the discard of failed units is discussed. The model includes the following properties of each unit: ordinary failure rate, fatal failure rate, repair rate, mean setting time, unit cost and mean setting time. In this model, a steady state availability is meaningless, therefore, a point-wise availability is discussed. The point-wise availability is obtained by a numerical method. But, in some cases, it is obtained analytically. Then optimal spare units allocation problems are discussed. The problems are formulated as nonlinear integer programming problems. Finally, some numerical examples are presented.

### 1. Introduction

In reliability theory, many efforts were devoted to maintainability analyses of various repairable systems. In the models analyzed before, analyses were based on the assumption that all the failed units were repaired. However, in some cases, the failed units may be discarded because of the technical problems or the repair cost. Thus the maintainability analysis considering the discard of failed units is an important and interesting problem in reliability theory.

The model considered in this paper is the system composed of  $M$  subsystems in series. Each subsystem consists of an operating unit, standby units and spare units. The model includes the system mission time and the following properties of each unit: ordinary failure rate, fatal failure rate, repair rate, mean setting time, unit cost and mean repair cost. In this model, the steady state availability always vanishes if each unit has a possibility of being discarded. Therefore, the point-wise availability is discussed.

In section 2, the model is defined and some notations are introduced. In section 3, the system equations are derived and the point-wise availabilities are obtained for a general case and special cases. In the general case, the equations are too complicated to solve. Therefore, the solution is obtained by a numerical method. The special cases are the cases that each subsystem is a 2-unit standby redundant system and a single unit system. In section 4, optimal spare units allocation problems are considered. The problems are formulated as nonlinear integer programming problems. In section 5, some numerical examples are presented.

## 2. Model and Assumption

In this paper, the following nomenclature and notations are used.

$i$ -subsystem: the system consisting of  $i$ -th standby redundant system and spare  $i$ -units; see Fig.1

$i$ -unit: the units in  $i$ -subsystem, one is an operating unit and the others are standby units and spare units

$i$ -th standby redundant system( $i$ -SRS): the redundant system consisting of an operating  $i$ -unit and standby  $i$ -units; see Fig.1

ordinary failure: repairable failure

fatal failure: unrepairable failure from the viewpoint of technical problems or repair cost

$M$ : number of the subsystems(all different) in the system

$i$ : index for subsystem,  $i=1,2,\dots,M$

$n_i$ : number of  $i$ -units allocated to  $i$ -SRS at time  $t=0$ , and at the same time the maximum number of  $i$ -units in  $i$ -SRS

$r_i$ : number of spare  $i$ -units allocated to  $i$ -SRS at time  $t=0$

$\lambda_i$ : constant ordinary failure rate of an  $i$ -unit

$\nu_i$ : constant fatal failure rate of an  $i$ -unit

$\mu_i$ : constant repair rate of an  $i$ -unit

$\epsilon_i$ : constant setting rate of a spare  $i$ -unit on  $i$ -SRS

$U_i$ : unit cost of an  $i$ -unit

$R_i$ : mean repair cost of an  $i$ -unit

$\mathbf{R}=(r_1, r_2, \dots, r_M)$ : allocation of spare units in the system at time  $t=0$

$A_s(\mathbf{R}, t)$ : system availability at time  $t$  for the allocation  $\mathbf{R}$

$C(\mathbf{R}, t)$ : mean system cost at time  $t$  for the allocation  $\mathbf{R}$

$T_0$ : system mission time

$A_0$ : lower bound of system availability

$C_0$ : upper bound of system cost

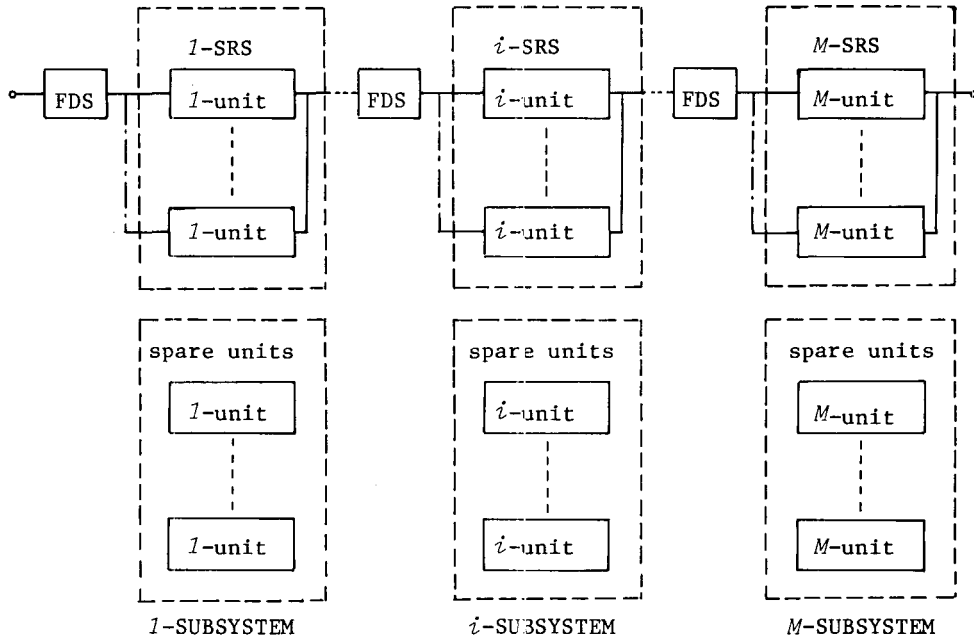


Fig.1. The system in the model.

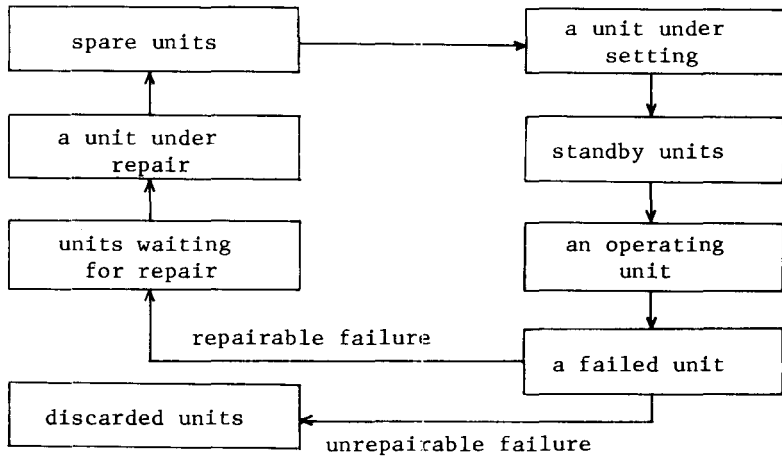


Fig.2. Flow of the units in the system.

The assumptions are as follows.

1. The concepts in the nomenclature above apply to the model.
2. The system consists of  $i$ -subsystem( $i=1,2,\dots,M$ ) in series; see Fig.1.
3. Occurrences of failures, completions of repairs and completions of settings are stochastically independent.
4. The failed units are repaired in the case of ordinary failures or discarded in the case of fatal failures.
5. Each  $i$ -subsystem has only one repair facility. The repair is done one by one. The repair is perfect and repaired units are stocked as spare units.
6. When an operating unit fails, one of standby units takes its place instantaneously by fault detection and switching(FDS), if available. FDS is perfect. The failed unit is removed from the system and it is sent to the repair facility in the case of ordinary failure or discarded in the case of fatal failure.
7. Whenever the number of  $i$ -units in  $i$ -SRS is less than  $n_i$ , one of spare units is set on  $i$ -SRS, if available.
8. Standby units and spare units do not fail.
9. Units are not replenished to the system in the period of mission time  $T_0$ .
10. At time  $t=0$ , all the units are perfect.

Fig.2 shows a flow of units in the system.

### 3. System Availability

#### 3.1. Analysis of the general case

Some new notations are introduced as follows.

$(j_i, k_i, l_i, m_i)$ : the possible states for  $i$ -subsystem;  $j_i + k_i + l_i + m_i = n_i + r_i$

$j_i$ : number of good  $i$ -units in  $i$ -SRS;  $0 \leq j_i \leq n_i$

$k_i$ : number of spare  $i$ -units;  $0 \leq k_i \leq n_i + r_i$

$l_i$ : number of failed  $i$ -units;  $0 \leq l_i \leq n_i + r_i$

$m_i$ : number of discarded  $i$ -units;  $0 \leq m_i \leq n_i + r_i$

$P_{j_i, k_i, l_i, m_i}^{j_i, k_i, l_i, m_i}(t)$ : probability for the state  $(j_i, k_i, l_i, m_i)$  at time  $t$

Fig.3 shows a part of the transition diagram of  $i$ -subsystem(index  $i$  is abbreviated).

From the assumptions, the detail 1 balance equations of  $i$ -subsystem are derived as follows(index  $i$  is abbreviated).

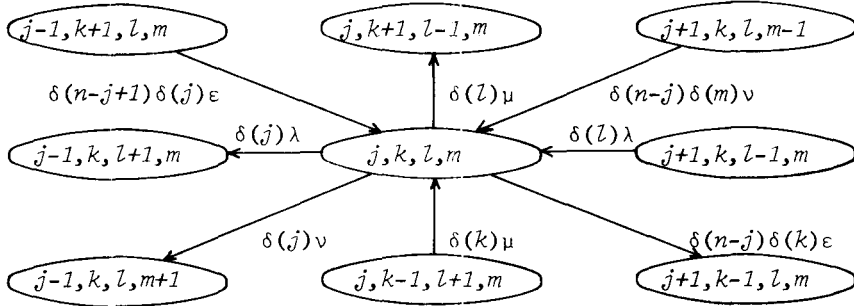


Fig.3. The transition diagram of the state  $(j, k, l, m)$ .

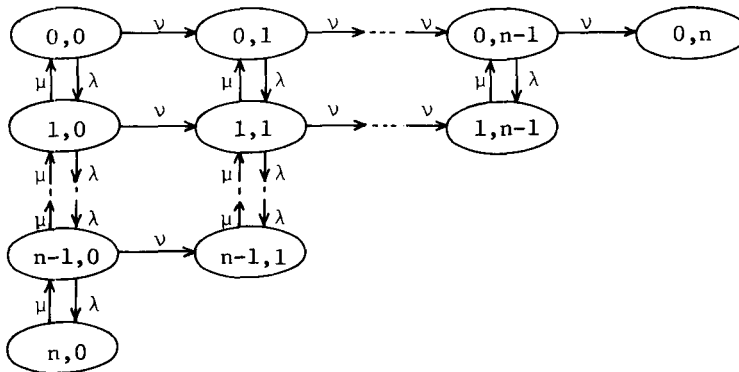


Fig.4. The transition diagram of  $i$ -subsystem in the case of  $r_i=0$ .

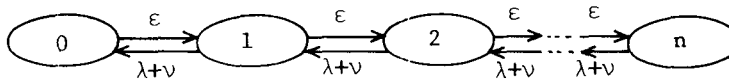


Fig.5. The transition diagram of  $i$ -subsystem in the case of  $r_i=\infty$ .

$$(1) \quad \frac{dP_{j,k,l,m}(t)}{dt} = -(\delta(j)(\lambda+\nu) + \delta(k)\delta(n-j)\epsilon + \delta(l)\mu)P_{j,k,l,m}(t) \\ + \lambda\delta(n-j)\delta(l)P_{j+1,k,l-1,m}(t) + \mu\delta(k)P_{j,k-1,l+1,m}(t) \\ + \epsilon\delta(j)\delta(n-j+1)P_{j-1,k+1,l,m}(t) + \nu\delta(n-j)\delta(m)P_{j+1,k,l,m-1}(t),$$

$$\text{where } \delta(x) = \begin{cases} 1 & (x > 0), \\ 0 & (x \leq 0) \end{cases}, \quad j+k+l+m=n+r, \quad 0 \leq j \leq n, \quad 0 \leq k \leq n+r, \quad 0 \leq l \leq n+r,$$

and the initial condition is

$$(2) \quad P_{j,k,l,m}(0) = \begin{cases} 1 & (j=n, k=r, l=m=0). \\ 0 & (\text{otherwise}) \end{cases}.$$

The equations are too complicated to obtain the solution analytically. Hence, a numerical method is used. Enumerating all the equations in (1) and applying an algorithm for solving simultaneous differential equations (e.g. Runge-Kutta-Gill method) to the equations with the initial condition (2), we obtain the solution. Then the availability of  $i$ -subsystem  $A_i(r_i, t)$  is

$$(3) \quad A_i(r_i, t) = \sum_{j_i \neq 0} P_{j_i, k_i, l_i, m_i}(t),$$

and system availability is

$$(4) \quad A_s(\mathbf{R}, t) = \prod_{i=1}^M A_i(r_i, t).$$

### 3.2. Analyses of the special cases

In many practical cases,  $n_i$  equals 1 (single unit system) or 2 (2-unit standby redundant system). And if the unit is complex system itself, the maintenance is usually done by only repair or exchange of the parts of the unit. In the former case,  $r_i=0$ , and in the latter case, the setting time can be neglected, hence,  $r_i=0$  (a spare unit can be regarded as a standby unit). In this section, the exact solutions are obtained for the four cases:  $(n_i, r_i) = (1, 0), (1, \infty), (2, 0), (2, \infty)$ . The cases of  $r_i=\infty$  are impractical, but as seen later, the solution for the case  $(n_i, r_i)$  converges rapidly to the solution for the case  $(n_i, \infty)$  as  $r_i$  increases. Therefore, they are very useful for the determinations of the upper bounds of the spare units in the optimal spare units allocation problems.

#### 3.2.1. The case of $r_i=0$

For simplicity, the indices of the parameters are abbreviated throughout this section. Some new notations are introduced.

$(a_i, b_i)$ : possible states for  $i$ -subsystem;  $0 \leq a_i + b_i \leq n_i$

$a_i$ : number of failed units in  $i$ -subsystem;  $0 \leq a_i \leq n_i$

$b_i$ : number of discarded units in  $i$ -subsystem;  $0 \leq b_i \leq n_i$

$P_{a_i, b_i}(t)$ : probability for the state  $(a_i, b_i)$  at time  $t$ ; state probability function

$A_i^!(n_i, t) = \sum_{a_i + b_i \neq n_i} P_{a_i, b_i}(t)$ : point-wise availability of  $i$ -subsystem without spare units

$$f^*(s) \equiv \int_0^\infty \exp(-st) f(t) dt$$

Fig.4 shows the transition diagram of  $i$ -subsystem. When  $n_i=1$ , the detailed balance equations of  $i$ -subsystem ( $i=1, 2, \dots, M$ ) are as follows, where the state probability functions are in the form of Laplace transforms.

$$(5) \quad (s+\lambda+\nu)P_{0,0}^*(s) = \mu P_{1,0}^*(s) + 1$$

$$(6) \quad sP_{0,1}^*(s) = \nu P_{0,0}^*(s)$$

$$(7) \quad (s+\mu)P_{1,0}^*(s) = \lambda P_{0,0}^*(s)$$

When  $n_i=2$ , the detailed balance equations are obtained as follows.

$$(8) \quad (s+\lambda+\nu)P_{0,0}^*(s) = \mu P_{1,0}^*(s) + 1$$

$$(9) \quad (s+\lambda+\nu)P_{0,1}^*(s) = \mu P_{1,1}^*(s) + \nu P_{0,0}^*(s)$$

$$(10) \quad sP_{0,2}^*(s) = \nu P_{0,1}^*(s)$$

$$(11) \quad (s+\lambda+\mu+\nu)P_{1,0}^*(s) = \lambda P_{0,0}^*(s) + \mu P_{2,0}^*(s)$$

$$(12) \quad (s+\mu)P_{1,1}^*(s) = \lambda P_{0,1}^*(s) + \nu P_{1,0}^*(s)$$

$$(13) \quad (s+\mu)P_{2,0}^*(s) = \lambda P_{1,0}^*(s)$$

Solving simultaneous equations (5)-(7) and applying inverse Laplace transform, we obtain  $A_i^!(1, t)$  as follows.

$$(14) \quad A_i^!(1, t) = ((s_1 + \mu) \exp(s_1 t) - (s_2 + \mu) \exp(s_2 t)) / (s_1 - s_2),$$

$$\text{where } s_1 = (-a + \sqrt{a^2 - 4\mu\nu}) / 2, \quad s_2 = -a - s, \quad a = \lambda + \mu + \nu.$$

Solving simultaneous equations (8)-(13), we obtain

$$(15) \quad A_i^!(2, s) = g(s) / \{(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)\},$$

$$\text{where } s_3 = -2r \cos(\phi/3) - b/3, \quad s_4 = 2r \cos((\pi - \phi)/3) - b/3, \quad s_5 = 2r \cos((\pi + \phi)/3) - b/3,$$

$$\phi = \cos^{-1}((2b^3 - 9bc + 27d) / 54r^3), \quad r = \text{sgn}(2b^3 - 9bc + 27d) \sqrt{(b^2 - 3c)/3},$$

$$b = 2a, \quad c = a^2 - \lambda\mu + \mu\nu, \quad d = \mu\nu a, \quad \text{sgn}(x) = \begin{cases} 1 & (x \geq 0) \\ -1 & (x < 0) \end{cases} \text{ and}$$

$$g(s) = s^4 + 3as^3 + (2a^2 + \lambda\mu + \mu^2 + 4\mu\nu)s^2 + \mu(a^2 + 4\lambda\nu + 5\mu\nu + 3\nu^2)s + 2\mu^2\nu a.$$

If  $s_j \neq s_k (j \neq k)$ ,  $A_i^!(2, t)$  are obtained as follows.

$$(16) \quad A_i^!(2, t) = \sum_{j=1}^5 (g(s_j) \exp(s_j t) / \prod_{k=1, \neq j}^5 (s_j - s_k))$$

Even if  $s_j = s_k$  for some  $j$  and  $k (\neq j)$ ,  $A_i^!(2, t)$  is obtained easily.

### 3.2.2 The case of $r_i = \infty$

When  $r_i = \infty$ , the transition diagram of  $i$ -subsystem is more simplified. Fig.5 shows the transition diagram, where state  $j$  is the state that the number of good  $i$ -units in  $i$ -SRS is  $j$ . Let  $P_j(t)$  be the state probability function for the state  $j$  and  $A_i''(n_i, t)$  be the point-wise availability of  $i$ -subsystem when  $r_i = \infty$

When  $n_i = 1$ , the detailed balance equations are as follows.

$$(17) \quad (s + \lambda + \nu)P_1^*(s) = \varepsilon P_0^*(s) + 1$$

$$(18) \quad (s + \varepsilon)P_0^*(s) = (\lambda + \nu)P_1^*(s)$$

When  $n_i = 2$ , the detailed balance equations are as follows.

$$(19) \quad (s + \lambda + \nu)P_2^*(s) = \varepsilon P_1^*(s) + 1$$

$$(20) \quad (s + \lambda + \nu + \varepsilon)P_1^*(s) = (\lambda + \nu)P_2^*(s) + \varepsilon P_0^*(s)$$

$$(21) \quad (s + \varepsilon)P_0^*(s) = (\lambda + \nu)P_1^*(s)$$

Solving the equations (17)-(21), we obtain  $A_i''(1, t)$  and  $A_i''(2, t)$  as follows.

$$(22) \quad A_i''(1, t) = \varepsilon / (\lambda + \nu + \varepsilon) + (\lambda + \nu) \exp(-(\lambda + \nu + \varepsilon)t) / (\lambda + \nu + \varepsilon)$$

$$(23) \quad A_i''(2, t) = \varepsilon(\lambda + \nu + \varepsilon) / ((\lambda + \nu)^2 + (\lambda + \nu)\varepsilon + \varepsilon^2) \\ - (\lambda + \nu) \sqrt{(\lambda + \nu)\varepsilon} (\exp(s_6 t) / s_6 - \exp(s_7 t) / s_7) / 2\varepsilon, \\ \text{where } s_6 = -(\lambda + \nu + \varepsilon) + \sqrt{(\lambda + \nu)\varepsilon}, \quad s_7 = -(\lambda + \nu + \varepsilon) - \sqrt{(\lambda + \nu)\varepsilon}.$$

## 4. Optimal Allocation of Spare Units

This section formulates the problem of optimally allocating spare units for the maximum availability as constrained by cost. The modified version is also formulated where cost is minimized as constrained by system availability. The total system cost is given in (24).

$$(24) \quad C(\mathbf{R}, t) = \sum_{i=1}^M \{(n_i + r_i)U_i + R_i \lambda_i t\},$$

where  $\lambda_i t$  approximates to the mean number of failures of  $i$ -units during  $t$  hours.

The problems are formulated as nonlinear integer programming problems shown below.

P1. Maximize  $A_s(\mathbf{R}, T_0)$ , subject to  $C(\mathbf{R}, T_0) \leq C_0$ .

P2. Minimize  $C(\mathbf{R}, T_0)$ , subject to  $A_s(\mathbf{R}, T_0) \geq A_0$ .



There are many algorithms for nonlinear integer programming problems. We use the algorithm presented in [1]. The algorithm is applied to the following nonlinear integer programming problem.

Minimize  $f_0(\mathbf{R})$ ,

subject to  $f_{j1}(\mathbf{R}) - f_{j2}(\mathbf{R}) \geq 0 \quad (j=1,2,\dots,N)$ ,

$\underline{r}_i \leq r_i \leq \bar{r}_i \quad (i=1,2,\dots,M)$ ,

where  $\mathbf{R}=(r_1, r_2, \dots, r_M)$ ,  $\mathbf{R}_{\max}=(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_M)$ ,  $\mathbf{R}_{\min}=(\underline{r}_1, \underline{r}_2, \dots, \underline{r}_M)$ ,

and  $f_0(\mathbf{R})$ ,  $f_{j1}(\mathbf{R})$ ,  $f_{j2}(\mathbf{R})$  are all nondecreasing functions in each of the variable  $r_i$  or are all nonincreasing functions in each of the variable  $r_i (j=1,2,\dots,N; i=1,2,\dots,M)$ .

## 5. Numerical Examples

### 5.1. System availability

Table 1 shows point-wise unavailabilities of  $i$ -subsystem at time  $t=500$  for the given parameters. As seen easily, unavailabilities of  $i$ -subsystem rapidly converge to  $A_i^u(n_i, t)$  as  $r_i$  increases. This property makes easy to solve the optimal spare units allocation problems as shown in the numerical examples presented later. The results in Table 1 are obtained by Runge-Kutta-Gill method.

Table 1. Unavailabilities of  $i$ -subsystem at time  $t=500$ .

	$(n_i, r_i)$	(1,0)	(1,1)	(1,2)	(1,3)	(1,∞)
$v=10^{-5}$ $\mu=0.1$ $\epsilon=1.0$	$\lambda=10^{-4}$	.0060756	.0001282	.0001100	.0001100	.0001100
	$\lambda=10^{-3}$	.0157622	.0011594	.0010104	.0010090	.0010090
$v=10^{-5}$ $\mu=0.01$ $\epsilon=1.0$	$\lambda=10^{-5}$	.0059817	.0000382	.0000200	.0000200	.0000200
	$\lambda=10^{-4}$	.0148427	.0002644	.0001114	.0001100	.0001100
$v=10^{-6}$ $\mu=0.1$ $\epsilon=1.0$	$\lambda=10^{-4}$	.0015976	.0001025	.0001010	.0001010	.0001010
	$\lambda=10^{-3}$	.0113659	.0010948	.0010009	.0010000	.0010000

5.2. Optimum spare units

Fig.6 shows a simplified scheme of a power generator system in a vessel. In this model, seriously damaged units are repaired after a mission(one voyage), therefore, this type of failures are considered to be unreparable failures. We use this model as the numerical example i.e.  $M=3$ ,  $n_1=2$  and  $n_2=n_3=1$ . Data of each unit are given in Table 2, and Table 3 shows availabilities of  $i$ -subsystem( $i=1,2,3$ ) for the given data.

Table 2. Data of each unit.

$i$	1(G)	2(ABT)	3(AQB)
$n_i$	2	1	1
$\lambda_i$	$10^{-4}$	.0005	$10^{-6}$
$\mu_i$	.005	.05	.02
$\epsilon_i$	1.0	1.0	10.0
$\nu_i$	$10^{-6}$	$10^{-6}$	$10^{-5}$
$U_i$	10000	5000	1000
$R_i$	5000	2000	2000
$C_0=50000 \quad A_0=0.99 \quad T_0=500$			

Table 3. Availabilities of  $i$ -subsystem at time  $t=500$ ( $i=1,2,3$ ).

$r \setminus i$	1	2	3
0	.99970976	.98911915	.99500743
1	.99999626	.99940007	.99998643
2	.99999995	.99949831	.99999888
3	.99999999	.99949924	.99999890
4	1.0	.99949925	.99999890
5	1.0	.99949925	.99999890

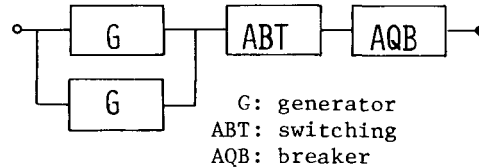


Fig.6. A power generator system in a vessel.

First, we solve P1. The problem is

Maximize  $A_s(\mathbf{R}, 500)$ , subject to  $C(\mathbf{R}, 500) \leq 50000$ .

From the data in Table 2,  $C((0,0,0), 500)=26751$ . Hence,  $(2,4,23)$  is one of upper bounds of  $\mathbf{R}$ . On the other hand, it is easily seen from Table 3 that  $(4,4,3)$  is also an upper bound of  $\mathbf{R}$ . Then the problem is rewritten as follows.

Maximize  $A_s(\mathbf{R}, 500)$ , subject to  $C(\mathbf{R}, 500) \leq 50000$ ,

where  $\mathbf{R}_{\max}=(2,4,3)$  and  $\mathbf{R}_{\min}=(0,0,0)$ .

The optimal solution is  $\mathbf{R}=(1,2,3)$ ,  $C(\mathbf{R}, 500)=49751$  and  $A_s(\mathbf{R}, 500)=0.99949$ .

Next, we solve P2. The problem is

Minimize  $C(\mathbf{R}, 500)$ , subject to  $A_s(\mathbf{R}, 500) \geq 0.9$ .

From the data in Table 3, it is easily seen that  $\mathbf{R}=(1,1,1)$  is one of feasible solutions. And  $C((1,1,1), 500)-C((0,0,0), 500)=16000$ . Hence,  $(1,3,16)$  is an upper

bound of  $\mathbf{R}$ . On the other hands,  $(4,4,3)$  is also an upper bound of  $\mathbf{R}$ . Therefore, the problem is rewritten as follows.

Minimize  $C(\mathbf{R},500)$ , subject to  $A_s(\mathbf{R},500) \geq 0.9$ ,

where  $\mathbf{R}_{\max}=(1,3,3)$  and  $\mathbf{R}_{\min}=(0,0,0)$ .

The optimal solution is  $\mathbf{R}=(0,1,1)$ ,  $A_s(\mathbf{R},500)=0.99910$  and  $C(\mathbf{R},500)=32751$ .

## 6. Concluding Remarks

This paper analyzed the complex system considering the discard of failed units. The assumption of exponential repair law is impractical in some cases. However, when repair is done by exchanges of assemblies, the assumption of exponential repair law is reasonable for the model. In this model, it is too difficult to analyze on the assumption of general repair law. Therefore, some modifications will be needed for the analysis.

When the setting time can be neglected, there is no difference between standby units and spare units. Therefore, this case results in the case of  $r_i=0$ .

## Reference

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