Journal of the Operations Research
Society of Japan
Vol. 22, No. 4, December 1979

OPTIMUM REPLACEMENT POLICIES FOR A USED UNIT

Toshio Nakagawa
Meijo University

(Received January 16, 1979; Revised July 5, 1979)

Abstract In many situations, it may be more economical to use a used unit than to do a new one. This paper considers an age replacement policy and a periodic replacement policy with minimal repair at failure for a used unit of age x. We discuss optimum replacement policies which minimize the expected cost rates for a specified age x. Further, we obtain an upper bound of an optimum time for an age replacement model and an optimum age of a used unit for a periodic replacement model when the replacement time is previously specified.

1. Introduction

Failure of a unit during actual operation is sometimes costly or dangerous. It is important to maintain the operating unit preventively before failure (e.g., inspection, overhaul, repair or replacement if needed). Many replacement and maintenance policies have been studied by many authors, e.g., [1, 5].

In the earlier contributions, almost all models discussed the optimum policies for a new unit, i.e., a new unit begins to operate at time 0, and any units which operate successively are as good as new after repair or replacement. However, it may be better to operate a used unit than to do a new one in the case where the cost of a used unit is much less than a new one. Of course, this would depend on the performance of a used unit. Bhat [2] considered the replacement policy by a new unit at regular intervals of time and by a used unit at failure.

This paper considers two replacement models for a used unit of age x, which is replaced by one of identical units of the same age x in the following policies [1]:

1. Age replacement; a unit is replaced at failure or at time t_0 after instal-

lation, whichever occurs first.

2. Periodic replacement with minimal repair; a unit is replaced at times kT and undergoes a minimal repair at failure.

In this paper, we adopt the "expected cost rate" as the appropriate objective function, introducing the cost $c_0(x)$ of a used unit of age x. We obtain optimum replacement times t_0^* and T^* which minimize the expected cost rates for a specified age x. Further, we obtain an upper bound of t_0^* for model 1, an optimum age x^* of a used unit for model 2 when T is specified, and moreover, we consider the modified model of periodic replacement suggested by Muth [4]. Several examples are presented.

2. Age replacement policy

Consider a used unit of age x ($0 \leq x < \infty$) which is replaced by a unit of the same age upon failure. Assume that the failure time distribution of a new unit of the same type is an arbitrary F(t) with finite mean λ. Then, the failure time distribution $F(t|x)$ of a used unit of age x is, from [1]

$$(1) \qquad F(t|x) \equiv \frac{F(t+x) - F(x)}{\overline{F}(x)} \qquad \text{for } F(x) < 1,\ t \geq 0,$$

where $\overline{F} \equiv 1 - F$, and the mean residual life $\lambda(x)$ of the unit is

$$(2) \qquad \lambda(x) \equiv \int_x^\infty \overline{F}(t)dt/\overline{F}(x) \qquad \text{for } F(x) < 1.$$

Let $c_0(x)$ be the acquisition cost of a used unit of age x and c_1 be all costs resulting from the failure. Then, the expected cost rate until failure is easily given by

$$(3) \qquad C(x) = [c_0(x) + c_1]/\lambda(x)$$

$$= \frac{[c_0(x) + c_1]\overline{F}(x)}{\int_x^\infty \overline{F}(t)dt} \qquad \text{for } 0 \leq x < \infty.$$

Next, consider an age replacement policy for a used unit of age x, where x is previously specified. The unit is replaced at failure or is exchanged when it operates for a planned replacement time t_0 ($0 < t_0 \leq \infty$) without failure. Then, by a similar argument of obtaining (3), the expected cost rate is

$$(4)\qquad C(t_0;x) = \frac{c_0(x) + c_1 F(t_0|x)}{\int_0^{t_0} \overline{F}(t+x)dt/\overline{F}(x)}$$

$$= \frac{c_0(x)\overline{F}(x) + c_1[F(t_0+x) - F(x)]}{\int_0^{t_0} \overline{F}(t+x)dt}.$$

It is evident that $\lim_{t_0\to 0} C(t_0;x) = \infty$ and $\lim_{t_0\to\infty} C(t_0;x) = C(x)$ in (3). Further, if $x = 0$ then $C(t_0;0)$ is coincident with that of [1, p. 87], by replacing $c_0(x)$ and c_1 in (4) into c_2 and $c_1 - c_2$, respectively.

We seek an optimum planned replacement time t_0^* which minimizes $C(t_0;x)$ in (4) for a fixed $x \geq 0$.

Suppose that the failure time distribution has a density f. Let $r(t) \equiv f(t)/\overline{F}(t)$ be the failure rate and there exists the limit of $r(\infty) \equiv \lim_{t\to\infty} r(t)$. Then, differentiating $C(t_0;x)$ with respect to t_0 and putting it equal to zero yield

$$(5)\qquad r(t_0+x)\int_0^{t_0} \overline{F}(t+x)dt - [F(t_0+x) - F(x)] = c_0(x)\overline{F}(x)/c_1.$$

Thus, if the failure rate $r(t)$ is monotonely increasing and

$$(6)\qquad r(\infty) > \frac{[1 + c_0(x)/c_1]\overline{F}(x)}{\int_x^\infty \overline{F}(t)dt},$$

i.e., $r(\infty) > C(x)/c_1$, then there exists a finite and unique t_0^* which satisfies (5), and the expected cost rate is

$$(7)\qquad C(t_0^*;x) = c_1 r(t_0^*+x).$$

Further, from the assumption that $r(t)$ is monotonely increasing, we easily have the inequality

$$(8)\qquad r(t_0+x)\int_0^{t_0} \overline{F}(t+x)dt - [F(t_0+x) - F(x)] > r(t_0+x)\int_x^\infty \overline{F}(t)dt - \overline{F}(x).$$

Thus, if there exists a $\overline{t}_0$ satisfying $r(t_0+x) = C(x)/c_1$, i.e.,

$$(9)\qquad r(t_0+x)\int_x^\infty \overline{F}(t)dt - \overline{F}(x) = c_0(x)\overline{F}(x)/c_1,$$

then $t_0^* < \bar{t}_0$, which is the upper limit of t_0^*. In this case, it is easily shown that $\bar{t}_0$ is finite and unique since $r(\infty) > C(x)/c_1$ and $r(x) < 1/\lambda(x) < C(x)/c_1$.

If $r(t)$ is monotonely increasing and $r(\infty) \le C(x)/c_1$, or $r(t)$ is non-increasing, then $C'(t_0;x) \le 0$ and hence, the optimum policy is $t_0^* \to \infty$.

Example 1.

Suppose that $\bar{F}(t) = (1 + \alpha t)e^{-\alpha t}$. Then, the failure rate $r(t)$ is monotonely increasing with $r(0) = 0$ and $r(\infty) = \alpha$. Thus, from the above results, if $c_0(x) \ge c_1/(1 + \alpha x)$ then we should make no planned replacement. If $c_0(x) < c_1/(1 + \alpha x)$, we should adopt a planned replacement time t_0^* which satisfies uniquely the following equation:

$$\frac{\alpha t_0 - (1 - e^{-\alpha t_0})}{1 + \alpha(t_0 + x)} = \frac{c_0(x)(1 - \alpha x)}{c_1},$$

and

$$t_0^* + x < \frac{(1 + \alpha x)[c_0(x) + c_1]}{\alpha[c_1 - (1 + \alpha x)c_0(x)]}.$$

3. Periodic replacement with minimal repair

In the first model, we have assumed that a unit is replaced when it fails before a planned replacement time t_0. However, for more complex systems, it is costly to replace or overhaul systems for any intervening failures. We should repair failures as quickly as possible. From the point of view, model 2 has the following assumptions [1]:

(i) A unit is replaced or overhauled at times kT ($k = 1, 2, \ldots; T > 0$).

(ii) A unit undergoes only minimal repair at failures between planned replacement, and the failure rate remains undisturbed by the minimal repair.

For instance, a complex system fails for failure of a single component in the system. The failed component is replaced and the system begins to operate again. In this case, the system after the replacement has the same failure rate as before the replacement, due to the aging of the other components. Holland and McLean [3] gave a practical procedure for the policy to pieces of equipments, as examples of large motors and small electrical parts.

The expected cost rate for a used unit of age x is, by the method similar to [1, p. 96], easily given by

(10) $$C(T;x) = [c_0(x) + c_2\int_x^{T+x} r(t)dt]/T,$$

where c_2 is the cost of minimal repair.

Suppose that x is constant and previously specified. Then, differentiating C(T;x) with respect to T and setting it equal to zero, we have

(11) $$\int_x^{T+x} (t - x)dr(t) = c_0(x)/c_2.$$

Thus, if r(t) is monotonely increasing and $\int_0^\infty tdr(t+x) > c_0(x)/c_2$, then there exists a T^* uniquely which minimizes C(T;x) in (10), and the expected cost rate is

(12) $$C(T^*;x) = c_2 r(T^*+x).$$

Next, consider the problem that it is the most economical to use a unit of what is the age. Suppose that x is a variable and inversely, T is constant, and $c_0(x)$ is differentiable. Then, differentiating C(T;x) with respect to x and setting it equal to zero imply

(13) $$r(T+x) - r(x) = -c_0'(x)/c_2,$$

which is a necessary condition that a finite x minimizes C(T;x) for a fixed T.

Example 2.

Suppose that $c_0(x) = c_0 e^{-\theta x}$ and $\bar{F}(t) = e^{-\alpha t^m}$ (m > 1), which is a Weibull distribution with a shape parameter m, and the failure rate is $r(t) = m\alpha t^{m-1}$. Then, from (11), we have

(14) $$mT(T + x)^{m-1} - (T + x)^m + x^m = [c_0/(\alpha c_2)]e^{-\theta x},$$

which is monotonely increasing in T, taking the values from 0 to infinity. Thus, an optimum T^* exists uniquely, which satisfies (14).

Further, from (13),

(15) $$[(T + x)^{m-1} - x^{m-1}]e^{\theta x} = [(c_0\theta)/(m\alpha c_2)],$$

which is monotonely increasing in x, taking the values from T^{m-1} to infinity.

Thus,

(i) if $T^{m-1} < (c_0\theta)/(m\alpha c_2)$ then an optimum x^* exists uniquely, which satisfies (15),

(ii) if $T^{m-1} \geq (c_0\theta)/(m\alpha c_2)$ then $x^* = 0$, i.e., we should use a new unit.

Next, consider the particular case of m = 2. Then, from (14) and (15), the respective optimum times T^* and x^* are given by, explicitly,

$$T^* = [\frac{c_0}{\alpha c_2} e^{-\theta x}]^{1/2},$$

and for $T < (c_0\theta)/(2\alpha c_2)$,

$$x^* = \frac{1}{\theta} \log \frac{c_0\theta}{2\alpha c_2 T} .$$

Further, suppose that both x and T are variables. Then, from (14) and (15),

$$\alpha c_2 T^2 = c_0 e^{-\theta x},$$

$$2\alpha c_2 T = c_0 \theta e^{-\theta x}.$$

Thus, $T^* = 2/\theta$ and

(i) if $4\alpha c_2 \geq c_0\theta^2$ then $x^* = 0$,

(ii) if $4\alpha c_2 < c_0\theta^2$ then $x^* = \frac{1}{\theta} \log \frac{c_0\theta^2}{4\alpha c_2}$.

We give the numerical examples where $c_0/c_2 = 5$, m = 2, and $1/\alpha = 10,000$. Table 1 shows the optimum replacement time T^* of a used unit of age x and the optimum age x^* for a planned replacement time T, when $1/\theta = 50$. Further, Table 2 shows the optimum replacement time T^* and the optimum age x^*, both of which are variables, for a discount factor θ. It is noted that, in particular case of $1/\theta = 50$, the results coincide with those of Table 1.

Finally, consider the following modification of periodic replacement with minimal repair suggested by Muth [4]:

(i) A unit is replaced when it fails for the first time after time T ($T \geq 0$).

(ii) A unit undergoes only minimal repair at failure before time T.

Then, the expected cost of a used unit of age x is easily given by

$$(16) \qquad C(T;x) = \frac{c_0(x) + c_1 + c_2\int_x^{T+x} r(t)dt}{T + \lambda(T+x)} ,$$

Table 1. Body of the table gives the optimum replacement time T^* for the age x of a used unit and the optimum age x^* of a used unit for a planned replacement time T, where $c_0/c_2 = 5$, $m = 2$, $1/\alpha = 10{,}000$, and $1/\theta = 50$.

age of unit x	optimum replacement time T^*	replacement time T	optimum age x^*
0	224	20	161
10	202	40	116
20	183	60	106
40	150	80	92
60	123	100	80
80	100	120	71
100	82	140	64
120	67	160	57
140	55	200	46

Table 2. Body of the table gives the optimum replacement time T^* and the optimum age x^* of a used unit for a discount factor $1/\theta$.

discount factor $1/\theta$	optimum replacement time T^*	optimum age x^*
20	40	69
40	80	82
50	100	80
60	120	75
80	160	54
100	200	22
120	240	0

where c_1 is the cost suffered for the failure and $\lambda(x)$ is defined in (2). It is evident that $C(0;x) = C(x)$ in (3).

Suppose that x is previously specified. Then, differentiating $C(T;x)$ with respect to T and setting it equal to zero, we have

$$(17) \qquad \frac{T}{\lambda(T+x)} - \int_x^{T+x} r(t)dt = \frac{c_0(x) + c_1 - c_2}{c_2}.$$

Assume that $c_0(x) + c_1 > c_2$ and $r(t)$ is monotonely increasing. Then, if a solution to (17) exists, it is unique. For, denoting the left side of (17) by $h(T)$ and differentiating it with respect to T, we have

$$(18) \qquad h'(T) = [\frac{1}{\lambda(T+x)} - r(T+x)][1 + \frac{T}{\lambda(T+x)}] > 0,$$

since $1/\lambda(T+x) > r(T+x)$ by the assumptions. If $c_0(x) + c_1 \leq c_2$ then $C'(T;x) \leq 0$, i.e., we should replace a unit only at failure.

4. Conclusions

We have considered two replacement models of a used unit and obtained the optimum policy which minimizes the expected cost rate of each model. In the example, we have obtained both optimum replacement time and optimum age of a used unit, when both are variables. These results would be useful in practical cases such that we have to serve a used unit or it is more economical to adopt a used unit than a new one.

In the examples, we have only considered two simple functions of $c_0(x)$, however, we could apply to more general functions of $c_0(x)$, e.g., $c_0(x) = c_0\exp(-\theta x^\beta)$ $(\beta \geq 1)$ and

$$c_0(x) = \begin{cases} c_0 - \theta x & \text{for } x < c_0/\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if we introduce a net resale value $e_0(x)$ of a used unit of age x, which is exchanged at a planned replacement, the equations (4) and (10) are rewritten as, respectively,

$$(19) \qquad C(t_0;x) = \frac{c_0(x)\bar{F}(x) - e_0(t_0+x)\bar{F}(t_0+x) + c_1[F(t_0+x) - F(x)]}{\int_0^{t_0} \bar{F}(t+x)dt},$$

$$(20) \qquad C(T;x) = \frac{c_0(x) - e_0(T+x) + c_2\int_x^{T+x} r(t)dt}{T}.$$

Further, we can consider other maintenance policies of a used unit, e.g., a preventive maintenance policy [5] and a block replacement policy [1, p. 95]. The discussions in this paper could be applied to such policies.

References

[1] Barlow, R.E. and Proschan, F.: *Mathematical Theory of Reliability*. John Wiley & Sons, New York, 1965.

[2] Bhat, B.R.: Used Item Replacement Policy. *J. Appl. Prob.*, Vol. 6, No. 2 (1969), 309-318.

[3] Holland, C.W. and McLean, R.A.: Applications of Replacement Theory. *AIIE Trans.*, Vol. 7, No. 1 (1975), 42-47.

[4] Muth, E.J.: An Optimal Decision Rule for Repair vs Replacement. *IEEE Trans. on Reliability*, Vol. R-26, No. 3 (1977), 179-181.

[5] Nakagawa, T.: Optimum Preventive Maintenance Policies for Repairable Systems. *IEEE Trans. on Reliability*, Vol. R-26, No. 3 (1977), 168-173.

Toshio NAKAGAWA
Department of Mathematics
Meijo University
Tenpaku-cho, Tenpaku-ku
Nagoya, 468, Japan