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THE EFFICIENCY OF TWO-STAGE SERIES LINES

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Abstract In this paper, we consider the procedure to estimate the various effects of the design factors in the two-stage series lines in which the first stage has an arbitrary service time. First, the system states are represented by the imbedded Markov chain and the steady state probabilities are solved by the Laplace transform with respect to the arbitrary service time distribution. Next, the efficiency, the idling time distribution, the blocking time distribution and the number of in-process works distribution are considered from the system states. Moreover the relations between the dual models are discussed.

1. Introduction

Many papers have been published concerning the two-stage series lines. Hunt and others estimate the efficiencies and the mean number of in-process works for the lines with exponential or erlang service times by the Markov model [6], [7], [8], [11], [12], [13]. However, most of the papers discuss only the efficiencies for the lines with the arbitrary service times by the approximation methods [1], [3], [10]. On the other hand, the queueing system M/G/1 with a finite waiting room which is related to the dual model considered in this paper is discussed by Hashida and others, and the various results are presented [4], [5].

In this paper we consider the procedure to estimate the effects of the design factors in the two-stage series lines in which the first stage has an arbitrary service time and the second stage has an exponential service time. First, we show that the system states can be represented by the imbedded Markov chain like GI/M/1 or M/G/1 queueing model and that the state probabilities in the steady state condition can be solved by the Laplace transform with respect to the arbitrary service time distribution. And from the state probabilities the efficiency, the idling time distribution, the blocking time distribution and the number of in-process works distribution are represented. Moreover, we discuss the relations between the dual models and show that the various properties for the dual model can be represented by the other model.

2. Model

The two-stage series line consists of two stages to operate the work and the buffer storage holding in-process works temporarily, as shown in Fig. 2.1.

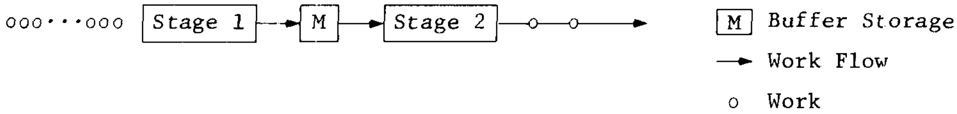

Fig. 2.1 Two-stage series line

Then the line is defined by the service time in stage i (i=1,2) represented by the probability density function (p.d.f.) $f_i(x)$ and the buffer capacity M.

Assume that there are infinite works ready to be operated in the first stage and there is infinite buffer capacity just behind the second stage, therefore idling due to lack of input works is never occurred in the first stage and blocking is never occurred in the second stage because the completed work is always ejected from the stage. And the service times are mutually independent.

Now we formulate the work flow of this model. This flow is defined by the output interval $O_{i,j}$ between the (j-1)-th work and the j-th work in stage i, which is represented by

$$(2.1) \qquad O_{i,j} = I_{i,j-1} + X_{i,j} + B_{i,j} \qquad (i=1,2;\ j=1,2,3,\ldots),$$

where $X_{i,j}$, $I_{i,j-1}$ and $B_{i,j}$ denote the service time for the j-th work in stage i, the idling time occurred when stage i waits the j-th work in stage i after ejecting the (j-1)-th work from the stage and the blocking time occurred when the service for the j-th work in stage i is completed but the work finds the buffer full and is held in the stage, respectively. In this equation $I_{i,j-1}$ and $B_{i,j}$ are unknown random variables, but if we have the initial conditions that the numbers of work are independently counted in each stage and that stages 1 and 2 begin to operate the first work at the same time t=0 and then there is no in-process work in the buffer, i.e. the j-th work in stage 1 is the (j+1)-th work in stage 2 and $I_{i,0}=0$, $I_{i,j-1}$ and $B_{i,j}$ are given by

$$(2.2) \qquad I_{2,j-1} = \left[\left(\sum_{n=1}^{j-2} O_{1,n} + X_{1,j-1}\right) - \sum_{n=1}^{j-1} O_{2,n}\right]^+$$

$$B_{1,j} = [\sum_{n=1}^{j-M} 0_{2,n} - (\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j})]^+$$

$$I_{1,j-1} = 0$$

$$B_{2,j} = 0,$$

where the notation $[Y]^+$ denotes

$$[Y]^+ = \mathrm{Max}\ (0,\ Y)$$

and $I_{1,j-1} = B_{2,j} = 0$ is given by the above assumption.

3. The imbedded Markov chain

We consider the system states at the time $t = \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} - 0$, i.e. the time just before the completion of service for the j-th work in stage 1. These states are defined by the state of the each stage and the number of in-process works in the buffer and are represented as follows;

$S_j(W_1, W_2|m)$: Stages 1 and 2 are in operating and m in-process works are in the buffer $(0 \leqq m \leqq M)$

$S_j(W_1, I_2|0)$: Stage 1 is in operating and Stage 2 is in idling and there is no in-process work in the buffer.

Furthermore the time in which the each state is occurred is given by the following range of time. The state $S_j(W_1, I_2|0)$ is occurred in

$$\sum_{n=1}^{j} 0_{2,n} \leqq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \leqq \sum_{n=1}^{j} 0_{2,n} + [X_{1,j} - X_{2,j}]^+,$$

the state $S_j(W_1, W_2|m)$ is occurred in

$$\sum_{n=1}^{j-1-m} 0_{2,n} \leqq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} < \sum_{n=1}^{j-m} 0_{2,n} \qquad (0 \leqq m \leqq M-1),$$

and the state $S_j(W_1, W_2|M)$ is occurred in

$$\sum_{n=1}^{j-M} 0_{2,n} - [X_{2,j-M} - X_{1,j}]^+ \leqq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} < \sum_{n=1}^{j-M} 0_{2,n}.$$

In Fig. 3.1 the above relations are described.

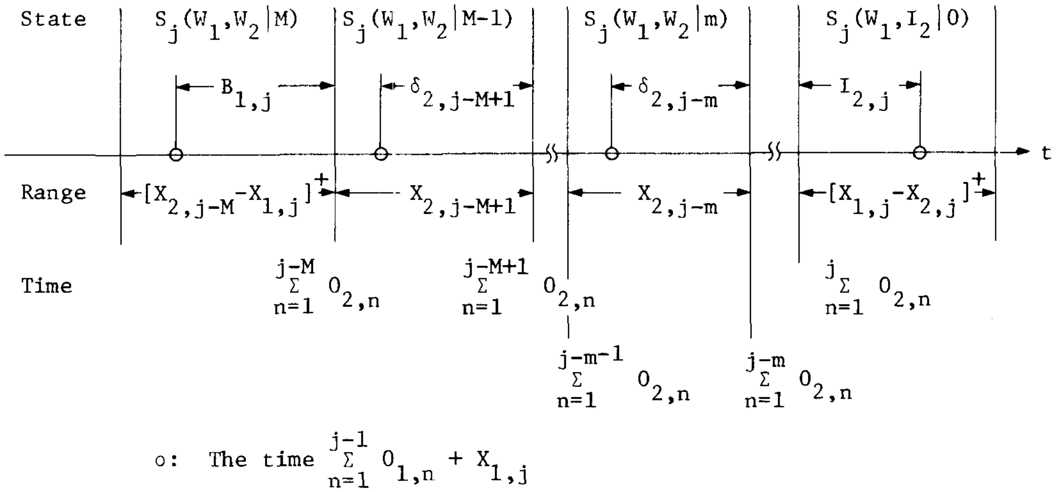

Fig. 3.2 The system state

In this figure $I_{2,j}$, $B_{1,j}$ and $\delta_{2,j-m}$ $(0 \leq m \leq M-1)$ denote the idling time occurred immediately after the ejection of the j-th work in stage 2, the blocking time occurred immediately after the completion of service for the j-th work in stage 1 and the rest part of service time for the (j-m)-th work which exists in stage 2 when the service for the j-th work in stage 1 is completed and also there are m in-process works in the buffer, and they are given by

$$I_{2,j} = [(\sum_{n=1}^{j-1} O_{1,n} + X_{1,j}) - \sum_{n=1}^{j} O_{2,n}]^+ \leq [X_{1,j} - X_{2,j}]^+$$

$$B_{1,j} = [\sum_{n=1}^{j-M} O_{2,n} - (\sum_{n=1}^{j-1} O_{1,n} + X_{1,j})]^+ \leq [X_{2,j-M} - X_{1,j}]^+$$

$$\delta_{2,j-m} = [\sum_{n=1}^{j-m} O_{2,n} - (\sum_{n=1}^{j-1} O_{1,n} + X_{1,j})]^+ < X_{2,j-m}.$$

In this model the system state at any time does not form the Markov chain but the state at $\sum_{n=1}^{j-1} O_{1,n} + X_{1,j} - 0$ forms the imbedded Markov chain, therefore this transition probabilities are determined by the number of works for which the services are completed in stage 2 during the service time $X_{1,j+1}$. In Table 3.1 the transition probabilities are represented. In this table $\delta_{2,j-m}$ is the unknown random variable, but the p.d.f. $h_m(\delta)$ can be calculated as follows.

We define the completed part of the service time $\Delta_{2,j-m}$ for the (j-m)-th work which exists in stage 2 when the (j+1)-th work in stage 1 is ejected, and denote the p.d.f. by $g_m(\Delta)$. Then $\Delta_{2,j-m}$ and $X_{2,j-m}$ are mutually independent and

Table 3.1 Transition probability

State at $j-1$ $\sum_{n=1}^{j-1} O_{1,n}+X_{1,j}-0$	State at j $\sum_{n=1}^{j} O_{1,n}+X_{1,j+1}-0$	Transition Probability
$S_j(W_1,W_2\|M)$	$S_{j+1}(W_1,W_2\|M)$	Prob. $(X_{1,j+1} \leq X_{2,j+1-M})$
	$S_{j+1}(W_1,W_2\|m)$	Prob. $(\sum_{n=m+1}^{M} X_{2,j+1-n} < X_{1,j+1} \leq \sum_{n=m}^{M} X_{2,j+1-n})$
	$S_{j+1}(W_1,I_2\|0)$	Prob. $(\sum_{n=0}^{M} X_{2,j+1-n} < X_{1,j+1})$
$S_j(W_1,W_2\|m)$	$S_{j+1}(W_1,W_2\|m+1)$	Prob. $(X_{1,j+1} \leq \delta_{2,j-m})$
$(0 \leq m \leq M-1)$	$S_{j+1}(W_1,W_2\|m)$	Prob. $(\delta_{2,j-m} < X_{1,j+1} \leq \delta_{2,j-m}+X_{2,j-m+1})$
$(0 \leq k \leq m-1)$	$S_{j+1}(W_1,W_2\|k)$	Prob. $(\delta_{2,j-m} + \sum_{n=k+1}^{m} X_{2,j+1-n} < X_{1,j+1} \leq \delta_{2,j-m} + \sum_{n=k}^{m} X_{2,j+1-n})$
	$S_{j+1}(W_1,I_2\|0)$	Prob. $(\delta_{2,j-m} + \sum_{n=0}^{m} X_{2,j+1-n} < X_{1,j+1})$
$S_j(W_1,I_2\|0)$	$S_{j+1}(W_1,W_2\|0)$	Prob. $(X_{1,j+1} \leq X_{2,j+1})$
	$S_{j+1}(W_1,I_2\|0)$	Prob. $(X_{2,j+1} < X_{1,j+1})$

$\delta_{2,j-m}$ is given by

(3.1) $$\delta_{2,j-m} = X_{2,j-m} - \Delta_{2,j-m},$$

therefore $h_m(\delta)$ is represented by

(3.2) $$h_m(\delta) = \frac{\int_0^\infty g_m(\Delta) f_2(\Delta+\delta)\,d\Delta}{\int_0^\infty \int_0^\infty g_m(\Delta) f_2(\Delta+\delta)\,d\Delta d\delta}$$

$$= \lambda_2 \exp(-\lambda_2 \delta),$$

where the p.d.f. $f_2(x)$ of the service time in stage 2 is

$$f_2(x) = \lambda_2 \exp(-\lambda_2 x).$$

Consequently it is appeared that $\delta_{2,j-m}$ has the same p.d.f. with that of $X_{2,j-m}$ and all the transition probabilities can be calculated and finally we can estimate the system state probabilities in the steady state condition.

Now we estimate the steady state probabilities. These transition probabilities can be rewritten by k_i

$$(3.3) \qquad k_i = \int_0^\infty \frac{(\lambda_2 x)^i}{i!} \exp(-\lambda_2 x)\ f_1(x)dx$$

and the system state equations in the steady state condition are given by

$$(3.4) \qquad P(W_1,W_2|M) = \{P(W_1,W_2|M) + P(W_1,W_2|M-1)\}k_0$$

$$P(W_1,W_2|M-1) = \{P(W_1,W_2|M) + P(W_1,W_2|M-1)\}k_1 + P(W_1,W_2|M-2)k_0$$

$$P(W_1,W_2|M-2) = \{P(W_1,W_2|M) + P(W_1,W_2|M-1)\}k_2 + P(W_1,W_2|M-2)k_1 + P(W_1,W_2|M-3)k_0$$

$$\vdots$$

$$P(W_1,W_2|0) = \{P(W_1,W_2|M) + P(W_1,W_2|M-1)\}k_M + P(W_1,W_2|M-2)k_{M-1} + \cdots\cdots + P(W_1,W_2|0)k_1 + P(W_1,I_2|0)k_0$$

$$P(W_1,I_2|0) = \{P(W_1,W_2|M) + P(W_1,W_2|M-1)\} \sum_{i=M+1}^{\infty} k_i + P(W_1,W_2|M-2) \sum_{i=M}^{\infty} k_i + \cdots\cdots + P(W_1,W_2|0) \sum_{i=2}^{\infty} k_i + P(W_1,I_2|0) \sum_{i=1}^{\infty} k_i$$

$P(\cdot)$: The steady state probability that the system state is $S(\cdot)$.

Therefore, if P_i denotes $P(W_1,W_2|M-i)$ and P_{M+1} denotes $P(W_1,I_2|0)$, these equations are rewritten by using $K(Z)$ and $F(Z)$

$$(3.5)\qquad K(Z) = \sum_{i=0}^{\infty} k_i Z^i$$

$$F(Z) = \sum_{i=0}^{\infty} P_i Z^i,$$

where $P_i=0$ $(i \geq M+2)$, and we have

$$(3.6)\qquad F(Z) = P_0 \cdot \frac{1-Z}{1-Z/K(Z)} + Z^{M+2}\, H(Z),$$

where H(Z) is the power series of Z.

If the Laplace transform of $f_1(x)$ is represented by $U_1(s)$, we have

$$K(Z) = U_1(\lambda_2(1-Z)),$$

so the above equation is

$$(3.7)\qquad F(Z) = P_0 \cdot \frac{1-Z}{1-Z/U_1(\lambda_2(1-Z))} + Z^{M+2}\, H(Z).$$

In this equation the steady state probabilities P_i $(0 \leq i \leq M+1)$ are given by the coefficient of Z^i and expressed in terms of P_0 if $U_1(\cdot)$ can be expanded in a power series. Moreover the root of the equation $1-Z/U_1(\lambda_2(1-Z))=0$ has always $Z=1$ and if we denote the other roots by $1/\alpha_i$, the equation (3.7) is

$$(3.8)\qquad F(Z) = P_0 \cdot \frac{1}{(1-\alpha_1 Z)(1-\alpha_2 Z)\cdots\cdots\cdots} + Z^{M+2}\, H(Z).$$

4. Various line characters

First we estimate the idling time $I_{2,j+1}$ and the blocking time $B_{1,j+1}$. In this model $B_{1,j+1}$ is occurred when the system state at $\sum_{n=1}^{j-1} 0_{1,n}+X_{1,j}-0$ is $S_j(W_1,W_2|M)$ and changes to $S_{j+1}(W_1,W_2|M)$ at $\sum_{n=1}^{j} 0_{1,n}+X_{1,j+1}-0$ or when the state is $S_j(W_1,W_2|M-1)$ and changes to $S_{j+1}(W_1,W_2|M)$, and it is not occurred in the other cases. Therefore $B_{1,j+1}$ is represented by

$$(4.1)\qquad B_{1,j+1} = [X_{2,j-M+1} - X_{1,j+1}]^+ \quad \text{or}$$

$$= [\delta_{2,j-M+1} - X_{1,j+1}]^+ \quad \text{or}$$

$$= 0.$$

Consequently the p.d.f. $w_B(x)$ $(x>0)$ for the blocking time $B_{1,j+1}$ is given by

$$(4.2)\qquad w_B(x) = P_j(W_1,W_2|M)\cdot P_T[S_j(W_1,W_2|M)\to S_{j+1}(W_1,W_2|M)]\cdot \frac{\bar{f}(x)}{\text{Prob. }(X_{2,j-M+1}\geqq X_{1,j+1})}$$

$$+\ P_j(W_1,W_2|M-1)\cdot P_T[S_j(W_1,W_2|M-1)\to S_{j+1}(W_1,W_2|M)]\cdot \frac{\bar{f}(x)}{\text{Prob. }(\delta_{2,j-M+1}\geqq X_{1,j+1})},$$

$P_j(\cdot)$: The probability that the system state at $\sum_{n=1}^{j-1} 0_{1,n}+X_{1,j}-0$ is $S_j(\cdot)$

$P_T[S_j(\cdot)\to S_{j+1}(\cdot)]$: The transition probability that the system state changes from $S_j(\cdot)$ to $S_{j+1}(\cdot)$

$\bar{f}(x)$: $\int_0^\infty f_1(y)f_2(y+x)dy$.

On the other hand $I_{2,j+1}$ is occurred when the system state at $\sum_{n=1}^{j-1} 0_{1,n}+X_{1,j}-0$ is $S_j(W_1,W_2|M)$ and changes to $S_{j+1}(W_1,I_2|0)$ at $\sum_{n=1}^{j} 0_{1,n}+X_{1,j+1}-0$, when the state is $S_j(W_1,W_2|m)$ $(0\leqq m\leqq M-1)$ and changes to $S_{j+1}(W_1,I_2|0)$ or when the state is $S_j(W_1,I_2|0)$ and changes to $S_{j+1}(W_1,I_2|0)$, and it is not occurred in the other cases. Therefore it is represented by

$$(4.3)\qquad I_{2,j+1} = [X_{1,j+1} - \sum_{n=0}^{M} X_{2,j+1-n}]^+ \quad \text{or}$$

$$= [X_{1,j+1} - \delta_{2,j-m} - \sum_{n=0}^{m} X_{2,j+1-n}]^+ \quad \text{or}$$

$$= [X_{1,j+1} - X_{2,j+1}]^+ \quad \text{or}$$

$$= 0.$$

Consequently the p.d.f. $w_I(x)$ $(x>0)$ for the idling time $I_{2,j+1}$ is given by

$$(4.4)\qquad W_I(x) = P_j(W_1,W_2|M)\cdot P_T[S_j(W_1,W_2|M)\to S_{j+1}(W_1,I_2|0)]\cdot \frac{\bar{f}_{M+1}(x)}{\text{Prob. }(X_{1,j+1} > \sum_{n=0}^{M} X_{2,j+1-n})}$$

$$+ \sum_{m=0}^{M-1} P_j(W_1,W_2|m)\cdot P_T[S_j(W_1,W_2|m)\to S_{j+1}(W_1,I_2|0)]\cdot$$

$$\frac{\bar{f}_{m+2}(x)}{\text{Prob. }(X_{1,j+1} > \delta_{2,j-m} + \sum_{n=0}^{m} X_{2,j+1-n})}$$

$$+ P_j(W_1, I_2|0)\cdot P_T[S_j(W_1, I_2|0) \to S_{j+1}(W_1, I_2|0)].$$

$$\frac{\bar{f}_1(x)}{\text{Prob. }(X_{1,j+1} > X_{2,j+1})},$$

$$\bar{f}_m(x): \quad \int_0^\infty f_2^{(m)}(y) f_1(y+x)\,dy,$$

where $f_2^{(m)}(x)$ denotes the m times convolution function of $f_2(x)$. From the Table 3.1 we have the next equalities, too

(4.5) $$P_T[S_j(W_1,W_2|M) \to S_{j+1}(W_1,W_2|M)] = \text{Prob. }(X_{2,j-M+1} \geqq X_{1,j+1})$$

$$P_T[S_j(W_1,W_2|M-1) \to S_{j+1}(W_1,W_2|M)] = \text{Prob. }(\delta_{2,j-M+1} \geqq X_{1,j+1})$$

$$P_T[S_j(W_1,W_2|M) \to S_{j+1}(W_1,I_2|0)] = \text{Prob. }(X_{1,j+1} > \sum_{n=0}^{M} X_{2,j+1-n})$$

$$P_T[S_j(W_1,W_2|m) \to S_{j+1}(W_1,I_2|0)] = \text{Prob. }(X_{1,j+1} > \delta_{2,j-m} + \sum_{n=0}^{m} X_{2,j+1-n})$$

$$P_T[S_j(W_1,I_2|0) \to S_{j+1}(W_1,I_2|0)] = \text{Prob. }(X_{1,j+1} > X_{2,j+1}),$$

so that in the steady state condition (4.2) and (4.4) are rewritten by

(4.6) $$w_B(x) = (P_0+P_1)\cdot\bar{f}(x)$$

$$w_I(x) = (P_0+P_1)\cdot\bar{f}_{M+1}(x) + \sum_{m=2}^{M+1} P_m\cdot\bar{f}_{M+2-m}(x).$$

In this model the efficiency ρ is represented by

(4.7) $$\rho = \frac{1}{\bar{X}_1+\bar{B}_1} = \frac{1}{\bar{X}_2+\bar{I}_2}$$

where $\bar{X}_i$, $\bar{B}_1$ and $\bar{I}_2$ denote the mean operation time in stage i, the mean blocking time in stage 1 and the mean idling time in stage 2. And the number of in-process works distribution at the time just before the completion of service in stage 1 is given by $P_i(0 \leqq i \leqq M+1)$ and the mean number $\bar{N}$ of the in-process works is given by

(4.8) $$\bar{N} = \sum_{m=0}^{M} m\cdot P(W_1,W_2|m) = \sum_{m=0}^{M} m\cdot P_{M-m}.$$

5. The dual model

In this section we discuss the dual model in which the service time in each stage is exchanged.

At first we consider the system states occurred at the time $t=\sum_{n=1}^{j} O_{2,n}-0$, i.e. the time just before the ejection of the j-th work in stage 2. These states are defined by

$S_j^*(W_1,W_2|m)$: Stages 1 and 2 are in operating and m in-process works are in the buffer $(0\leq m\leq M)$

$S_j^*(B_1,W_2|M)$: Stage 1 is in blocking, Stage 2 is in operating and M in-process works are in the buffer,

and the relation between the system states and the range of time in which each state is occurred are described in Fig. 5.1. In this figure $I_{2,j}^*$, $B_{1,j+M}^*$ and $\delta_{1,j+m}^*$ $(1\leq m\leq M)$ denote the idling time occurred immediately after the ejection of the j-th work in stage 2, the blocking time occurred immediately after the completion of service for the (j+M)-th work in stage 1 and the rest part of service time for the (j+m)-th work which exists in stage 1 when the service for the j-th work in stage 2 is completed and also there are m in-process works in the buffer.

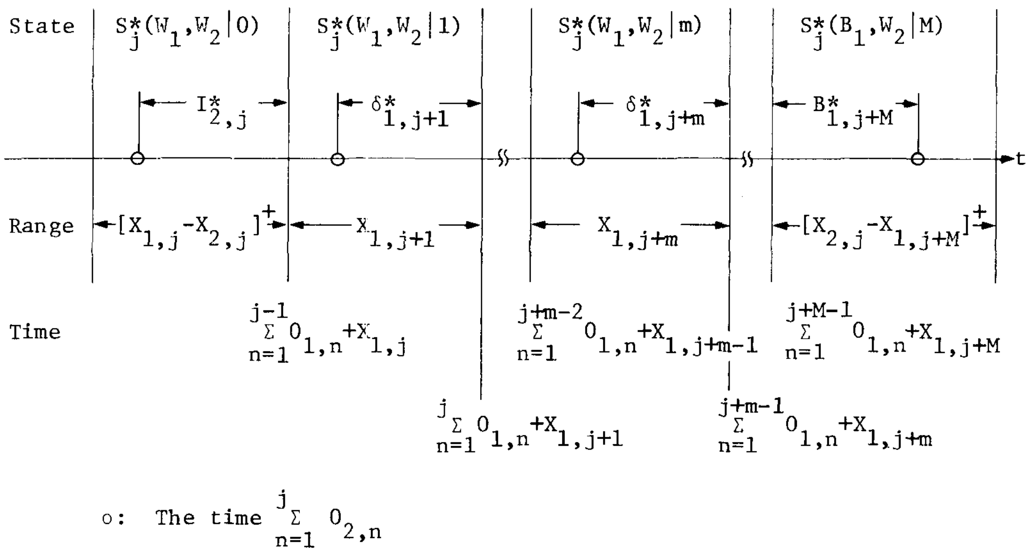

Fig. 5.1 The system state

In this model the state at $\sum_{n=1}^{j} 0_{2,n}-0$ forms the imbedded Markov chain and these transition probabilities are determined by the number of works for which the services are completed in stage 1 during the service time $X_{2,j+1}$. In Table 5.1 these probabilities are represented, where $\delta^*_{1,j+m}$ has the same p.d.f. with the p.d.f. for the service time in stage 1 and it is represented by

(5.1) $\quad f_1^*(x) = \lambda_1 \exp(-\lambda_1 x).$

Table 5.1 Transition probability

State at $\sum_{n=1}^{j} 0_{2,n}-0$	State at $\sum_{n=1}^{j+1} 0_{2,n}-0$	Transition Probability
$S_j^*(W_1,W_2\|0)$	$S_{j+1}^*(W_1,W_2\|0)$	Prob. $(X_{2,j+1} \leqq X_{1,j+1})$
	$S_{j+1}^*(W_1,W_2\|m)$	Prob. $(\sum_{n=1}^{m} X_{1,j+n} < X_{2,j+1} \leqq \sum_{n=1}^{m+1} X_{1,j+n})$
	$S_{j+1}^*(B_1,W_2\|M)$	Prob. $(\sum_{n=1}^{M+1} X_{1,j+n} < X_{2,j+1})$
$S_j^*(W_1,W_2\|m)$	$S_{j+1}^*(W_1,W_2\|m-1)$	Prob. $(X_{2,j+1} \leqq \delta^*_{1,j+m})$
$(1 \leqq m \leqq M)$	$S_{j+1}^*(W_1,W_2\|m)$	Prob. $(\delta^*_{1,j+m} < X_{2,j+1} \leqq \delta^*_{1,j+m}+X_{1,j+m+1})$
$(m+1 \leqq k \leqq M)$	$S_{j+1}^*(W_1,W_2\|k)$	Prob. $(\delta^*_{1,j+m} + \sum_{n=m+1}^{k} X_{1,j+n} < X_{2,j+1} \leqq \delta^*_{1,j+m} + \sum_{n=m+1}^{k+1} X_{1,j+n})$
	$S_{j+1}^*(B_1,W_2\|M)$	Prob. $(\delta^*_{1,j+m} + \sum_{n=m+1}^{M+1} X_{1,j+n} < X_{2,j+1})$
$S_j^*(B_1,W_2\|M)$	$S_{j+1}^*(W_1,W_2\|M)$	Prob. $(X_{2,j+1} \leqq X_{1,j+M+1})$
	$S_{j+1}^*(B_1,W_2\|M)$	Prob. $(X_{1,j+M+1} < X_{2,j+1})$

From this Table we can estimate the steady state probabilities $P^*(W_1,W_2|m)$ and $P^*(B_1,W_2|M)$, and if we denote

$$(5.2)\qquad P_i^* = P^*(W_1,W_2|i) \qquad (0 \leqq i \leqq M)$$

$$P_{M+1}^* = P^*(B_1,W_2|M)$$

and

$$(5.3)\qquad F^*(Z) = \sum_{i=0}^{\infty} P_i^* Z^i,$$

we have the following system equation, where $P_i^*=0$ $(i \geqq M+2)$,

$$(5.4)\qquad F^*(Z) = P_0^* \cdot \frac{1-Z}{1-Z/U_2^*(\lambda_1(1-Z))} + Z^{M+2} H(Z)$$

$U_2^*(s)$: Laplace transform in terms of $f_2^*(x)$.

In the dual models we have

$$U_2^*(\lambda_1(1-Z)) = U_1(\lambda_2(1-Z))$$

$$\lambda_1 = \lambda_2,$$

so the following equalities are obtained

$$(5.5)\qquad P_i^* = P_i$$

and

$$(5.6)\qquad P^*(W_1,W_2|m) = P(W_1,W_2|M-m) \qquad (0 \leqq m \leqq M)$$

$$P^*(B_1,W_2|M) = P(W_1,I_2|0).$$

The idling time $I_{2,j+1}^*$ and the blocking time $B_{1,j+M+1}^*$ are estimated by using the system state at $\sum_{n=1}^{j} O_{2,n}-0$ as follows. The idling time is occurred when the system state is $S_j^*(W_1,W_2|0)$ and changes to $S_{j+1}^*(W_1,W_2|0)$ or when the system state is $S_j^*(W_1,W_2|1)$ and changes to $S_{j+1}^*(W_1,W_2|0)$, and it is not occurred in the other cases. Therefore $I_{2,j+1}^*$ is represented by

$$(5.7)\qquad I^*_{2,j+1} = [X_{1,j+1}-X_{2,j+1}]^+ \text{ or}$$
$$= [\delta^*_{1,j+1}-X_{2,j+1}]^+ \text{ or}$$
$$= 0.$$

And the blocking time $B^*_{1,j+M+1}$ is occurred when the system state is $S^*_j(W_1,W_2|0)$ and changes to $S^*_{j+1}(B_1,W_2|M)$, when the state is $S^*_j(W_1,W_2|m)$ $(1 \le m \le M)$ and changes to $S^*_{j+1}(B_1,W_2|M)$ or when the state is $S^*_j(B_1,W_2|M)$ and changes to $S^*_{j+1}(B_1,W_2|M)$, and it is not occurred in the other cases, therefore it is represented by

$$(5.8)\qquad B^*_{1,j+M+1} = [X_{2,j+1} - \sum_{n=1}^{M+1} X_{1,j+n}]^+ \text{ or}$$
$$= [X_{2,j+1} - \delta^*_{1,j+m} - \sum_{n=m+1}^{M+1} X_{1,j+n}]^+ \text{ or}$$
$$= [X_{2,j+1} - X_{1,j+M+1}]^+ \text{ or}$$
$$= 0.$$

Therefore by using the same procedure in Section 4 the idling time distribution $w^*_I(x)$ $(x > 0)$ and the blocking time distribution $w^*_B(x)$ $(x > 0)$ in the steady state condition are given by

$$(5.9)\qquad W^*_I(x) = (P^*_0 + P^*_1)\cdot\bar{f}^*(x)$$
$$w^*_B(x) = (P^*_0 + P^*_1)\cdot\bar{f}^*_{M+1}(x) + \sum_{m=2}^{M+1} P^*_m\cdot\bar{f}^*_{M+2-m}(x),$$
$$\bar{f}^*(x):\quad \int_0^\infty f^*_2(y)f^*_1(y+x)\,dy$$
$$\bar{f}^*_m(x):\quad \int_0^\infty f^{*(m)}_1(y)f^*_2(y+x)\,dy,$$

where $f^{*(m)}_1(x)$ denotes the m times convolution function of $f^*_1(x)$. From (4.6), (5.5) and (5.9) we have the following equalities,

$$(5.10)\qquad w^*_I(x) = w_B(x)$$
$$w^*_B(x) = w_I(x).$$

Finally the mean number $\bar{N}^*$ of the in-process works is represented by

$$(5.11)\qquad \bar{N}^* = \sum_{m=0}^{M} m\cdot P^*(W_1,W_2|m) + M\cdot P^*(B_1,W_2|M)$$

and from (4.8), (5.6) and (5.11) we have

(5.12) $\bar{N}^* = M - \bar{N}$.

6. Conclusion

We have discussed the procedure to estimate the effects of the design factors in the two-stage series lines in which the first stage has the arbitrary service time. And the various line characters are obtained and the relations between the dual models are presented. For example, we represent the mean idling times for the various two-stage series lines in Table 6.1 and 2.

Table 6.1 Numerical results of $\bar{B}_1$ for the balancing line (λ_2=1.0)

M	Service Time Distribution $f_1(x)$					
	K-Erlang				Uniform	Constant
	K=1	K=2	K=4	K=5	$0.5 \leq x \leq 1.5$	x=1.0
0	0.5000	0.4444	0.4096	0.4019	0.3834	0.3679
1	0.3333	0.2807	0.2497	0.2428	0.2268	0.2141
2	0.2500	0.2045	0.1785	0.1731	0.1600	0.1500
3	0.2000	0.1607	0.1389	0.1343	0.1235	0.1154
4	0.1667	0.1324	0.1136	0.1098	0.1006	0.0938
5	0.1429	0.1125	0.0962	0.0928	0.0848	0.0790
6	0.1250	0.0978	0.0833	0.0804	0.0734	0.0682
7	0.1111	0.0865	0.0735	0.0709	0.0646	0.0600
8	0.1000	0.0776	0.0658	0.0634	0.0577	0.0536

Table 6.2 Numerical results of $\bar{B}_1$ for the unbalancing line (λ_1=0.5, λ_2=1.0)

M	Service Time Distribution $f_1(x)$					
	K-Erlang				Uniform	Constant
	K=1	K=2	K=4	K=5	$1.0 \leqq x \leqq 3.0$	x=2.0
0	0.3333	0.2500	0.1975	0.1859	0.1590	0.1353
1	0.1429	0.0833	0.0530	0.0471	0.0346	0.0251
2	0.0667	0.0303	0.0154	0.0129	0.0081	0.0050
3	0.0323	0.0114	0.0046	0.0036	0.0019	0.0010
4	0.0159	0.0043	0.0014	0.0010	0.0005	0.0002
5	0.0079	0.0016	0.0004	0.0003	0.0001	
6	0.0039	0.0006	0.0001	0.0001		
7	0.0020	0.0002				
8	0.0010	0.0001				

λ_i: The operation rate in stage i

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