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ALGORITHMS FOR A VARIANT OF THE RESOURCE ALLOCATION PROBLEM

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Abstract In this paper we consider the following variant of the resource allocation problem:

$$\text{Maximize } \sum_{i=1}^{n} x_i \text{ subject to } \sum_{i=1}^{n} f_i(x_i) \leq R, \text{ and } x_i\text{: nonnegative integers,}$$

where f_i's are nondecreasing convex functions defined over $[0, \infty)$ and satisfy $\sum_{i=1}^{n} f_i(0) \leq R$ and $\lim_{x_i \to \infty} f_i(x_i) = \infty$. R is a real number. We present three algorithms for this problem. The first one requires $O(N^* \log n + n)$ time, where N^* denotes the optimal objective value. The second one requires $O(n^2 (\log \bar{N})^2)$ time, where $\bar{N}$ denotes a given upper bound of the optimal value. The third one requires $O(b(n, R) + n \log n)$ time, where $b(n, R)$ denotes the computational time required to solve the continuous problem obtained from the original one by dropping the integrality condition on x_i's.

1. Introduction

The following resource allocation problem has been studied well.

$$(1.1) \quad \begin{aligned} P\text{: minimize } & z(x) = \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to } & \sum_{i=1}^{n} x_i = N \text{ and } x_i\text{: nonnegative integers,} \end{aligned}$$

where f_i's are convex functions defined over $[0, N]$ and N is a positive integer. This problem is also known as the distribution of efforts problem in production planning. Its properties have been discussed in many papers over the last years (Johnson [5], Karush [7], Veinott [13], Fox [4], Weinstein and Yu [16], Shih [12], Mjelde [9], Kao [6], Proll [10], Dunstan [3], Katoh, Ibaraki and Mine [8]; Veinott [14] contains additional references). In addition to the standard procedure of dynamic programming (e.g., [7] and textbooks [2, 15]), three more efficient algorithms have been proposed. The first one is based on the incremental analysis [13, 4, 12]. If each evaluation of $f_i(x_i)$

is done in constant time, an appropriate implementation of the incremental method requires $O(N \log n+n)$ running time. Two other methods are based on the Lagrange multiplier method [8, 16]; one algorithm requires $O(n^2(\log N)^2)$ time† and the other requires $O(c(n, N)+n \log n)$ time, where $c(n, N)$ is the time required to solve the continuous problem P' obtained from P by dropping the integrality condition on x_i's.

In this paper, we consider the following variant of the resource allocation problem.

$$\text{(1.2)}\quad \begin{aligned} &Q:\ \text{maximize } \sum_{i=1}^{n} x_i \\ &\quad\text{subject to } z(x) = \sum_{i=1}^{n} f_i(x_i) \le R \text{ and } x_i\text{: nonnegative integers,} \end{aligned}$$

where f_i's are nondecreasing convex functions defined over $[0, \infty)$, and satisfy $\sum_{i=1}^{n} f_i(0) \le R$ and $\lim_{x_i \to \infty} f_i(x_i) = \infty$ (these guarantee the existence of an optimal solution). This variant Q seems to be equally meaningful in most practical situations where the resource allocation problem P plays a crucial role.

We present three algorithms for solving Q, which are based on the incremental method and two versions of the Lagrange multiplier method for P. They require $O(N^* \log n+n)$, $O(n^2(\log \bar{N})^2)$ and $O(b(n, R)+n \log n)$ time, respectively††, where N^* denotes the optimal objective value, $\bar{N}$ is an upper bound of N^*, and $b(n, R)$ is the time to solve the continuous version of Q (i.e., integrality condition is dropped). Algorithm 1 is recommended if N^* is not very large compared with n; otherwise Algorithms 2 and 3 are preferred. Algorithm 3 is superior to Algorithm 2 if $b(n, R) < O(n^2(\log \bar{N})^2)$, which are possible for various important $f_i(x_i)$'s such as $f_i(x_i) = a_i x_i^k$.

2. Notations and Basic Concepts

Let $x^* = (x_1^*, \ldots, x_n^*)$ denote an optimal solution of Q, and let $x' = (x_1', \ldots, x_n')$ denote an optimal solution of Q', which is the same as Q except that

† Added in proof. The recent paper (Z. Galil and N. Megiddo: A Fast Selection Algorithm and the Problem of Optimum Distribution of Effort, J. of ACM, Vol. 26 (1979), pp. 58-64) has shown that this time bound can be further reduced to $O(n(\log N)^2)$ by using a sofisticated data handling method.

†† The second time bound for Algorithm 2 can be reduced to $O(n(\log \bar{N})^2)$ if we use Galil and Megiddo's result in place of the algorithm of [8], as the starting algorithm for constructing Algorithm 2.

the integrality condition on x_i's has been dropped. Let

$$(2.1) \qquad d_i(x_i) = f_i(x_i+1) - f_i(x_i),$$

where x_i is a nonnegative integer. Here $d_i(-1)=0$ is assumed for convenience. Note that $0 = d_i(-1) \le d_i(0) \le d_i(1) \le \dots$ holds since $f_i(x_i)$ is convex and non-decreasing. We assume throughout this paper that each $f_i(x_i)$ can be evaluated in constant time. Also we assume that $\sum_{i=1}^{n} f_i(0) < R$ and $\lim_{x_i\to\infty} f_i(x_i) = \infty$ for all i. Therefore, Q and Q' are feasible and have bounded optimal solutions. Furthermore, we sometimes assume in the subsequent discussion that $\sum_{i=1}^{n} f_i(0) < R$; if $\sum_{i=0}^{n} f_i(0) = R$ then $x_i' = \max\{x_i \mid f_i(x_i) = f_i(0)\}$ $(i=1, 2, \dots, n)$ obviously give an optimal solution of Q'.

The subgradient of f_i at x_i is a set of real numbers defined by

$$(2.2) \qquad \frac{\partial f_i(x_i)}{\partial x_i} = \{p \mid f_i(z) - f_i(x_i) \ge p(z - x_i) \text{ for any } z \text{ in the domain of } f_i\}.$$

If $f_i(x_i)$ is differentiable at x_i, the subgradient $\frac{\partial f_i(x_i)}{\partial x_i}$ is equivalent to the ordinary derivative $\frac{df_i(x_i)}{dx_i}$. The following well-known property in the convex analysis (e.g., Rockafellar [11]) provides a basis of our algorithms.

Proposition 2.1. A solution $x' = (x_1', \dots, x_n')$ of Q' is optimal if and only if there exists λ' satisfying

$$(2.3) \qquad 0 \in \frac{\partial L(x', \lambda')}{\partial x_i} \quad \text{for all } i,$$

$$(2.4) \qquad x_i' \ge 0 \quad \text{for all } i, \text{ and}$$

$$(2.5) \qquad \sum_{i=1}^{n} f_i(x_i') = R,$$

where $L(x, \lambda) = \sum_{i=1}^{n} x_i + \lambda(\sum_{i=1}^{n} f_i(x_i) - R)$.

3. Algorithm 1

In this section we propose an algorithm for solving Q, which is based on the incremental property used in [13, 4, 12]: Let $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ be an optimal solution of P under the constraint $\sum_{i=1}^{n} x_i = k$ with $0 \le k \le N$ and let j_k be the index satisfying

$$(3.1) \qquad d_{j_k}(x_{j_k}^{(k)}) = \min_{1 \le i \le n} d_i(x_i^{(k)}).$$

Then an optimal solution $x^{(k+1)}$ is given by $(x_1^{(k)}, \ldots, x_{j_k-1}^{(k)}, x_{j_k}^{(k)}+1, x_{j_k+1}^{(k)}, \ldots, x_n^{(k)})$.

Lemma 3.1. Let $x^{(k)}$ be an optimal solution of P under the constraint $\sum_{i=1}^{n} x_i = k$, and let $z^{(k)} = \sum_{i=1}^{n} f_i(x_i^{(k)})$. Then $x^{(k)}$ is an optimal solution of Q under the constraint $\sum_{i=1}^{n} f_i(x_i) \leq r$ for any r with $z^{(k)} \leq r < z^{(k+1)}$.

Proof: It is easy to see that $x^{(k)}$ is a feasible solution of Q under the constraint $\sum_{i=1}^{n} f_i(x_i) \leq r$ satisfying $z^{(k)} \leq r < z^{(k+1)}$. The lemma immediately follows since any solution x with $\sum_{i=1}^{n} x_i \geq k+1$ does not satisfy $\sum_{i=1}^{n} f_i(x_i) < z^{(k+1)}$ by the minimality of $z^{(k+1)}$ and the monotonicity of f_i. □

Now we give Algorithm 1, which obtains $x^{(0)}, x^{(1)}, \ldots, x^{(k)}$ by applying the above incremental method until the condition $\sum_{i=1}^{n} f_i(x_i^{(k)}) > R$ is first satisfied; $x^{(k-1)}$ is an optimal solution of Q by Lemma 3.1.

Algorithm 1

Step 1: Let $x^{(0)} \leftarrow (\overbrace{0, 0, \ldots, 0}^{n})$, $k \leftarrow 0$, $z^{(k)} \leftarrow \sum_{i=1}^{n} f_i(0)$.

Step 2: With j_k satisfying $d_{j_k}(x_{j_k}^{(k)}) = \min_{1 \leq i \leq n} d_i(x_i^{(k)})$, let
$x^{(k+1)} \leftarrow (x_1^{(k)}, \ldots, x_{j_k-1}^{(k)}, x_{j_k}^{(k)}+1, x_{j_k+1}^{(k)}, \ldots, x_n^{(k)})$.

Step 3: Let $z^{(k+1)} \leftarrow z^{(k)} + d_{j_k}(x_{j_k}^{(k)})$ and $k \leftarrow k+1$. If $z^{(k)} > R$, then $x^* \leftarrow x^{(k-1)}$ and stop (an optimal solution is found). Else return to Step 2.

Theorem 3.2. Algorithm 1 generates an optimal solution x^* of Q in $O(N^* \log n + n)$ steps, where N^* denotes the optimal value $N^* = \sum_{i=1}^{n} x_i^*$ of Q.

Proof: Correctness is obvious by the above discussion and Lemma 3.1. Step 1 requires $O(n)$ steps. If $k = 0$, Step 2 requires $O(n)$ steps since $x^{(1)} = (0, \ldots, 1, 0, \ldots, 0)$ is obtained in $O(n)$ steps by computing $d_i(0)$ for all i and then finding $d_{j_0}(0) = \min_{1 \leq i \leq n} d_i(0)$. For $k \geq 1$, Step 2 requires $O(\log n)$ steps since the index j_k of Step 2 is computed in $O(\log n)$ steps if data $d_i(x_i^{(k)})$ ($i = 1, 2, \ldots, n$) are appropriately manipulated (e.g., using the technique of heap [1]) by using the fact that $d_i(x_i^{(k)}) = d_i(x_i^{(k-1)})$ holds for all $i \neq j_{k-1}$ and the evaluation of each $d_i(x_i^{(k)})$ requires constant time by assumption. Since Steps 2 and 3 are repeated N^*+1 times, Algorithm 1 requires $O(N^* \log n + n)$ steps in total. □

4. Algorithm 2

We assume in this section that an upper bound $\overline{N}$ of the optimal objective value of Q is given in advance. $\overline{N}$ is typically given by $\sum_{i=1}^{n}\max\{x_i \mid f_i(x_i) \le R,$ x_i: integer$\}$ or by the optimal value of Q'. Let $\hat{f}_i(x_i)$ be the convex function which is the piecewise linear approximation of $f_i(x_i)$ on integer lattice points:

$$\text{(4.1)}\quad \begin{aligned}\hat{f}_i(x_i) &= f_i(x_i) \text{ if } x_i \text{ is an integer,}\\ &= f_i(\lfloor x_i\rfloor) + d_i(\lfloor x_i\rfloor)(x_i - \lfloor x_i\rfloor) \text{ otherwise,}\end{aligned}$$

where $\lfloor x_i\rfloor$ denotes the integer part of x_i. Let $\hat{Q}$ and $\hat{Q}'$ be the same as Q and Q' except that $f_i(x_i)$ is replaced by $\hat{f}_i(x_i)$ for all $i = 1, 2, \ldots, n$, and let $\hat{x}^*$ and $\hat{x}'$ be optimal solutions of $\hat{Q}$ and $\hat{Q}'$, respectively. Note that $\hat{x}^* = x^*$ holds as $\hat{Q}$ and Q are identical as far as integer solutions are concerned. As shown below, $\hat{x}^*$ is computed in $O(n^2(\log \overline{N})^2)$ steps by applying the method of [8] with some modification.

By (2.2) and (4.1),

$$\text{(4.2)}\quad \begin{aligned}\frac{\partial \hat{f}_i(x_i)}{\partial x_i} &= \{p \mid d_i(x_i-1) \le p \le d_i(x_i)\} \text{ if } x_i \text{ is an integer with } 0 \le x_i \le \overline{N}.\\ &= d_i(\lfloor x_i\rfloor) \text{ if } x_i \text{ is not an integer.}\end{aligned}$$

Letting

$$\hat{L}(x, \lambda) = \sum_{i=1}^{n} x_i + \lambda\left(\sum_{i=1}^{n} \hat{f}_i(x_i) - R\right),$$

we obtain the following result by Proposition 2.1.

Proposition 4.1. $\hat{x}' = (\hat{x}_1', \hat{x}_2', \ldots, \hat{x}_n')$ is an optimal solution of $\hat{Q}'$ if and only if there is $\hat{\lambda}'$ satisfying

$$\text{(4.3)}\quad 0 \in \frac{\partial \hat{L}(\hat{x}', \hat{\lambda}')}{\partial x_i} \text{ for all } i,$$

$$\text{(4.4)}\quad \hat{x}_i' \ge 0 \text{ for all } i, \text{ and}$$

$$\text{(4.5)}\quad \sum_{i=1}^{n} \hat{f}_i(\hat{x}_i') = R.$$

We remark here that (4.3) is equivalent to

$$\text{(4.6)}\quad \begin{aligned}&-d_i(\hat{x}_i') \le 1/\hat{\lambda}' \le -d_i(\hat{x}_i' - 1), \text{ if } \hat{x}_i' \text{ is an integer,}\\ &1/\hat{\lambda}' = -d_i(\lfloor \hat{x}_i'\rfloor), \text{ if } \hat{x}_i' \text{ is not an integer,}\end{aligned}$$

which can be easily proved from (4.2). Note that $\hat{\lambda}' < 0$ follows from (4.6).

Lemma 4.2. There exists an optimal Lagrange multiplier $\hat{\lambda}'$ of $\hat{Q}'$ which is equal to $-1/d_i(x_i)$ for some i and integer x_i with $1 \le i \le n$ and $0 \le x_i \le \overline{N}$. ($-1/d_i(x_i)$ is considered to be $-\infty$ if $d_i(x_i) = 0$.)

Proof: Let $\hat{x}'$ be an optimal solution of $\hat{Q}'$. If $\hat{x}'$ is not an integer solution, $1/\hat{\lambda}' = -d_i(\lfloor \hat{x}_i' \rfloor)$ holds by (4.6) for some i with non-integer $\hat{x}_i'$. Thus assume that $\hat{x}'$ is an integer solution. Let $1/\tilde{\lambda} = -d_j(\hat{x}_j') = \max\{-d_i(\hat{x}_i') \mid 1 \le i \le n\}$ (i.e., $\tilde{\lambda} = 1/-d_j(\hat{x}_j')$ and $\hat{x}_j'$ is an integer). It will be shown that this $\tilde{\lambda}$ is optimal. First $-d_i(\hat{x}_i') \le 1/\tilde{\lambda}$ holds for all i by definition. Assume $1/\tilde{\lambda} > -d_i(\hat{x}_i' - 1)$ for some i. Then $-d_i(\hat{x}_i') \le -d_i(\hat{x}_i' - 1) < -d_j(\hat{x}_j')$ $(=1/\tilde{\lambda})$. This shows that the Lagrange multiplier $\hat{\lambda}'$ satisfying (4.6) for all i does not exist, a contradiction to the optimality of $\hat{x}'$ by Proposition 4.1. Thus $1/\tilde{\lambda} \le -d_i(\hat{x}_i' - 1)$ for all i, and $\tilde{\lambda}$ is optimal by Proposition 4.1 and (4.6). □

Lemma 4.3. From any optimal solution $\hat{x}'$ of $\hat{Q}'$, we can obtain an optimal solution $\hat{x}^*$ of $\hat{Q}$ satisfying $\sum_{i=1}^{n} \hat{x}_i^* = \lfloor \sum_{i=1}^{n} \hat{x}_i' \rfloor$. This $\hat{x}^*$ is optimal to Q.

Proof: Let $\hat{x}' = (\hat{x}_1', \hat{x}_2', \ldots, \hat{x}_n')$ be an optimal solution of $\hat{Q}'$ which satisfies (4.3) - (4.5). Let $I = \{i \mid \hat{x}_i'$ is not an integer$\}$. Assume $I \neq \phi$, since otherwise the lemma is proved. By (4.3) (i.e., (4.6)),

$$(4.7) \qquad 1/\hat{\lambda}' = -d_i(\lfloor \hat{x}_i' \rfloor)$$

holds for all $i \in I$. Let $I = \{i_1, i_2, \ldots, i_m\}$, and let

$$(4.8) \qquad \begin{aligned} \tilde{x}_j &= \hat{x}_j' \text{ for } j \notin I, \\ \tilde{x}_i &= \lfloor \hat{x}_i' \rfloor + 1 \text{ for } i \in I_1, \\ \tilde{x}_i &= \lfloor \hat{x}_i' \rfloor \text{ for } i \in I_2, \end{aligned}$$

where I_1 and I_2 are subsets of I satisfying $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \phi$ and $I_2 = \{i_1, \ldots, i_k\}$. Here $k = \max\{\ell \mid \sum_{j=1}^{\ell} (\lfloor \hat{x}_{i_j}' \rfloor + 1) + \sum_{j=\ell+1}^{m} \lfloor \hat{x}_{i_j}' \rfloor \le \sum_{i \in I} \hat{x}_i' \}$. Then

$$\begin{aligned} \sum_{i \in I} \hat{f}_i(\tilde{x}_i) &= \sum_{i \in I_1} (\hat{f}_i(\lfloor \hat{x}_i' \rfloor) + d_i(\lfloor \hat{x}_i' \rfloor)) + \sum_{i \in I_2} \hat{f}_i(\lfloor \hat{x}_i' \rfloor) \\ &= \sum_{i \in I} \hat{f}_i(\lfloor \hat{x}_i' \rfloor) - (1/\hat{\lambda}') \sum_{i \in I} (\tilde{x}_i - \lfloor \hat{x}_i' \rfloor) \quad \text{(by (4.7))} \\ &\le \sum_{i \in I} \hat{f}_i(\lfloor \hat{x}_i' \rfloor) - (1/\hat{\lambda}') \sum_{i \in I} (\hat{x}_i' - \lfloor \hat{x}_i' \rfloor) \\ &\qquad \text{(by } \sum_{i \in I} (\tilde{x}_i - \lfloor \hat{x}_i' \rfloor) = |I_1| \le \sum_{i \in I} (\hat{x}_i' - \lfloor \hat{x}_i' \rfloor) \text{ and } -1/\hat{\lambda}' \ge 0) \\ &= \sum_{i \in I} \hat{f}_i(\lfloor \hat{x}_i' \rfloor) + \sum_{i \in I} d(\lfloor \hat{x}_i' \rfloor)(\hat{x}_i' - \lfloor \hat{x}_i' \rfloor) \\ &= \sum_{i \in I} \hat{f}_i(\hat{x}_i). \end{aligned}$$

Therefore, $\tilde{x}$ is feasible in $\hat{Q}$. Furthermore, $\lfloor \sum_{i=1}^{n} \hat{x}_i' \rfloor = \sum_{i=1}^{n} \tilde{x}_i$ holds from the definition of $\tilde{x}$. Thus $\tilde{x}$ is an optimal solution of $\hat{Q}$. Let $\hat{x}^* = \tilde{x}$. $\hat{x}^*$ is optimal to Q also, as discussed at the beginning of this section. □

We now describe Algorithm 2. The basic idea is to search an optimal multiplier λ among set $\{-1/d_i(x_i) \mid i=1, 2, \ldots, n, x_i=0, 1, \ldots, \bar{N}\}$ (see Lemma 4.2). For each λ thus generated the existence of $\hat{x}'$ satisfying conditions (4.3) - (4.5) is tested. If such $\hat{x}'$ exists, it provides an optimal solution of $\hat{Q}$ (and hence of Q) by Lemma 4.3. The test is done by computing the largest integer $x_i=\bar{x}_i$ and the smallest integer $x_i=\underline{x}_i$ satisfying $-d_i(x_i) \leq 1/\lambda \leq -d_i(x_i-1)$. A solution $\hat{x}'$ exists if and only if $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i)$ holds, as proved in Theorem 4.6. $\bar{x}_i$ and $\underline{x}_i$ are respectively computed by subroutines FMAX(i, λ) and FMIN(i, λ) in $O(\log \bar{N})$ steps by using the binary search technique (e.g., see Aho et al.[1]) over the integer set $\{0, 1, 2, \ldots, \bar{N}\}$. When $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i)$ is satisfied, an optimal solution x^* of $\hat{Q}$ (i.e., Q) is constructed from the obtained $\bar{x}_i$ and $\underline{x}_i$ in a way similar to the proof of Lemma 4.3.

The search of λ is also carried out by binary search. Initially m is set to 1, and λ is searched among the set of values $\{-1/d_m(0), -1/d_m(1), \ldots, -1/d_m(\bar{N})\}$ (note that $-1/d_m(0) \leq -1/d_m(1) \leq \ldots \leq -1/d_m(\bar{N})$ holds); starting with the middle one $-d_m(\lfloor \bar{N}/2 \rfloor)$, λ is decreased if $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i) > R$, and is increased if $\sum_{i=1}^{n} \hat{f}_i(\bar{x}_i) < R$. If none of the λ's generated for the current m satisfies $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i)$, m is increased by one and the same procedure is repeated. By Lemma 4.2, λ satisfying $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i)$ will be obtained before the search is completed for $m=1, 2, \ldots, n$.

Some notations are now introduced.

λ_k: The k-th Lagrange multiplier generated during computation.

$\underline{t}$, $\bar{t}$: Variables used to find the next choice of λ by the rule $\lambda = -1/d_m(\lfloor \frac{\underline{t}+\bar{t}}{2} \rfloor)$ according to binary search. Initially $\underline{t}$ is set to 0 and $\bar{t}$ is set to $\bar{N}$.

Algorithm 2 (It is assumed that $\sum_{i=1}^{n} \hat{f}_i(0) < R$; if $\sum_{i=1}^{n} \hat{f}_i(0) = R$, Q is easily solved as mentioned in Section 2.)

Step 1 [Initialization]: Set $k \leftarrow 1$, $m \leftarrow 1$.

Step 2: Set $\underline{t} \leftarrow 0$, $\bar{t} \leftarrow \bar{N}$, $t_k \leftarrow \lfloor \frac{\bar{t}+\underline{t}}{2} \rfloor$, $\lambda_k \leftarrow 1/-d_m(t_k)$.

Step 3 [Test for the optimality of λ_k]:

(1) For $i=1, 2, \ldots, n$, call FMIN(i,λ_k) to find $\underline{x}_i^k$ and call FMAX(i, λ_k) to find $\bar{x}_i^k$.

(2) If $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) > R$, then set $\bar{t} \leftarrow t_k$. Go to Step 4.

(3) Else if $\sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k) < R$, then set $\underline{t} \leftarrow t_k$. Go to Step 4.

(4) Else set $\hat{\lambda}' \leftarrow \lambda_k$. (In this case $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k)$ holds and an optimal Lagrange multiplier $\hat{\lambda}'$ is found.) Go to Step 5.

Step 4 [Find λ_{k+1}]: Set $k \leftarrow k+1$, $t_k \leftarrow \lfloor \frac{\underline{t}+\bar{t}}{2} \rfloor$. If $t_k = t_{k-1}$, then set $m \leftarrow m+1$ and return to Step 2. Else set $\lambda_k \leftarrow -1/d_m(t_k)$ and return to Step 3.

Step 5 [Find x^*]: Set $j \leftarrow \max\{\ell \mid \sum_{i=1}^{\ell} \hat{f}_i(\bar{x}_i^k) + \sum_{i=\ell+1}^{n} \hat{f}_i(\underline{x}_i^k) \le R\}$ (i.e., $j = n$ or $\sum_{i=1}^{j+1} \hat{f}_i(\bar{x}_i^k) + \sum_{i=j+2}^{n} \hat{f}_i(\underline{x}_i^k) > R$). For each i with $1 \le i \le j$, set $x_i^* \leftarrow \bar{x}_i^k$. For each i with $j+2 \le i \le n$, set $x_i^* \leftarrow \underline{x}_i^k$. Set $x_{j+1}^* \leftarrow \max\{x_{j+1} \mid x_{j+1}$ is an integer such that $\hat{f}_{j+1}(x_{j+1}) \le R - (\sum_{i=1}^{j} \hat{f}_i(\bar{x}_i^k) + \sum_{i=j+2}^{n} \hat{f}_i(\underline{x}_i^k))\}$.

Subroutine FMIN(i, λ_k): This routine finds

$$\underline{x}_i^k = \min\{x_i \in \{0, 1, \ldots, \bar{N}\} \mid -d_i(x_i) \le 1/\lambda_k \le -d_i(x_i - 1)\}$$

by binary search over the set $[0, \bar{N}]$, where $0 = -d_i(-1) \ge -d_i(0) \ge \ldots \ge -d_i(\bar{N})$ holds. The detail is not given here because this can be done by a direct application of the standard binary search technique. The running time is $O(\log \bar{N})$.

Subroutine FMAX(i, λ_k): This routine finds

$$\bar{x}_i^k = \max\{x_i \in \{0, 1, \ldots, \bar{N}\} \mid -d_i(x_i) \le 1/\lambda_k \le -d_i(x_i - 1)\}$$

by binary search in $O(\log \bar{N})$ steps.

Lemma 4.4. Algorithm 2 always finds a multiplier $\lambda = \lambda_k$ such that the associated $\underline{x}_i^k$ and $\bar{x}_i^k$ ($i = 1, 2, \ldots, n$) satisfy

$$\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) \le R \le \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k).$$

Proof is done in a manner similar to Lemma 4.1 of [8] by using Lemmas 4.2 and 4.3.

Lemma 4.5. Steps 3 and 4 of Algorithm 2 are repeated $O(\log \bar{N})$ times for each m (tested during computation) before $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) \le R \le \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k)$ holds in Step 3 or $t_k = t_{k-1}$ holds in Step 4.

Proof: Obvious since this is a straightforward application of binary search. □

Theorem 4.6. Algorithm 2 generates an optimal solution x^* of Q in $O(n^2 (\log \bar{N})^2)$ steps.

Proof: [Correctness] By Lemma 4.4, the algorithm correctly obtains λ_k for which $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) \le R \le \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k)$ is satisfied. This $\lambda_k = \hat{\lambda}'$ satisfies (4.3) and (4.4) of Proposition 4.1 for any x_i with $\underline{x}_i^k \le x_i \le \bar{x}_i^k$, and hence any solution $\hat{x}'$ satisfying $\sum_{i=1}^{n} \hat{f}_i(\hat{x}_i') = R$ (i.e., (4.5)) and $\underline{x}_i^k \le \hat{x}_i' \le \bar{x}_i^k$ $(1 \le i \le n)$ is optimal in $\hat{Q}'$. Such $\hat{x}'$ is given for example by $\hat{x}_i' = \bar{x}_i^k$, $i = 1, 2, \ldots, j$, $\hat{x}_i' = \underline{x}_i^k$, $i = j+2, \ldots, n$ and $\hat{x}_{j+1}' = \max\{x_{j+1} \mid \hat{f}_{j+1}(x_{j+1}) = R - (\sum_{i=1}^{j} \hat{f}_i(\bar{x}_i^k) + \sum_{i=j+2}^{n} \hat{f}_i(\underline{x}_i^k))\}$. Therefore the solution x^* constructed in Step 5 is optimal in $\hat{Q}$ (and hence in Q) since it is the same as $\tilde{x}$ constructed from the above $\hat{x}'$ in the proof of Lemma 4.3 (note that $I = \{j\}$ holds in this case).

[Computational Complexity] Steps 1 and 2 require constant time. By Lemma 4.5, Steps 3 and 4 are repeated $O(\log \bar{N})$ times for each tested m before

$t_k = t_{k-1}$ or $\sum_{i=1}^{n} \hat{f}_i(\underline{x}_i^k) \leq R \leq \sum_{i=1}^{n} \hat{f}_i(\bar{x}_i^k)$ holds. Since Step 3 calls FMAX and FMIN n times, it requires $O(n \log \bar{N})$ steps. Step 4 also requires constant steps (recall that each evaluation of $\hat{f}_i(x_i)$ is assumed to require constant time). Therefore, the loop consisting of Steps 3 and 4 requires $O(n(\log \bar{N})^2)$ steps for each m. Finally, since m is increased one by one from $m = 1$ and Step 5 is eventually reached for some m with $m \leq n$ (by Lemma 4.4), the loop consisting of Steps 2, 3 and 4 is repeated at most n times. Therefore, this loop requires $O(n^2(\log \bar{N})^2)$ steps in total. Step 5 requires $O(n + \log \bar{N})$ steps since an index j is obtained in $O(n)$ steps, x_i^* for $i \neq j+1$ are obtained in $O(n)$ steps and x^*_{j+1} is computed in $O(\log \bar{N})$ steps by using the binary search technique again. □

5. Algorithm 3

Since $f_i(x_i)$ is a nondecreasing convex function, the subgradient satisfies (e.g., [11])

(5.1) $$p_a \leq p_b \text{ for any } p_a \in \frac{\partial f_i(x_i^a)}{\partial x_i} \text{ and } p_b \in \frac{\partial f_i(x_i^b)}{\partial x_i} \text{ with } x_i^a < x_i^b,$$

(5.2) $$d_i(\lfloor x_i' \rfloor - 1) \leq p \text{ for any } p \in \frac{\partial f_i(\lfloor x_i' \rfloor)}{\partial x_i} \text{ with } x_i' \geq 1.$$

Let x' be an optimal solution of Q'. Then we have $0 \in \frac{\partial L(x', \lambda')}{\partial x_i}$ for all i by (2.3). This implies $1 + \lambda' p' = 0$ (i.e., $p' = -1/\lambda'$) holds for some $p' \in \frac{\partial f_i(x_i')}{\partial x_i}$, and hence for all i

(5.3) $$d_i(x_i) \leq d_i(\lfloor x_i' \rfloor - 1) \leq p \text{ (by (5.2))} \leq p' \text{ (by (5.1))} = -1/\lambda'$$
for any x_i with $0 \leq x_i \leq \lfloor x_i' \rfloor - 1$ (let $p = p'$ if x_i' is an integer).

Similarly, the following relation also holds for all i.

(5.4) $$d_i(x_i) \geq -1/\lambda' \text{ for any } x_i \text{ with } x_i \geq \lfloor x_i' \rfloor + 1.$$

An integer solution $\tilde{x}$ of Q (possibly infeasible) is now constructed from x'. Let I and J be a partition of the index set $\{1, 2, \ldots, n\}$ satisfying the following conditions.

(5.5) $$d_i(\lfloor x_i' \rfloor) \geq -1/\lambda' \text{ for } i \in I,$$

(5.6) $$d_j(\lfloor x_j' \rfloor) < -1/\lambda' \text{ for } j \in J.$$

Then $\tilde{x}$ and $\tilde{F}$ are given by

(5.7) $\quad \tilde{x}_i = \lfloor x'_i \rfloor$ for $i \in I$,

(5.8) $\quad \tilde{x}_j = \lfloor x'_j \rfloor + 1$ for $j \in J$,

(5.9) $\quad \tilde{R} = \sum_{i=1}^{n} f_i(\tilde{x}_i)$.

Lemma 5.1. $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ is an optimal solution of Q with the constraint $\sum_{i=1}^{n} f_i(x_i) \le \tilde{R}$.

Proof: Suppose that $\tilde{x}$ is not optimal. Then there exists an optimal solution x^* such that $\sum_{i=1}^{n} x_i^* > \sum_{i=1}^{n} \tilde{x}_i$.

Case 1: $x_{j_1}^* > \tilde{x}_{j_1}$ holds for some $j_1 \in J$. If $x_i^* \ge \tilde{x}_i$ for all i with $1 \le i \le n$, x^* is not feasible since $d_{j_1}(x_{j_1}^*) \ge d_{j_1}(\tilde{x}_{j_1}) \ge -1/\lambda' > d_{j_1}(\lfloor x'_{j_1} \rfloor) \ge 0$ by (5.4) and (5.6), and hence

$$z(x^*) - \tilde{R} = z(x^*) - z(\tilde{x}) = \sum_{i=1}^{n} (f_i(x_i^*) - f_i(\tilde{x}_i)) \ge f_{j_1}(x_{j_1}^*) - f_{j_1}(\tilde{x}_{j_1}) \ge d_{j_1}(x_{j_1}^*) > 0$$

Thus consider the following two cases.

Case 1A: Some $i_1 \in I$ satisfies $x_{i_1}^* < \tilde{x}_{i_1}$. Let x'' be the solution such that $x''_{i_1} = x_{i_1}^* + 1$, $x''_{j_1} = x_{j_1}^* - 1$ and $x''_k = x_k^*$ for $k \ne i_1, j_1$. Since $d_{i_1}(x_{i_1}^*) \le d_{i_1}(\tilde{x}_{i_1} - 1) = d_i(\lfloor x'_i \rfloor - 1) \le -1/\lambda'$ by (5.3) and $d_{j_1}(x_{j_1}^* - 1) \ge d_{j_1}(\tilde{x}_{j_1}) = d_{j_1}(\lfloor x'_{j_1} \rfloor + 1) \ge -1/\lambda'$ by (5.4), we have $z(x^*) - z(x'') = d_{j_1}(x_{j_1}^* - 1) - d_{i_1}(x_{i_1}^*) \ge 0$, i.e., x'' is feasible. Thus x'' is also optimal since x^* and x'' have the same objective value. Repeating this operation, an optimal solution satisfying the lemma will be eventually obtained.

Case 1B: Some $j_2 \in J$ satisfies $x_{j_2}^* < \tilde{x}_{j_2}$. Let x'' be a solution defined by $x''_{j_1} = x_{j_1}^* - 1$, $x''_{j_2} = x_{j_2}^* + 1$ and $x''_k = x_k^*$ for $k \ne j_1, j_2$. Since $d_{j_1}(x_{j_1}^* - 1) \ge d_{j_1}(\tilde{x}_{j_1}) = d_{j_1}(\lfloor x'_{j_1} \rfloor + 1) \ge -1/\lambda'$ by (5.4) and $d_{j_2}(x_{j_2}^*) \le d_{j_2}(\tilde{x}_{j_2} - 1) = d_{j_2}(\lfloor x'_{j_2} \rfloor) < -1/\lambda'$ by (5.6), $z(x^*) - z(x'') = d_{j_1}(x_{j_1}^* - 1) - d_{j_2}(x_{j_2}^*) > 0$. Thus x'' is feasible. Apply the same argument as Case 1A to x''.

Case 2: $x_{i_1}^* > \tilde{x}_{i_1}$ holds for some $i_1 \in I$. First note that it is not possible to have $x_i^* \ge \tilde{x}_i$ for all i with $1 \le i \le n$, as shown next. If $d_{i_1}(\tilde{x}_{i_1}) > 0$, we have $z(x^*) - \tilde{R} = z(x^*) - z(\tilde{x}) = \sum_{i=1}^{n} (f_i(x_i^*) - f_i(\tilde{x}_i)) \ge d_{i_1}(\tilde{x}_{i_1}) > 0$. Thus x^* is infeasible. On the other hand, if $d_i(\tilde{x}_i) = 0$, the solution $x^{\#} = (x'_1, \ldots, \lfloor x'_{i_1} \rfloor + 1, \ldots, x'_n)$ is also feasible in Q' (since $z(x^{\#}) - z(x') = \sum_{i=1}^{n} f_i(x_i^{\#}) - \sum_{i=1}^{n} f_i(x'_i) = f_{i_1}(x_{i_1}^{\#}) - f_{i_1}(x'_{i_1}) \le d_{i_1}(\tilde{x}_{i_1}) = 0$) and satisfies $\sum_{i=1}^{n} x_i^{\#} > \sum_{i=1}^{n} x'_i$, a contradiction to the optimality of x'. Thus consider the following two cases.

Case 2A: Some $j_1 \in J$ satisfies $x_{j_1}^* < \tilde{x}_{j_1}$. Let x'' be a solution defined by $x''_{i_1} = x_{i_1}^* - 1$, $x''_{j_1} = x_{j_1}^* + 1$ and $x''_k = x_k^*$ for $k \ne i_1, j_1$. Since $d_{i_1}(x_{i_1}^* - 1) \ge d_{i_1}(\tilde{x}_{i_1}) \ge -1/\lambda'$ and $d_{j_1}(x_{j_1}^*) \le d_{j_1}(\tilde{x}_{j_1} - 1) = d_{j_1}(\lfloor x'_{j_1} \rfloor) < -1/\lambda'$, $z(x^*) - z(x'') = d_{i_1}(x_{i_1}^*) - d_{j_1}(x_{j_1}^*) > 0$. Thus x'' is also feasible. Apply the argument of Case 1A to x''.

Case 2B: Some $i_2 \in I$ satisfies $x^*_{i_2} < \tilde{x}_{i_2}$. Let x'' be a solution defined by $x''_{i_1} = x^*_{i_1} - 1$, $x''_{i_2} = x^*_{i_2} + 1$ and $x''_k = x^*_k$ for $k \neq i_1, i_2$. Since $d_{i_1}(x^*_{i_1} - 1) \geq d_{i_1}(\tilde{x}_{i_1}) = d_{i_1}(\lfloor x'_{i_1} \rfloor) \geq -1/\lambda'$ and $d_{i_2}(x^*_{i_2}) \leq d_{i_2}(\tilde{x}_{i_2} - 1) = d_{i_2}(\lfloor x'_{i_2} \rfloor - 1) \leq -1/\lambda'$ by (5.3) and (5.6). Thus x' is also feasible. Apply the argument of Case 1A to x''. □

Now we give the outline of Algorithm 3. It first solves the continuous Problem Q' in $b(n, R)$ steps (e.g., $b(n, R) = O(n)$ if $f_i(x_i) = a_i x_i^k$, as easily shown). If x' is an integer solution, it is an optimal solution of Q. Otherwise it computes I, J, $\tilde{x}$ and $\tilde{R}$. This obviously requires $O(n)$ time. If $R = \tilde{R}$, $\tilde{x}$ is an optimal solution of Q by Lemma 5.1. Finally, two cases remain.

(i) $\tilde{R} < R$. Then $\tilde{x}$ is an optimal solution to problem Q with R replaced by $\tilde{R}$. Now apply Algorithm 1 after assigning $x^{(0)}$ and $z^{(0)}$ as follows.

(5.10) $\quad x^{(0)} \leftarrow \tilde{x},\ z^{(0)} \leftarrow \tilde{R}.$

(ii) $\tilde{R} > R$. Then, starting with $x^{(0)} = \tilde{x}$ and $z^{(0)} = \tilde{R}$, apply Algorithm 1 in the reverse way according to the following modified algorithm.

Modified Algorithm 1

Step 1: Let $x^{(0)} \leftarrow \tilde{x}$, $k \leftarrow 0$, $z^{(k)} \leftarrow \tilde{R}$.

Step 2: With j_k satisfying $d_{j_k}(x^{(k)}_{j_k} - 1) = \max\{d_i(x^{(k)}_i - 1) \mid 1 \leq i \leq n,\ x^{(k)}_i \geq 1\}$, let

$$x^{(k+1)} \leftarrow (x^{(k)}_1, \ldots, x^{(k)}_{j_k-1}, x^{(k)}_{j_k} - 1, x^{(k)}_{j_k+1}, \ldots, x^{(k)}_n).$$

Step 3: Let $z^{(k+1)} \leftarrow z^{(k)} - d_{j_k}(x^{(k)}_{j_k} - 1)$ and $k \leftarrow k+1$. If $z^{(k)} \leq R$, then $x^* \leftarrow x^{(k)}$ and stop (an optimal solution is found). Else return to Step 2.

Algorithm 3

Step 1: Given a problem Q of (1.2), obtain an optimal solution x' and the associated optimal Lagrange multiplier λ' of the continuous problem Q'. Then compute

$$\tilde{x}_i = \lfloor x'_i \rfloor \quad \text{if } d_i(\lfloor x'_i \rfloor) \geq -1/\lambda'$$
$$= \lfloor x'_i \rfloor + 1 \quad \text{otherwise,}$$
$$\tilde{R} = \sum_{i=1}^{n} f_i(\tilde{x}_i).$$

Step 2: (1) If $\tilde{R} = R$, then $x^* \leftarrow \tilde{x}$. Halt.

(2) If $\tilde{R} < R$, apply Algorithm 1 after replacing the initial $x^{(0)}$ and $z^{(0)}$ in Step 1 by $\tilde{x}$ and $\tilde{R}$ respectively.

(3) If $\tilde{R} > R$, apply the modified Algorithm 1.

Theorem 5.2. Algorithm 3 generates an optimal solution x^* of Q in $O(b(n, R) + n \log n)$ steps, where $b(n, R)$ is the computational time required to solve Q'.

Proof: The correctness is first proved. By Lemma 5.1, $\tilde{x}$ is an optimal solution of Q with R replaced by $\tilde{R}$. If $R=\tilde{R}$, therefore, x^* obtained in Step 2 (1) is obviously optimal. If $\tilde{R}<R$, the argument of Section 3 ((3.1) and Lemma 3.1) can be started from $\tilde{x}$. The optimality of x^* obtained in Step 2 (2) can be similarly proved. Finally if $\tilde{R}>R$, the argument of Section 3 need be reversed. We omit the details, however, since it is exactly parallel to the argument of Section 3.

Next we analyze the computational time. Step 1 requires $b(n, R)$ time to compute x' and $O(n)$ time to obtain $\tilde{x}$ and $\tilde{R}$. Next note that $|\Sigma x'_i - \Sigma\tilde{x}_i| \leq \Sigma|x'_i - \tilde{x}_i| \leq n$ holds by $|x'_i - \tilde{x}_i| \leq 1$. Furthermore $\Sigma x'_i \geq \Sigma x^*_i \geq \Sigma\lfloor x'_i\rfloor > \Sigma x'_i - n$; the first inequality holds because Q' is a relaxation problem of Q, and the second inequality follows from the fact that $\lfloor x'\rfloor = (\lfloor x'_1\rfloor, \ldots, \lfloor x'_n\rfloor)$ is a feasible solution of Q. Thus $|\Sigma x^*_i - \Sigma\tilde{x}_i| < 2n$. This implies that Step 2 (2) or (3) requires at most $O(n)$ iterations of the loop consisting of Steps 2 and 3 of Algorithm 1 or the modified Algorithm 1. Since each iteration is done in $O(\log n)$ time as mentioned in Section 3, Step 2 of Algorithm 2 requires $O(n \log n)$ time. Thus the total computational time of Algorithm 3 is $O(b(n, R) + n\log n)$. □

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